Chapter 1: Interval computation

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1 Notions on set theory

Exercise: If $\mathbb{X} = \{a, b, c, d\}$ and $\mathbb{Y} = \{b, c, x, y\}$, then

$$\begin{aligned} \mathbb{X} \cap \mathbb{Y} &= \text{?} \\ \mathbb{X} \cup \mathbb{Y} &= \text{?} \\ \mathbb{X} \setminus \mathbb{Y} &= \text{?} \\ \mathbb{X} \times \mathbb{Y} &= \text{?} \end{aligned}$$

Exercise: If $\mathbb{X} = \{a, b, c, d\}$ and $\mathbb{Y} = \{b, c, x, y\}$, then

$$\begin{aligned}
\mathbb{X} \cap \mathbb{Y} &= \{b, c\} \\
\mathbb{X} \cup \mathbb{Y} &= \{a, b, c, d, x, y\} \\
\mathbb{X} \setminus \mathbb{Y} &= \{a, d\} \\
\mathbb{X} \times \mathbb{Y} &= \{(a, b), (a, c), (a, x), (a, y), \\
&\quad \ldots, (d, b), (d, c), (d, x), (d, y)\}
\end{aligned}$$

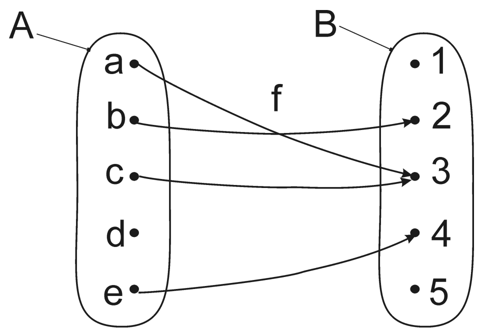
The *direct image* of $\mathbb{X}$ by f is

$$f(\mathbb{X}) \triangleq \{f(x) \mid x \in \mathbb{X}\}.$$

The *reciprocal image* of $\mathbb{Y}$ by f is

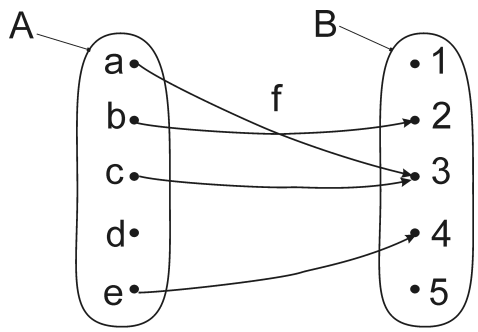
$$f^{-1}(\mathbb{Y}) \triangleq \{x \in \mathbb{X} \mid f(x) \in \mathbb{Y}\}.$$

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= ?. \\ f^{-1}(B) &= ?. \\ f^{-1}(f(A)) &= ? \\ f^{-1}(f(\{b, c\})) &= ?. \end{aligned}$$

Exercise: If f is defined as follows



$$\begin{aligned} f(A) &= \{2,3,4\} = \mathsf{Im}(f). \\ f^{-1}(B) &= \{a,b,c,e\} = \mathsf{dom}(f). \\ f^{-1}(f(A)) &= \{a,b,c,e\} \subset A \\ f^{-1}(f(\{b,c\})) &= \{a,b,c\}. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= ? \\ f^{-1}([4,9]) &= ?. \end{aligned}$$

Exercise: If $f(x) = x^2$, then

$$\begin{aligned} f([2,3]) &= [4,9] \\ f^{-1}([4,9]) &= [-3,-2] \cup [2,3]. \end{aligned}$$

This is consistent with the property

$$f^{-1}\left(f\left(\mathbb{Y}\right)\right) \supset \mathbb{Y}.$$

2 Interval arithmetic

If $\diamond \in \{+, -, \cdot, /, \max, \min\}$

$$[x] \diamond [y] \quad = \quad [\ \{x \diamond y \mid x \in [x], y \in [y]\}\ ].$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A} \subset \mathbb{R}$.

Exercise.

$$\begin{aligned}
[-1,3]+[2,5] &= [?,?],\\
[-1,3]\cdot[2,5] &= [?,?],\\
[-2,6]/[2,5] &= [?,?].
\end{aligned}$$

Solution.

$$
\begin{aligned}
[-1, 3] + [2, 5] &= [1, 8], \\
[-1, 3].[2, 5] &= [-5, 15], \\
[-2, 6]/[2, 5] &= [-1, 3].
\end{aligned}
$$

Exercise. Compute

$$[-2,2]/[-1,1] \quad = [?,?].$$

Solution.

$$[-2, 2]/[-1, 1] \quad = [-\infty, \infty].$$

$$
\begin{aligned}
[x^-, x^+] + [y^-, y^+] = \quad & [x^- + y^-, x^+ + y^+], \\
[x^-, x^+] \cdot [y^-, y^+] = \quad & [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+, \\
& x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+],
\end{aligned}
$$

If $f \in \{$cos, sin,sqr, sqrt, log, exp, $\ldots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Exercise.

$$\begin{aligned} \operatorname{sin}([0,\pi]) &= ?, \\ \operatorname{sqr}([-1,3]) &= [-1,3]^2 = ?, \\ \operatorname{abs}([-7,1]) &= ?, \\ \operatorname{sqrt}([-10,4]) &= \sqrt{[-10,4]} = ?, \\ \operatorname{log}([-2,-1]) &= ?. \end{aligned}$$

Solution.

$$\begin{aligned}
\sin([0,\pi]) &= [0,1],\\
\operatorname{sqr}([-1,3]) &= [-1,3]^2 = [0,9],\\
\operatorname{abs}([-7,1]) &= [0,7],\\
\operatorname{sqrt}([-10,4]) &= \sqrt{[-10,4]} = [0,2],\\
\log([-2,-1]) &= \emptyset.
\end{aligned}$$

3 Boxes

A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

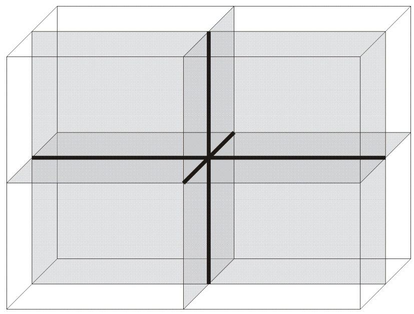
$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

The *width* $w([\mathbf{x}])$ is the length of the largest side.

$$w([1,2]\times[-1,3]) = 4$$

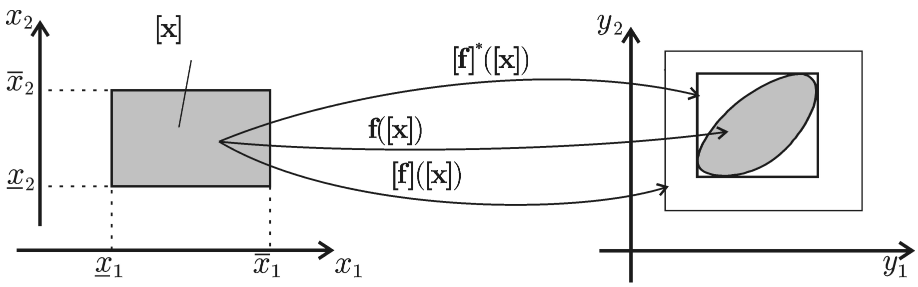
The *principal plane* of $[\mathbf{x}]$ is symmetric and perpendicular to the largest side.



4 Inclusion function

$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* of $\mathbf{f}$ if

$$\forall\, [\mathbf{x}] \in \mathbb{IR}^n, \quad \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

The inclusion function $[\mathbf{f}]$ is

monotonic	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([\mathbf{f}]([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{y}]))$
minimal	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [\mathbf{f}]([\mathbf{x}]) = [\mathbf{f}([\mathbf{x}])]$
thin	if	$w([\mathbf{x}]) = 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) = 0$
convergent	if	$w([\mathbf{x}]) \to 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) \to 0.$

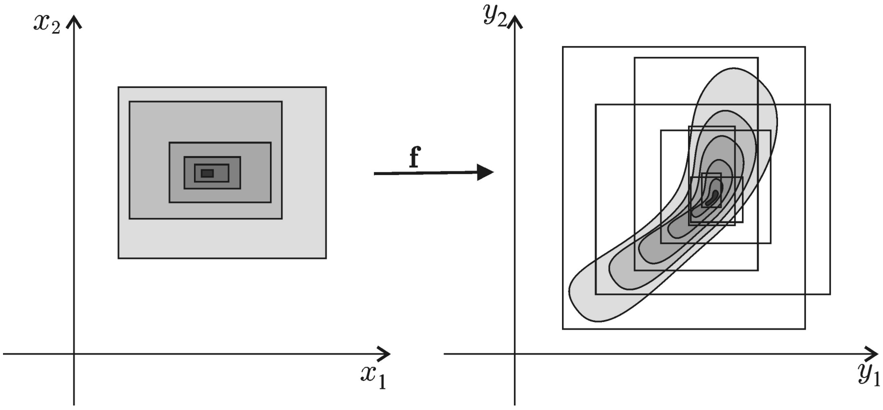
Exercise

The figure provides a nested sequence of boxes $[\mathbf{x}](k)$, their image $\mathbf{f}([\mathbf{x}])$ by a function $\mathbf{f}$ and the image by an inclusion function $[\mathbf{f}]$.

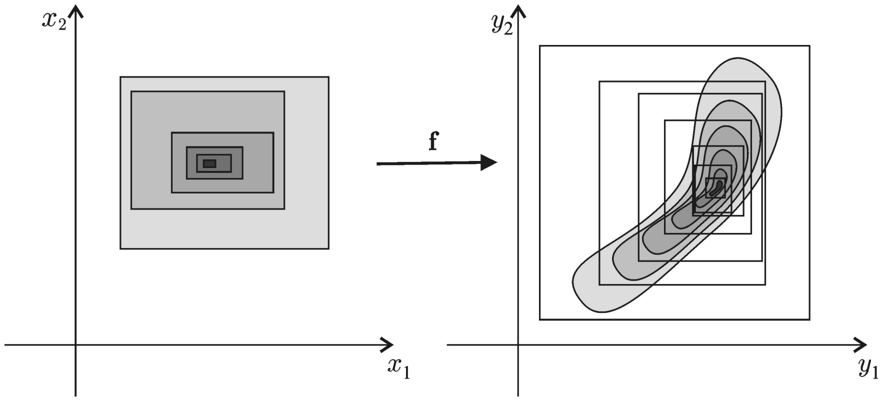
a) $[\mathbf{f}]$ is convergent.

b) $[\mathbf{f}]$ is monotonic

c) $[\mathbf{f}]$ is minimal.



Solution. $[\mathrm{f}]$ is convergent, non-monotonic, non-minimal.



Convergent ? monotonic ?

Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3, 4]$, we have

$$\begin{aligned}
[f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\
&= [0,16] + [-6,8] + 4 \\
&= [-2,28].
\end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{lcl} \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{lcl} \mathbb{IR}^2 & \rightarrow & \mathbb{IR}^3 \\ ([x_1], [x_2]) & \rightarrow & \left([x_1] * [x_2], [x_1]^2, [x_1] - [x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by

Algorithm $\mathbf{f}$(in: $\mathbf{x} = (x_1, x_2, x_3)$, out: $\mathbf{y} = (y_1, y_2)$)
$z := x_1$;
for $k := 0$ to 100
$z := x_2(z + k \cdot x_3)$;
next;
$y_1 := z$;
$y_2 := \sin(z x_1)$;

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in: $[\mathbf{x}]$, out: $[\mathbf{y}]$)

$[z] := [x_1]$;
for $k := 0$ to 100
    $[z] := [x_2] * ([z] + k \cdot [x_3])$;
next;
$[y_1] := [z]$;
$[y_2] :=$ sin$([z] * [x_1])$;

Is $[\mathbf{f}]$ convergent? thin? monotonic?

5 Boolean intervals

A *Boolean number* is an element of

$$\mathbb{B} \triangleq \{\mathit{false}, \mathit{true}\} = \{0, 1\}\,.$$

If we define the relation $\leq$ as

$$0 \leq 0, \qquad 0 \leq 1, \qquad 1 \leq 1,$$

then, the set $(\mathbb{B}, \leq)$ is a lattice for which intervals can be defined.

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{?, ?, ?, ?\},$$

Exercise: The set of *Boolean interval* is

$$\mathbb{IB} = \{\emptyset, 0, 1, [0, 1]\},$$

Boolean interval arithmetic

$$
\begin{aligned}
[a] \vee [b] &= \{a \vee b \mid a \in [a]\,, b \in [b]\}, \\
[a] \wedge [b] &= \{a \wedge b \mid a \in [a]\,, b \in [b]\}, \\
\neg\,[a] &= \{\neg a \mid a \in [a]\}.
\end{aligned}
$$

Exercise: Compute

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = ?$$

Solution: We have

$$([0,1] \vee 1) \wedge ([0,1] \wedge 1) = 1 \wedge [0,1] = [0,1].$$