Chapter 2: Subpavings

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1 Subpavings

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.

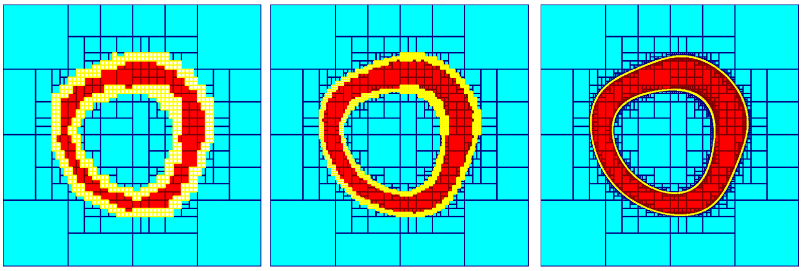
Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Exercise. The set

$$\mathbb{X} = \{(x_1, x_2) \,\big|\, x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}$$

are approximated by $\mathbb{X}^-$ and $\mathbb{X}^+$ for different accuracies. Denote by $\mathbb{R}, \mathbb{Y}, \mathbb{B}$ the union of red, yellow, blue boxes. Denote by $\partial\mathbb{X}$ the boundary of $\mathbb{X}$.



$\mathbb{X}^- \cap \mathbb{B} = \emptyset$	yes or no
$\mathbb{X} \cap \mathbb{B} \neq \emptyset$	yes or no
$\mathbb{X}^+ = \mathbb{R} \cup \mathbb{Y}$	yes or no
$\partial\mathbb{X} \supset \mathbb{Y}$	yes or no
$\mathbb{X} \backslash (\mathbb{R} \cup \mathbb{B}) = \mathbb{Y} \cap \mathbb{X}$	yes or no

Solution. We have

$$\begin{array}{lll}
\mathbb{X}^- \cap \mathbb{B} = \emptyset & \longrightarrow & \text{Yes} \\
\mathbb{X} \cap \mathbb{B} = \emptyset & \longrightarrow & \text{No} \\
\mathbb{X}^+ = \mathbb{R} \cup \mathbb{Y} & \longrightarrow & \text{Yes} \\
\partial\mathbb{X} \supset \mathbb{Y} & \longrightarrow & \text{No. Instead, we have } \partial\mathbb{X} \subset \mathbb{Y} \\
\mathbb{X} \backslash (\mathbb{R} \cup \mathbb{B}) = \mathbb{Y} \cap \mathbb{X} & \longrightarrow & \text{Yes}
\end{array}$$

Set operations such as $\mathbb{Z} := \mathbb{X} + \mathbb{Y}$, $\mathbb{X} := \mathbf{f}^{-1}(\mathbb{Y})$, $\mathbb{Z} := \mathbb{X} \cap \mathbb{Y} \dots$ can be approximated by subpaving operations.

Exercise. Consider the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4\}.$$

Find two subpavings $\mathbb{X}^-$ and $\mathbb{X}^+$, both made with a single box, such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Solution. We can take

$$\mathbb{X}^- = \left\{ \left[-\sqrt{2}, \sqrt{2} \right]^{\times 2} \right\} \text{ and } \mathbb{X}^+ = \left\{ [-2, 2]^{\times 2} \right\}.$$

Here, subpavings are singletons, but it is not the case usually.

2 Set inversion

If $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$.

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

Exercise. Define the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 x_2 + \sin x_2 \leq 0 \text{ and } x_1 - x_2 = 1\}.$$

Show that it is a set inversion problem.

Solution. We have

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

with

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 x_2 + \sin x_2 \\ x_1 - x_2 \end{pmatrix} \text{ and } \mathbb{Y} = [-\infty, 0] \times \{1\}\,.$$

$$\begin{array}{llll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

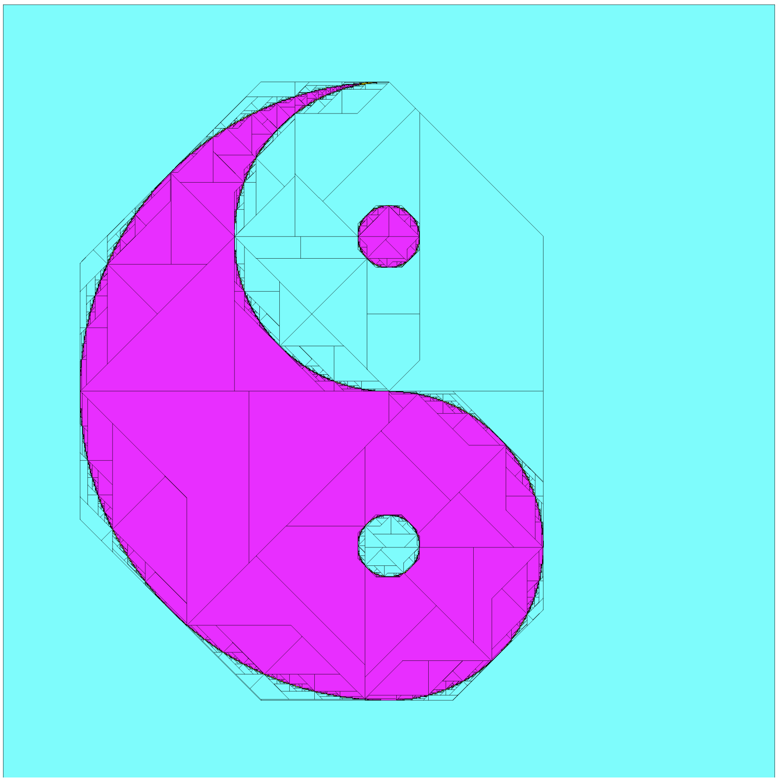
Algorithm Sivia(in: $[\mathbf{x}](0), \mathbf{f}, \mathbb{Y}$)

1 $\mathcal{L} := \{[\mathbf{x}](0)\}$;
2 pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$, 'red');
4 elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$, 'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$, 'yellow')};
6 else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$;
7 if $\mathcal{L} \neq \emptyset$, go to 2

If $\mathbf{\Delta}\mathbb{X}$ denotes the union of yellow boxes and if $\mathbb{X}^-$ is the union of red boxes then :

$$\mathbb{X}^- \subset \mathbb{X} \subset \underbrace{\mathbb{X}^- \cup \mathbf{\Delta}\mathbb{X}}_{\mathbb{X}^+}.$$

Sivia works with other abstract domains (or wrappers).



Sivia with octogones (made by D. Massé)

3 Image evaluation

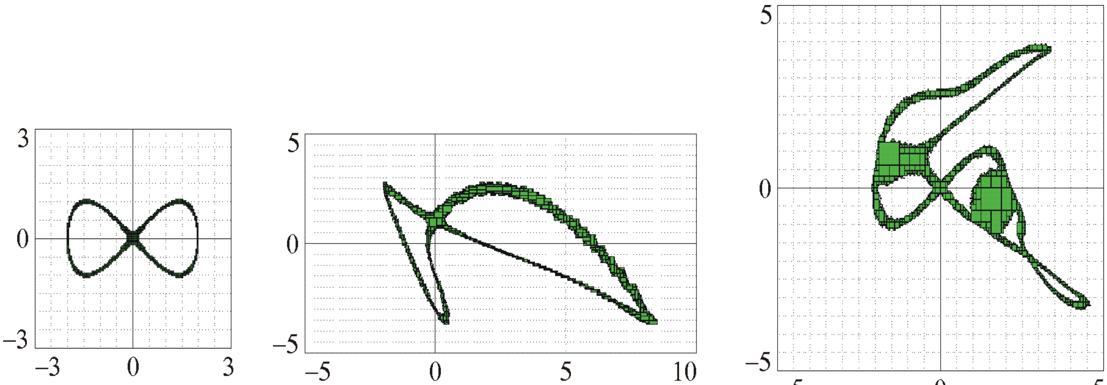
Define

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} (x_1 - 1)^2 - 1 + x_2 \\ -x_1^2 + (x_2 - 1)^2 \end{pmatrix},$$

and

$$\mathbb{X}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^4 - x_1^2 + 4x_2^2 \in [-0.1, 0.1] \right\}.$$

We shall compute $\mathbb{X}_1$, $\mathbf{f}(\mathbb{X}_1)$ and $\mathbf{f}^{-1} \circ \mathbf{f}(\mathbb{X}_1)$.



(a): $\mathbb{X}_1$ (b): $\mathbf{f}(\mathbb{X}_1)$ (c): $\mathbf{f}^{-1}(\mathbf{f}(\mathbb{X}_1))$

4 Bounded-error estimation

Exercise. Consider a parabola of the form

$$\phi(\mathbf{p}, t) = p_1 t^2 + p_2 t + p_3.$$

where $\mathbf{p} = (p_1, p_2, p_3)^\mathsf{T}$ is an unkown parameter vector. Assume that

$$\phi(\mathbf{p}, 1) \in [2, 3], \ \phi(\mathbf{p}, 4) \in [5, 6], \ \phi(\mathbf{p}, 7) \in [8, 9].$$

Show that the set $\mathbb{P}$ of all feasible $\mathbf{p}$ can be defined as a set inversion problem.

Solution. We have

$$\mathbb{P} = \mathbf{f}^{-1}(\mathbb{Y}),$$

where

$$\mathbf{f}(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p}, 1) \\ \phi(\mathbf{p}, 4) \\ \phi(\mathbf{p}, 7) \end{pmatrix} = \begin{pmatrix} p_1 + p_2 + p_3 \\ 16p_1 + 4p_2 t + p_3 \\ 49p_1 + 7p_2 + p_3 \end{pmatrix}$$

and

$$\mathbb{Y} = [2, 3] \times [5, 6] \times [8, 9].$$

Model : $\phi(\mathbf{p}, t) = p_1 e^{-p_2 t}$.

Prior feasible box for the parameters : $[\mathbf{p}] \subset \mathbb{R}^n$

Measurement times : $t_1, t_2, \ldots, t_m$

Data bars : $[y_1^-, y_1^+], [y_2^-, y_2^+], \ldots, [y_m^-, y_m^+]$

$\mathbb{S} = \{\mathbf{p} \in [\mathbf{p}], \phi(\mathbf{p}, t_1) \in [y_1^-, y_1^+], \ldots, \phi(\mathbf{p}, t_m) \in [y_m^-, y_m^+]$

If

$$\boldsymbol{\phi}(\mathbf{p}) = \left(\begin{array}{c} \phi(\mathbf{p}, t_1) \\ \\ \phi(\mathbf{p}, t_m) \end{array} \right)$$

and

$$[\mathbf{y}] = [y_1^-, y_1^+] \times \cdots \times [y_m^-, y_m^+]$$

then

$$\mathbb{P} = [\mathbf{p}] \cap \boldsymbol{\phi}^{-1}([\mathbf{y}]) .$$

5 Robust stability

The *stability domain* $\mathbb{S}_{\mathrm{p}}$ of the polynomial

$$P(s, \mathbf{p}) = s^n + a_{n-1}(\mathbf{p})s^{n-1} + \ldots + a_1(\mathbf{p})s + a_0(\mathbf{p})$$

is the set of all $\mathbf{p}$ such that $P(s, \mathbf{p})$ is stable.

If $P(s, \mathbf{p})$ is given by

$$s^3+(p_1+p_2+2)s^2+(p_1+p_2+2)s+2p_1p_2+6p_1+6p_2+2.25.$$

Its Routh table is given by

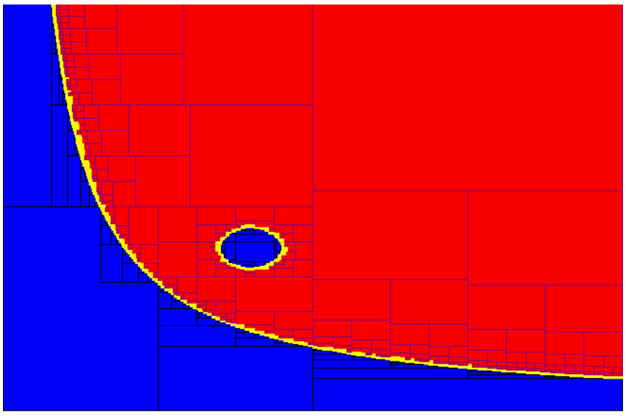
1	p_1+p_2+2
p_1+p_2+2	$2p_1p_2+6p_1+6p_2+2.25$
$\frac{(p_1-1)^2+(p_2-1)^2-0.25}{p_1+p_2+2}$	0
$2(p_1+3)(p_2+3)-15.75$	0

Its stability domain is thus defined by

$$\mathbb{S}_{\mathsf{p}} \triangleq \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{r}(\mathbf{p}) > \mathbf{0}\} = \mathbf{r}^{-1}\left( ]0, +\infty[^{\times n} \right).$$

where

$$\mathbf{r}(\mathbf{p}) = \begin{pmatrix} p_1 + p_2 + 2 \\ (p_1 - 1)^2 + (p_2 - 1)^2 - 0.25 \\ 2(p_1 + 3)(p_2 + 3) - 15.75 \end{pmatrix}.$$



Stability domain $\mathbb{S}_p$ generated by Proj2d