Chapter 4: Robust parameter estimation

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Exercise. A robot measures its own distance to three marks. The distances and the coordinates of the marks are

mark	x_i	y_i	d_i
1	0	0	$[22, 23]$
2	10	10	$[10, 11]$
3	30	-30	$[53, 54]$

1) Define the set $\mathbb{X}$ al all feasible positions.
2) Build the contractor associated with $\mathbb{X}$.
2) Build the contractor associated with $\overline{\mathbb{X}}$.

Solution.

$$\mathbb{X} = \bigcap_{i \in \{1,2,3\}} \underbrace{\left\{(x,y) \;\middle|\; (x-x_i)^2 + (y-y_i)^2 \in \left[d_i^-, d_i^+\right]\right\}}_{\mathbb{X}_i}$$

$$
\begin{aligned}
\overline{\mathbb{X}} &= \overline{\bigcap_{i \in \{1,2,3\}} \mathbb{X}_i} = \bigcup_{i \in \{1,2,3\}} \overline{\mathbb{X}_i} \\
&= \bigcup_{i \in \{1,2,3\}} \Big\{(x, y) \;\mid\; (x - x_i)^2 + (y - y_i)^2 \in \Big[-\infty, d_i^-\Big] \\
&\qquad\qquad \cup \Big\{(x, y) \;\mid\; (x - x_i)^2 + (y - y_i)^2 \in \Big[d_i^+, \infty\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= \bigcap_{i\in\{1,2,3\}} \mathcal{D}_{\left[d_i^-,d_i^+\right]} \\
\overline{\mathcal{C}} &= \bigcup_{i\in\{1,2,3\}} \left(\mathcal{D}_{\left[-\infty,d_i^-\right]}\right) \cup \left(\mathcal{D}_{\left[d_i^+,\infty\right]}\right)
\end{aligned}
$$

1 Relaxed intersection

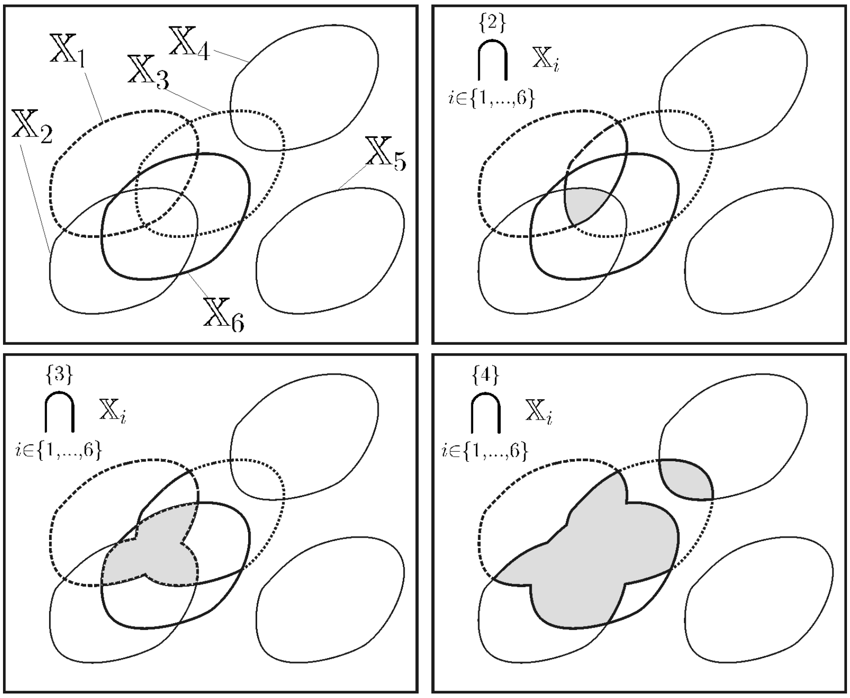
Dealing with outliers

$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Consider m sets $\mathbb{X}_{\mathbf{1}}, \ldots, \mathbb{X}_m$ of $\mathbb{R}^n$. The q-relaxed intersection $\bigcap^{\{q\}} \mathbb{X}_i$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most.

We have

$$\mathbf{x} \in \bigcap^{\{q\}} \mathbb{X}_i \Leftrightarrow \# \{i|\mathbf{x} \in \mathbb{X}_i\} \geq m - q$$



$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\bigcap_{i\in\{1,\ldots,6\}}^{\{2\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\ldots,6\}}^{\{3\}} \mathbb{X}_i$
$\bigcap_{i\in\{1,\ldots,6\}}^{\{4\}} \mathbb{X}_i$

Exercise. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{1\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?$$

$$\bigcap^{\{6\}} \mathbb{X}_i = ?$$

Solution. we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset \\
\bigcap^{\{1\}} \mathbb{X}_i &= \emptyset \\
\bigcap^{\{5\}} \mathbb{X}_i &= \bigcup \mathbb{X}_i \\
\bigcap^{\{6\}} \mathbb{X}_i &= \mathbb{R}^2
\end{aligned}$$

Exercise. Consider 8 intervals: $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$. Compute

$$\bigcap^{\{0\}} \mathbb{X}_i = ?, \quad \bigcap^{\{1\}} \mathbb{X}_i = ?, \quad \bigcap^{\{2\}} \mathbb{X}_i = ?,$$

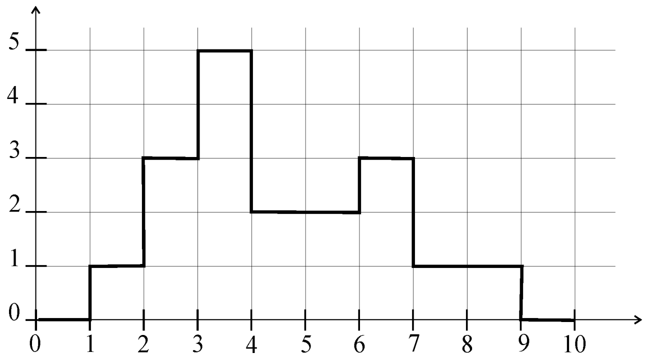
$$\bigcap^{\{3\}} \mathbb{X}_i = ?, \quad \bigcap^{\{4\}} \mathbb{X}_i = ?,$$

$$\bigcap^{\{5\}} \mathbb{X}_i = ?, \quad \bigcap^{\{6\}} \mathbb{X}_i = ?.$$

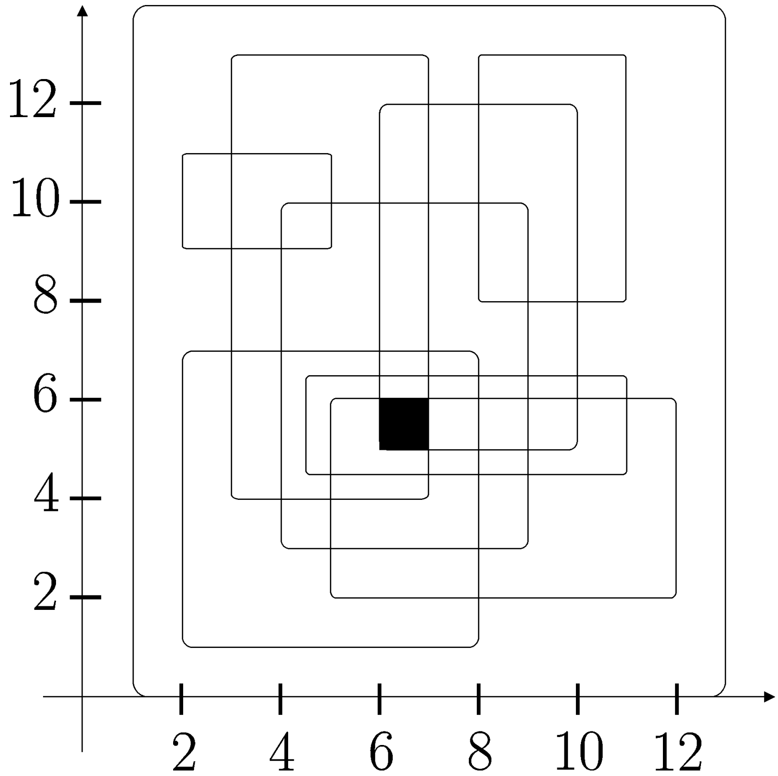
Solution. For $\mathbb{X}_1 = [1,4]$, $\mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4], \mathbb{X}_6 = [3,7]$, we have

$$\begin{aligned}
\bigcap^{\{0\}} \mathbb{X}_i &= \emptyset, \ \bigcap^{\{1\}} \mathbb{X}_i = [3,4], \ \bigcap^{\{2\}} \mathbb{X}_i = [3,4], \\
\bigcap^{\{3\}} \mathbb{X}_i &= [2,4] \cup [6,7], \ \bigcap^{\{4\}} \mathbb{X}_i = [2,7], \\
\bigcap^{\{5\}} \mathbb{X}_i &= [1,9], \ \bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}.
\end{aligned}$$

If $\mathbb{X}_i$'s are intervals, the relaxed intersection can be computed with a complexity of $m \log m$.



Computing the q relaxed intersection of m boxes is tractable.



The black box is the 2-intersection of 9 boxes

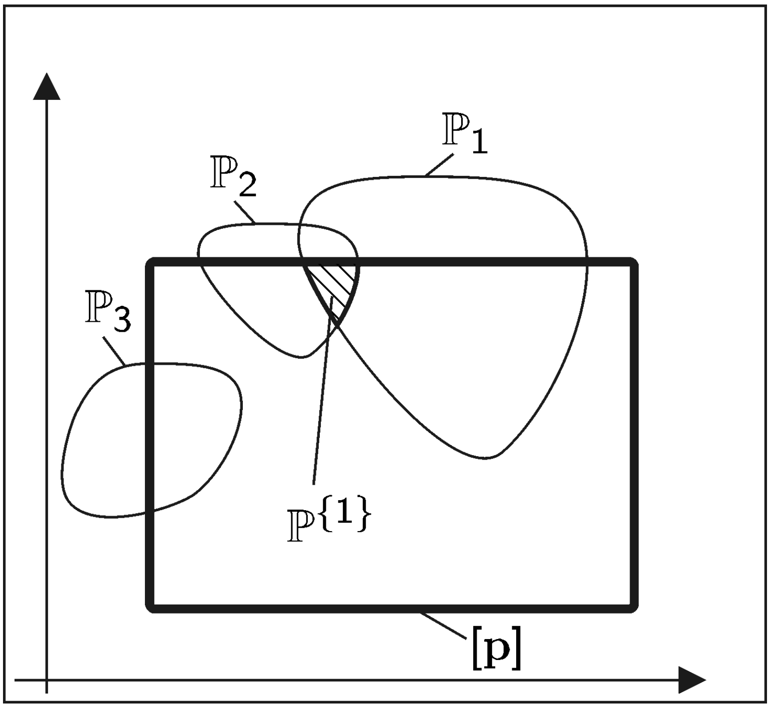
De Morgan's law

$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} = \bigcap^{\{m-q-1\}} \overline{\mathbb{X}_i}$$

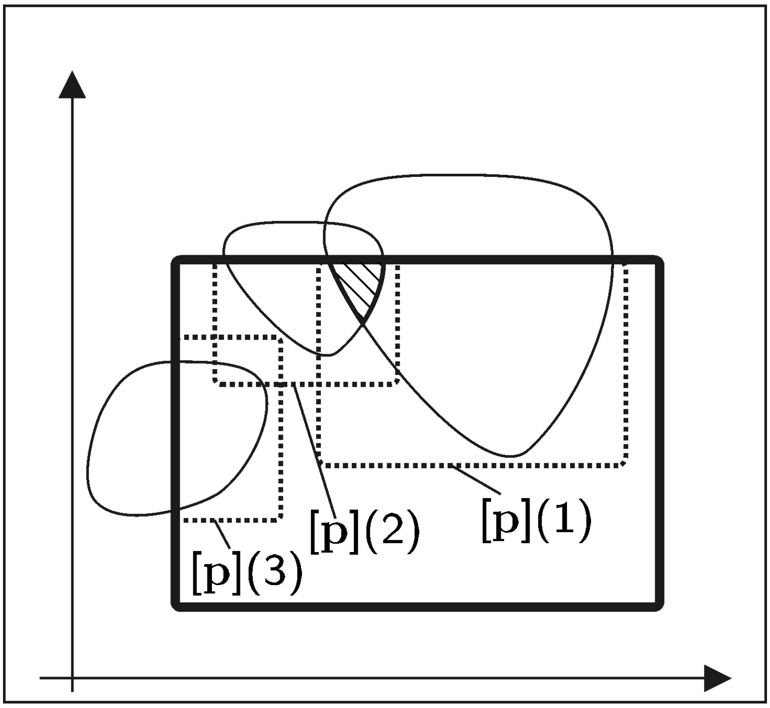
Relaxation of contractors

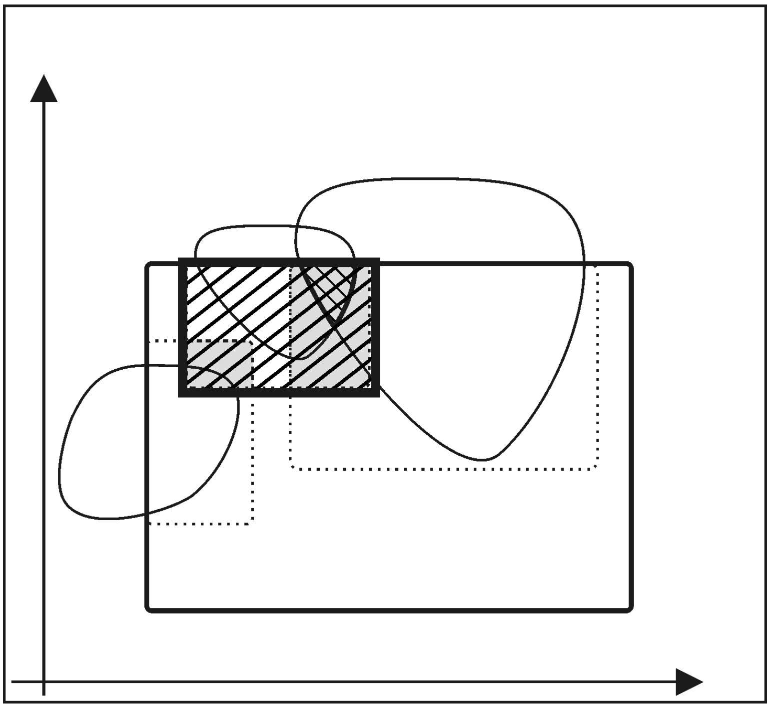
We define the q-relaxed intersection between m contractors

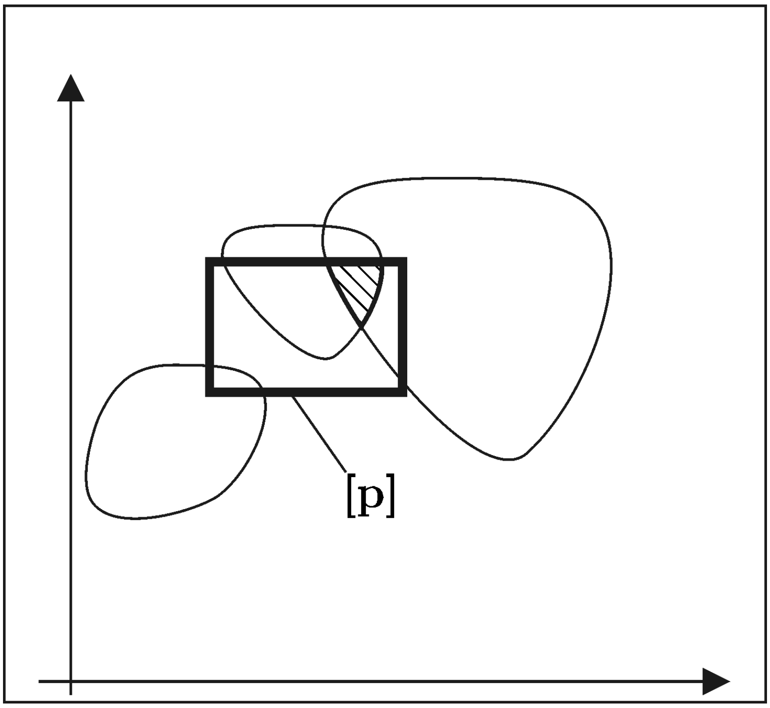
$$\mathcal{C} = \left(\bigcap_{i \in \{\mathbf{1},\dots,m\}}^{\{q\}} \mathcal{C}_i \right) \Leftrightarrow \forall \, [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\left([\mathbf{x}]\right) = \bigcap^{\{q\}} \mathcal{C}_i \left([\mathbf{x}]\right).$$

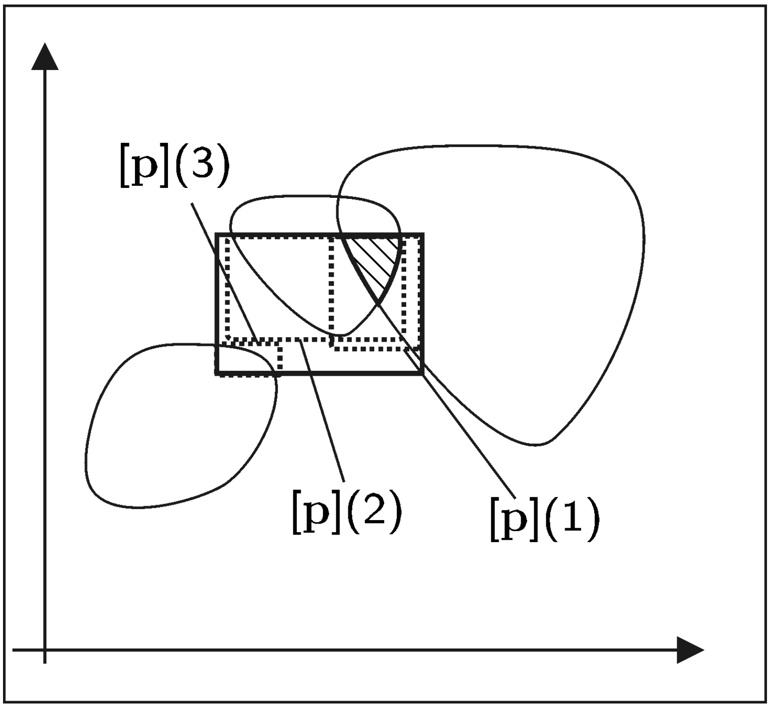


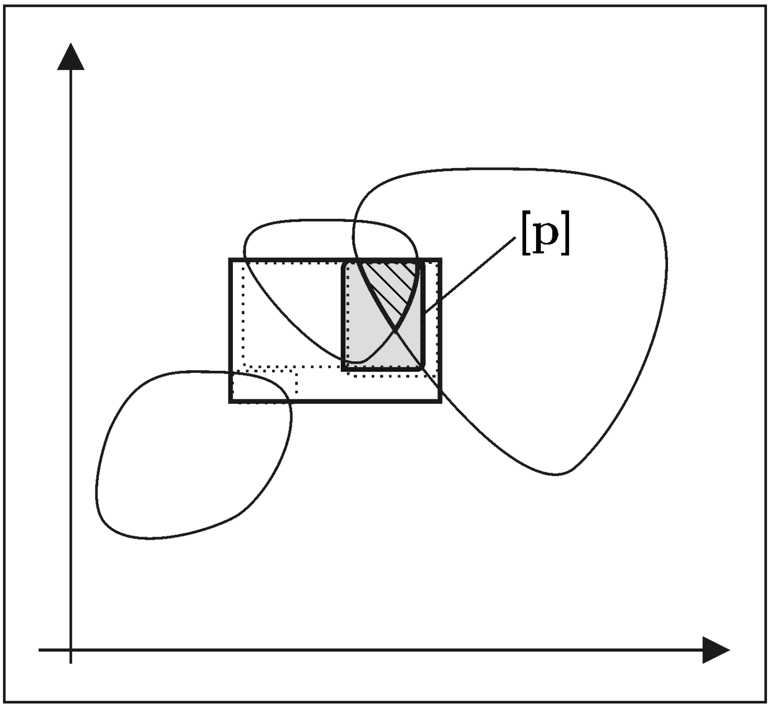
$\mathbb{P}_1$
$\mathbb{P}_2$
$\mathbb{P}_3$
$\mathbb{P}^{\{1\}}$
$[p]$


[p](2)
[p](1)
[p](3)

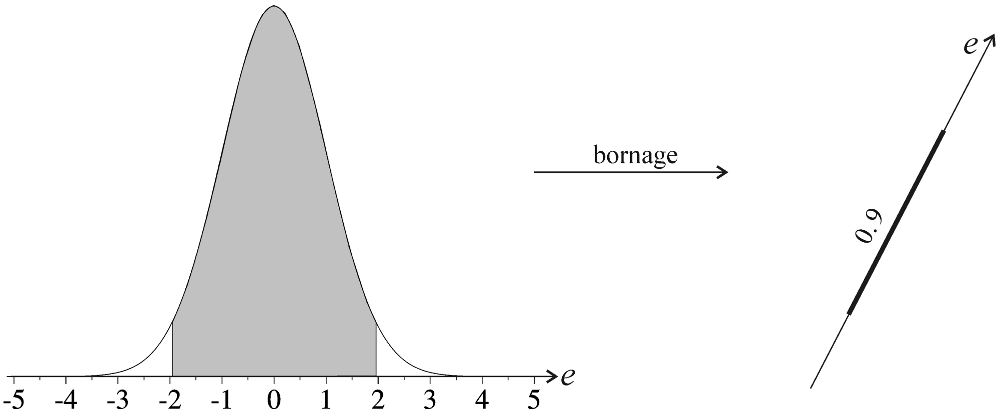



[p]

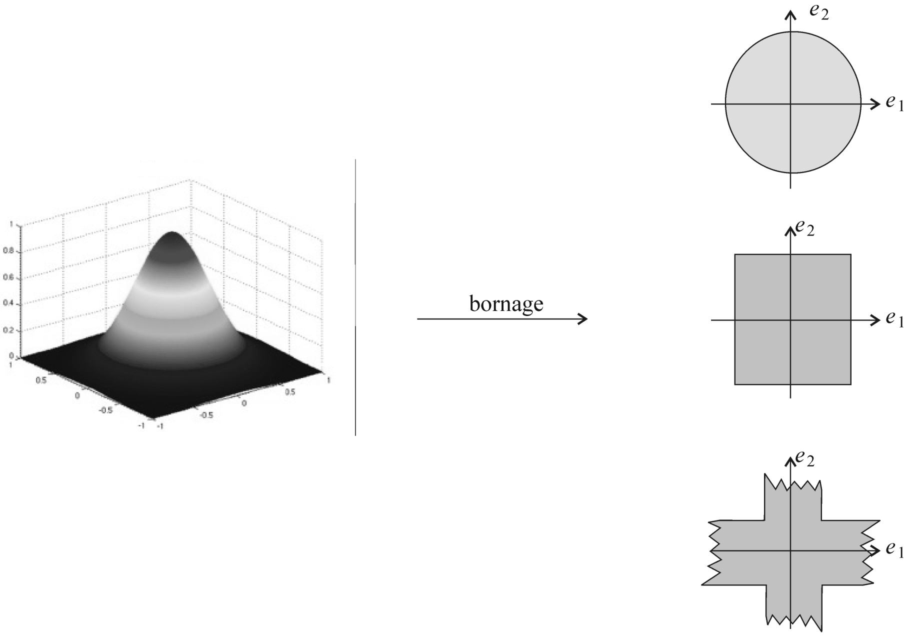

[p](3)
[p](2)
[p](1)


[p]

2 Probabilistic motivation



e
-5 -4 -3 -2 -1 0 1 2 3 4 5
bornage
e
0.9


bornage
e2
e1
e2
e1
e2
e1

Consider the error model

$$\mathbf{e} = \underbrace{\mathbf{y} - \boldsymbol{\psi}(\mathbf{p})}_{\mathbf{f}(\mathbf{y},\mathbf{p})}.$$

y_i is an *inlier* if $e_i \in [e_i]$ and an *outlier* otherwise. We assume that

$$\forall i, \ \Pr(e_i \in [e_i]) = \pi$$

and that all e_i's are independent.

Equivalently,

$$\begin{cases} f_1(\mathbf{y}, \mathbf{p}) \in [e_1] & \text{with a probability } \pi \\ \vdots & \vdots \\ f_m(\mathbf{y}, \mathbf{p}) \in [e_m] & \text{with a probability } \pi \end{cases}$$

The probability of having more than q outliers is thus

$$\gamma\left(q, m, \pi\right) \stackrel{\text{def}}{=} \sum_{k=0}^{m-q-1} \frac{m!}{k!\left(m-k\right)!} \pi^{k}.\left(1-\pi\right)^{m-k}.$$

Example. If $m = 1000$, $q = 900, \pi = 0.2$, we get $\gamma\left(q, m, \pi\right) = 7.04 \times 10^{-16}$. Thus having more than 900 outliers can be seen as a rare event.

3 Robust bounded error estimation

$$\mathbb{S} = \bigcap_{i}^{\{q\}} \{\mathbf{p} \in \mathbb{R}^n \mid f_i(\mathbf{p}) \in [y_i]\}$$

We build the following contractors

$$\begin{array}{rcl} \mathcal{C}_i & : & f_i\left(\mathbf{p}\right) \in [y_i] \\ \overline{\mathcal{C}_i} & : & f_i\left(\mathbf{p}\right) \notin [y_i] \\ \mathcal{C} & = & \bigcap_i^{\{q\}} \mathcal{C}_i \\ \overline{\mathcal{C}} & = & \overline{\bigcap_i^{\{q\}} \mathcal{C}_i} = \bigcap^{\{n-q-1\}} \overline{\mathcal{C}_i} \end{array}$$

Then we call a paver with $\overline{\mathcal{C}}$ and $\mathcal{C}$.

4 Application to localization

A robot measures distances to three beacons.

beacon	x_i	y_i	$[d_i]$
1	1	3	$[1, 2]$
2	3	1	$[2, 3]$
3	-1	-1	$[3, 4]$

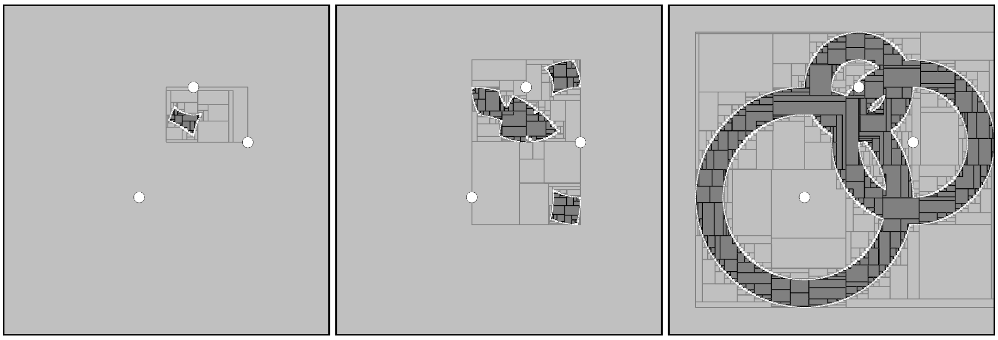
The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{\mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5]\right.$$

where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$.

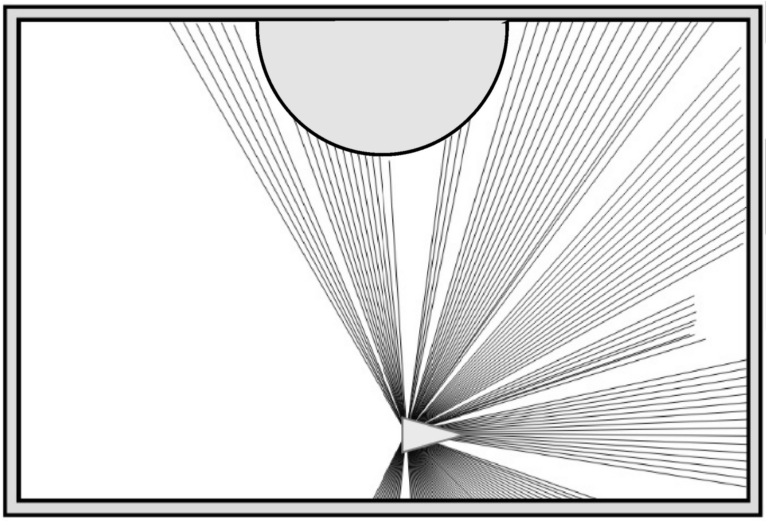
$$
\begin{aligned}
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{0\}}\right) &= \mathsf{0.729} \\
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{1\}}\right) &= \mathsf{0.972} \\
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{2\}}\right) &= \mathsf{0.999}
\end{aligned}
$$



Probabilistic sets $\mathbb{P}^{\{0\}}, \mathbb{P}^{\{1\}}, \mathbb{P}^{\{2\}}$.

5 With real data

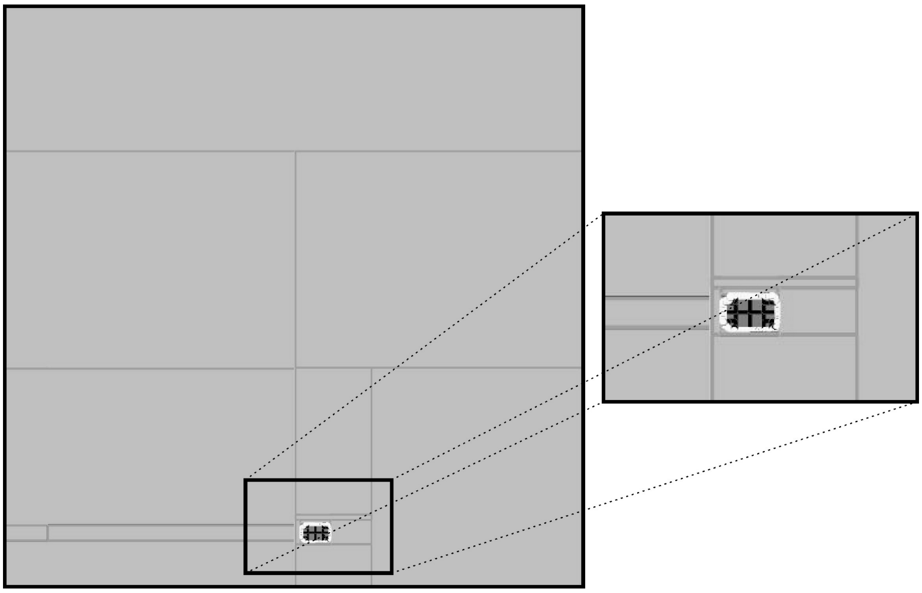
Robot equipped with a laser rangefinder and a compass.



143 distances collected by the rangefinder $\pm 10cm$

For $q = 16, m = 143, \pi = 0.95$, the probability of being wrong is

$$\alpha = \gamma(q, m, \pi) = 8.46 \times 10^{-4}.$$



$\mathbb{P}^{\{16\}}$ contains $\mathbf{p}^*$ with a probability $1 - \alpha = 0.99915$.