# Chapter 3: Contractors

Luc Jaulin,

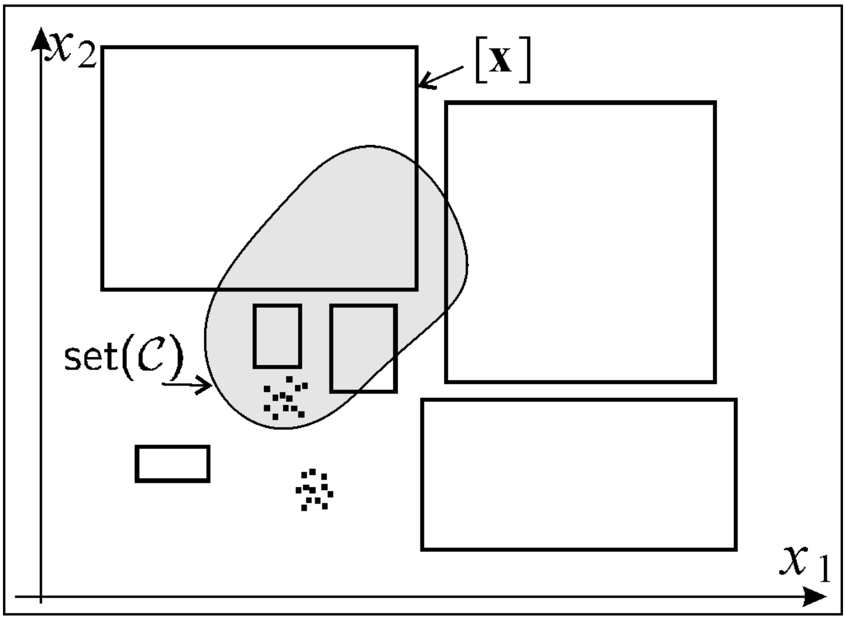
ENSTA-Bretagne, Brest, France.

http://www.ensta-bretagne.fr/jaulin/

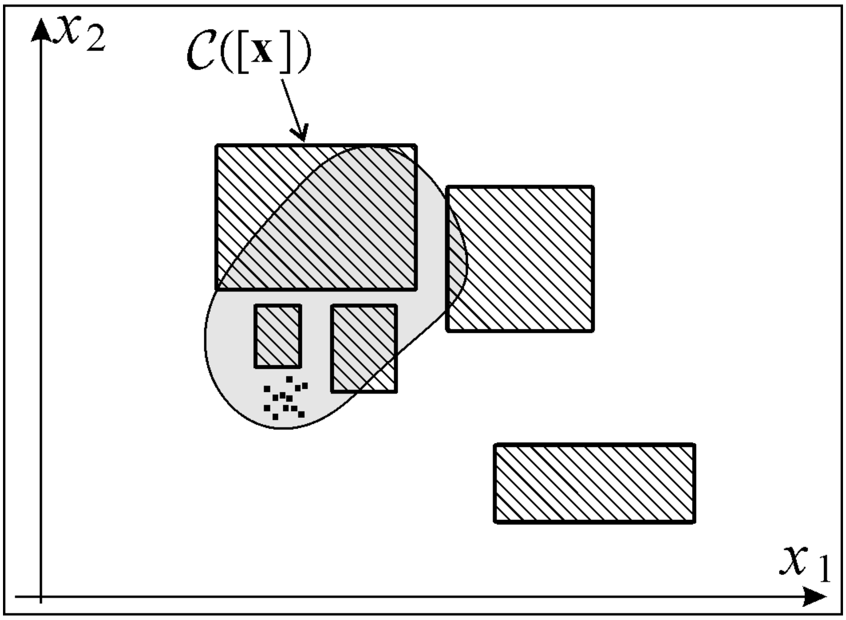
# 1 Definition

The operator $\mathcal{C}_{\mathbb{X}} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a *contractor* for $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)}, \\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness)}. \end{cases}$$



$x_2$
$[\mathbf{x}]$
set$(\mathcal{C})$
$x_1$



$x_2$
$\mathcal{C}([\mathbf{x}])$
$x_1$

The operator $\mathcal{C} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall[\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) \end{cases}$$

| | |
|---|---|
| $\mathcal{C}_{\mathbb{X}}$ is *monotonic* if | $[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset \mathcal{C}_{\mathbb{X}}([\mathbf{y}])$ |
| $\mathcal{C}_{\mathbb{X}}$ is *minimal* if | $\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) = [[\mathbf{x}] \cap \mathbb{X}]$ |
| $\mathcal{C}_{\mathbb{X}}$ is *thin* if | $\forall \mathbf{x} \in \mathbb{R}^n,\ \mathcal{C}_{\mathbb{X}}(\{\mathbf{x}\}) = \{\mathbf{x}\} \cap \mathbb{X}$ |
| $\mathcal{C}_{\mathbb{X}}$ is *idempotent* if | $\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}_{\mathbb{X}}(\mathcal{C}_{\mathbb{X}}([\mathbf{x}])) = \mathcal{C}_{\mathbb{X}}([\mathbf{x}])$ |
| $\mathcal{C}_{\mathbb{X}}$ is *convergent* if | $[\mathbf{x}](k) \to \mathbf{x} \quad \Rightarrow \mathcal{C}_{\mathbb{X}}([\mathbf{x}](k)) \to \{\mathbf{x}\} \cap$ |

**Exercise**. Replace the symbol $\bowtie$ either by $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$.

$$
\begin{array}{lcl}
\mathcal{C}_{\mathbb{X}} \text{ minimal} & \bowtie & \mathcal{C}_{\mathbb{X}} \text{ idempotent} \\
\mathcal{C}_{\mathbb{X}} \text{ thin} & \bowtie & \mathcal{C}_{\mathbb{X}} \text{ minimal} \\
\mathcal{C}_{\mathbb{X}} \text{ minimal} & \bowtie & \mathcal{C}_{\mathbb{X}} \text{ monotonic} \\
\mathcal{C}_{\mathbb{X}} \text{ thin} & \bowtie & \mathcal{C}_{\mathbb{X}} \text{ convergent}
\end{array}
$$

**Solution**. We have

$$\begin{array}{lcl}
\mathcal{C}_{\mathbb{X}} \text{ minimal} & \Rightarrow & \mathcal{C}_{\mathbb{X}} \text{ idempotent} \\
\mathcal{C}_{\mathbb{X}} \text{ thin} & \Leftarrow & \mathcal{C}_{\mathbb{X}} \text{ minimal} \\
\mathcal{C}_{\mathbb{X}} \text{ minimal} & \Rightarrow & \mathcal{C}_{\mathbb{X}} \text{ monotonic} \\
\mathcal{C}_{\mathbb{X}} \text{ thin} & \Leftarrow & \mathcal{C}_{\mathbb{X}} \text{ convergent}
\end{array}$$

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|---|---|
| intersection | $(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) = \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$ |
| union | $(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$ |
| composition | $(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$ |
| repetition | $\mathcal{C}^\infty \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \ldots$ |
| repeat intersection | $\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$ |
| repeat union | $\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^\infty$ |

**Exercise**. If $\mathcal{C}_1$,$\mathcal{C}_2$,$\mathcal{C}_3$,$\mathcal{C}$ are contractors, do we always have

| | |
|---|---|
| $\mathcal{C}^\infty$ is idempotent | (yes/no) |
| $(\mathcal{C}_1 \cap \mathcal{C}_2)^\infty = (\mathcal{C}_1 \circ \mathcal{C}_2)^\infty$ | (yes/no) |
| $\mathcal{C}_1$ minimal and $\mathcal{C}_2$ minimal $\Rightarrow \mathcal{C}_1 \cup \mathcal{C}_2$ minimal | (yes/no) |
| $\mathcal{C}_1$ minimal and $\mathcal{C}_2$ minimal $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ idempotent | (yes/no) |

**Solution**.

| | |
|---|---|
| $\mathcal{C}^\infty$ is idempotent | Yes (if Sc |
| $(\mathcal{C}_1 \cap \mathcal{C}_2)^\infty = (\mathcal{C}_1 \circ \mathcal{C}_2)^\infty$ | Yes |
| $\mathcal{C}_1$ minimal and $\mathcal{C}_2$ minimal $\Rightarrow \mathcal{C}_1 \cup \mathcal{C}_2$ minimal | Yes |
| $\mathcal{C}_1$ minimal and $\mathcal{C}_2$ minimal $\Rightarrow \mathcal{C}_1 \cap \mathcal{C}_2$ idempotent | No |

# 2 Projection of constraints

**Exercice**. Let $x, y, z$ be 3 variables such that

$$\begin{array}{rcl} x & \in & [-\infty, 5], \\ y & \in & [-\infty, 4], \\ z & \in & [6, \infty], \\ z & = & x + y. \end{array}$$

Contract the intervals for $x, y, z$.

**Solution.**

$$
\begin{aligned}
[x] &= [2, 5] \\
[y] &= [1, 4] \\
[z] &= [6, 9]
\end{aligned}
$$

To *project* a constraint (here, $z = x + y$), is to compute the smallest intervals which contains all consistent values.

For our example, this amounts to project onto $x, y$ and $z$ the set

$$\mathbb{S} = \{(x, y, z) \in [-\infty, 5] \times [-\infty, 4] \times [6, \infty] \mid z = x + y\}\,.$$

# 3 Numerical method for projection

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$
\begin{aligned}
z = x + y \Rightarrow \quad z \in \; & [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) \\
& = [6, \infty] \cap [-\infty, 9] = [6, 9]. \\
x = z - y \Rightarrow \quad x \in \; & [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) \\
& = [-\infty, 5] \cap [2, \infty] = [2, 5]. \\
y = z - x \Rightarrow \quad y \in \; & [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) \\
& = [-\infty, 4] \cap [1, \infty] = [1, 4].
\end{aligned}
$$

The contractor associated with $z = x + y$ is.

| **Algorithm** pplus(inout: $[z], [x], [y]$) |
|---|
| 1 $[z] := [z] \cap ([x] + [y])$ ; |
| 2 $[x] := [x] \cap ([z] - [y])$ ; |
| 3 $[y] := [y] \cap ([z] - [x])$ . |

The projection procedure can be extended to other ternary constraints such as mult: $z = x \cdot y$, or equivalently

$$\text{mult} \triangleq \left\{(x, y, z) \in \mathbb{R}^3 \mid z = x \cdot y\right\}.$$

The resulting projection procedure becomes

| **Algorithm** pmult(inout: $[z], [x], [y]$) |
|---|
| 1 $\quad [z] := [z] \cap ([x] \cdot [y])$ ; |
| 2 $\quad [x] := [x] \cap ([z] \cdot 1/[y])$ ; |
| 3 $\quad [y] := [y] \cap ([z] \cdot 1/[x])$ . |

For the binary constraint

$$\mathsf{exp} \triangleq \{(x, y) \in \mathbb{R}^n | y = \mathsf{exp}\,(x)\}\,,$$

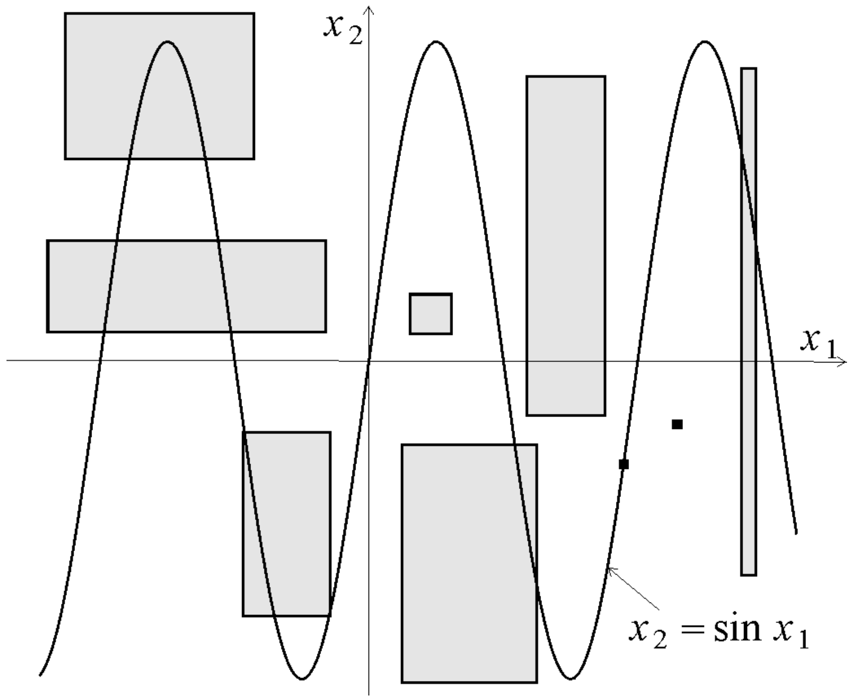
the associated contractor is

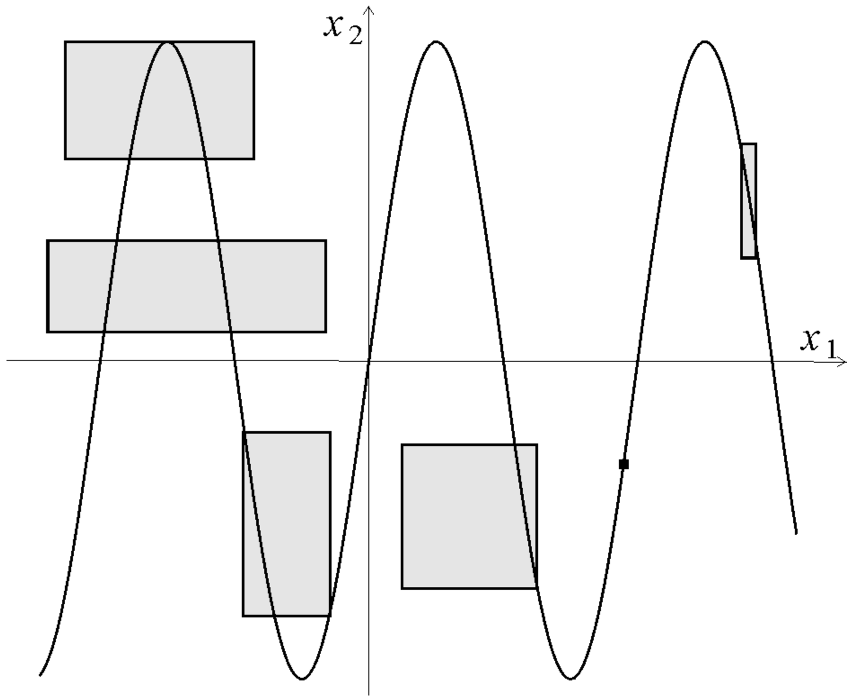
| **Algorithm** pexp(inout: $[y], [x]$) |
|---|
| 1 $[y] := [y] \cap \mathsf{exp}\,([x])\,;$ |
| 2 $[x] := [x] \cap \mathsf{log}\,([y])\,.$ |

Any constraint for which such a projection procedure is available will be called a *primitive constraint*.

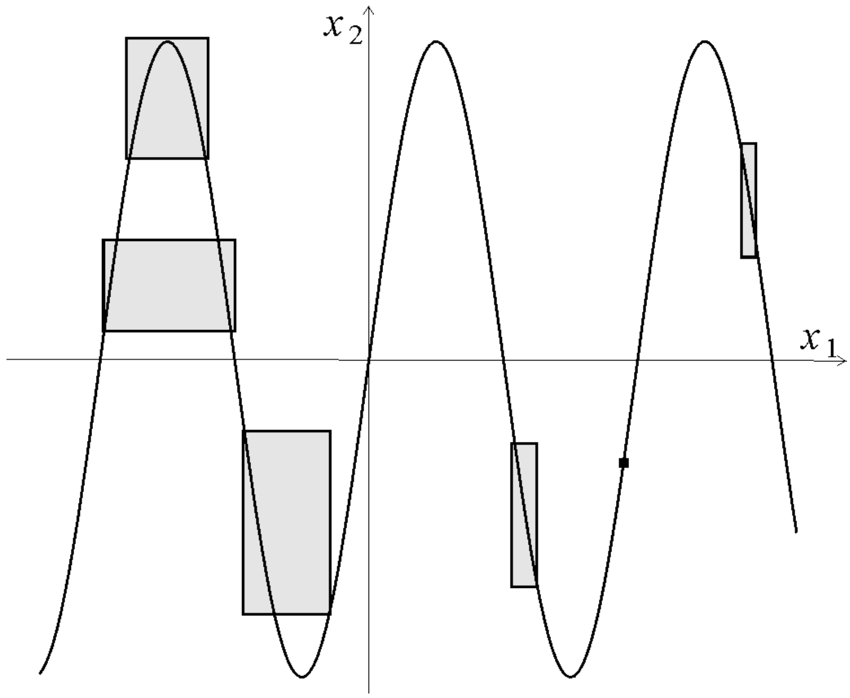
**Example**. Consider the primitive equation:

$$x_2 = \sin x_1.$$


$x_2$
$x_1$
$x_2 = \sin x_1$




Forward contraction




Backward contraction

## Decomposition

$$
\begin{aligned}
&x + \mathsf{sin}(xy) \leq 0, \\
&x \in [-1, 1], y \in [-1, 1]
\end{aligned}
$$

## Decomposition

$$x + \mathsf{sin}(xy) \leq 0,$$
$$x \in [x], y \in [y]$$

can be decomposed into

$$\left\{\begin{array}{l} a = xy \\ b = \mathsf{sin}(a) \\ c = x + b \end{array}\right. , \qquad \begin{array}{ll} x \in [x] & a \in [-\infty, \infty] \\ y \in [y] & b \in [-\infty, \infty] \\ & c \in [-\infty, 0] \end{array}$$

## Forward-backward contractor (HC4 revise)

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2]\,,$$

we have the following contractor:

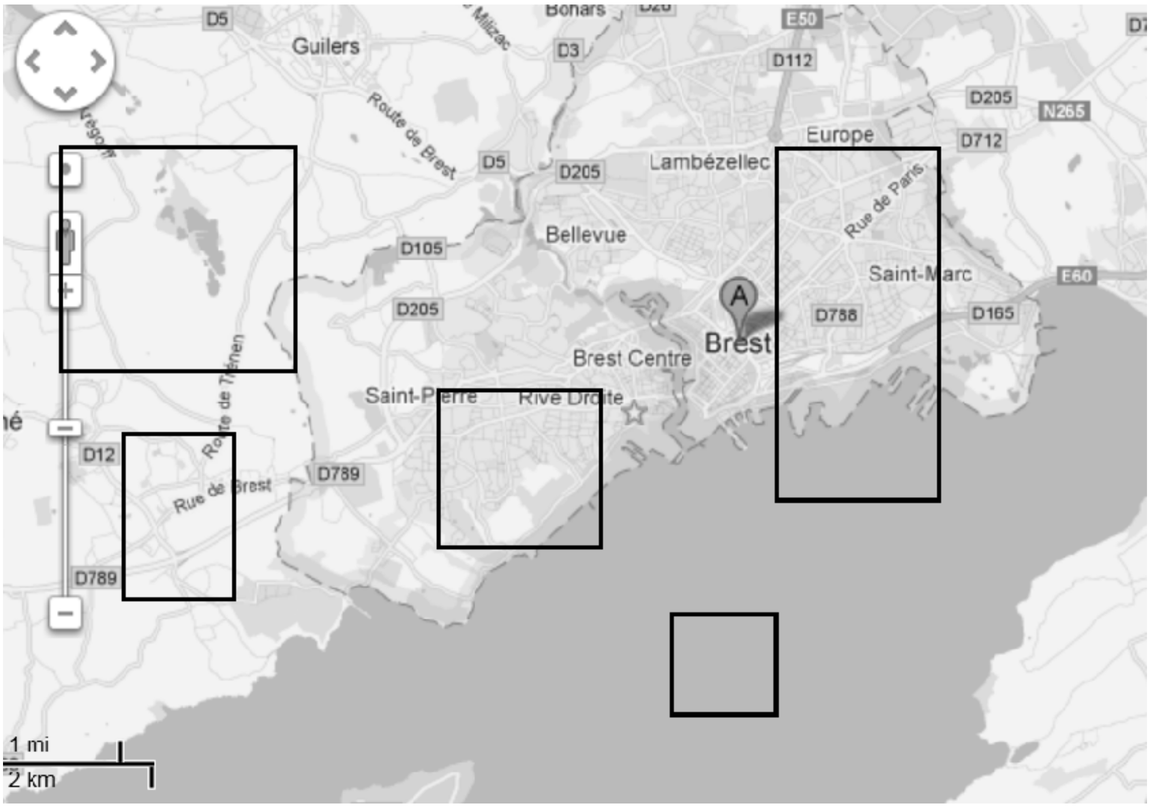
| algorithm $\mathcal{C}$ (inout $[x_1]\,,[x_2]\,,[x_3]$) | |
|---|---|
| $[a] = [x_1] + [x_2]$ | $//\ a = x_1 + x_2$ |
| $[b] = [a] \cdot [x_3]$ | $//\ b = a \cdot x_3$ |
| $[b] = [b] \cap [1, 2]$ | $//\ b \in [1, 2]$ |
| $[x_3] = [x_3] \ \cap\ \frac{[b]}{[a]}$ | $//\ x_3 = \frac{b}{a}$ |
| $[a] = [a] \ \cap\ \frac{[b]}{[x_3]}$ | $//\ a = \frac{b}{x_3}$ |
| $[x_1] = [x_1] \ \cap \ \ [a] - [x_2]$ | $//\ x_1 = a - x_2$ |
| $[x_2] = [x_2] \ \cap \ \ [a] - [x_1]$ | $//\ x_2 = a - x_1$ |

## Properties

$$\begin{array}{rcl}
(\mathcal{C}_1^\infty \cap \mathcal{C}_2^\infty)^\infty & = & (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty \\
(\mathcal{C}_1 \cap (\mathcal{C}_2 \cup \mathcal{C}_3)) & \supset & (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_1 \cap \mathcal{C}_3) \\
\left\{ \begin{array}{l} \mathcal{C}_1 \text{ minimal} \\ \mathcal{C}_2 \text{ minimal} \end{array} \right. & \Rightarrow & \mathcal{C}_1 \cup \mathcal{C}_2 \text{ minimal}
\end{array}$$

## Contractor on images

The robot with coordinates $(x_1, x_2)$ is in the water.



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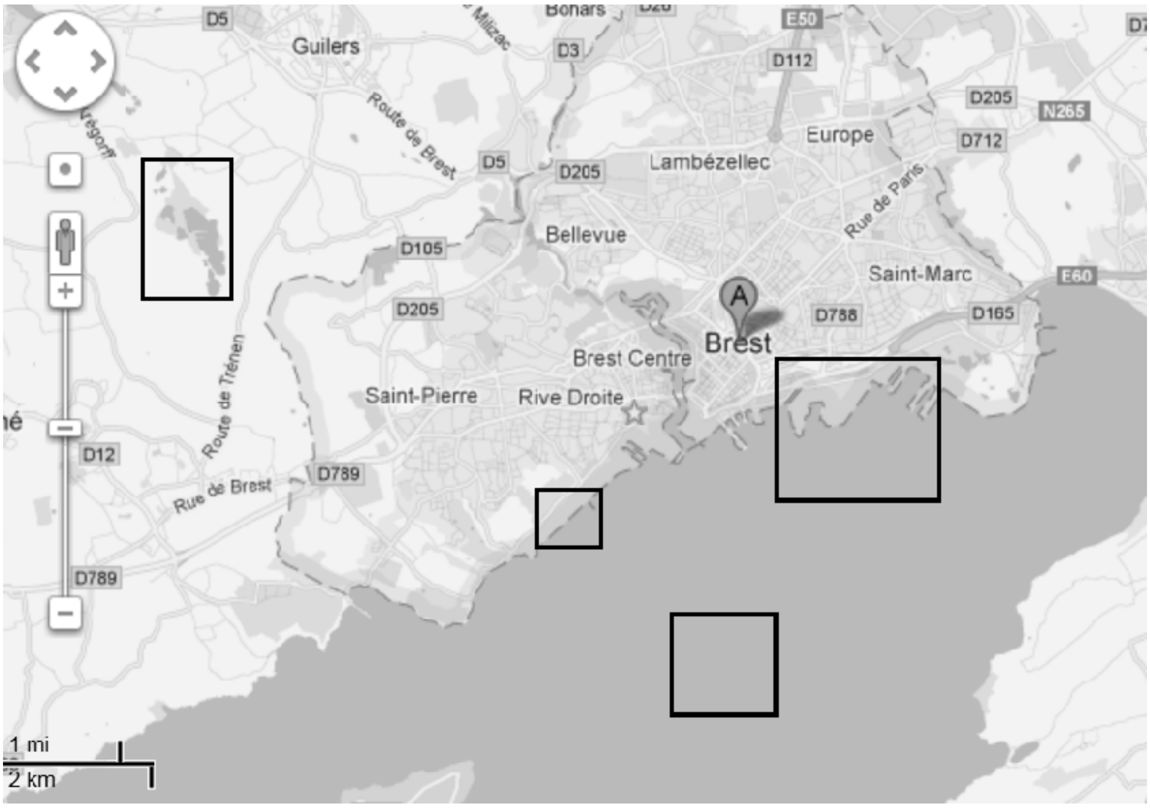
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Rue de Brest
D789
1 mi
2 km

# 4 Propagation

A CN (Constraint Network) is composed of
1) a set of variables $\mathcal{V} = \{x_1, \ldots, x_n\}$ ,
2) a set of constraints $\mathcal{C} = \{c_1, \ldots, c_m\}$ and
3) a set of interval domains $\{[x_1], \ldots, [x_n]\}$.

Principle of propagation techniques: contract $[\mathbf{x}] = [x_1] \times \cdots \times [x_n]$ as follows:

$$((((((([\mathbf{x}] \sqcap c_1) \sqcap c_2) \sqcap \ldots) \sqcap c_m) \sqcap c_1) \sqcap c_2) \ldots,$$

until a steady box is reached.

## 4.1 Example 1

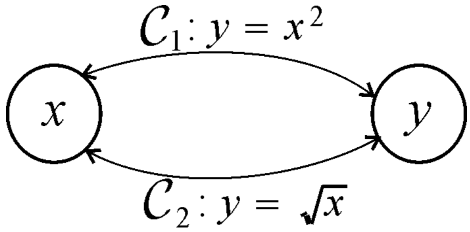
Consider the system of two equations.

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

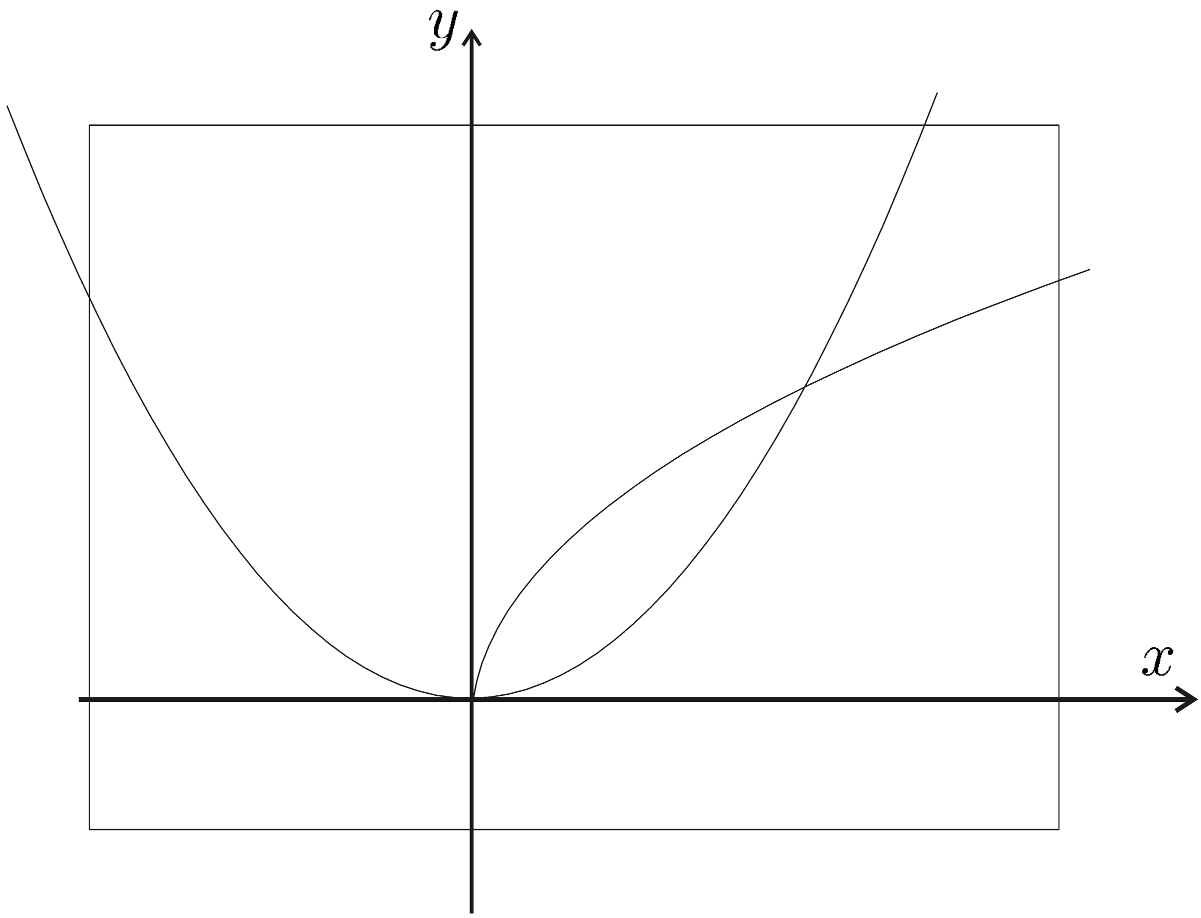
We can build two contractors

$$\mathcal{C}_1 : \left\{ \begin{array}{l} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{array} \right. \quad \text{associated to } y = x^2$$

$$\mathcal{C}_2 : \left\{ \begin{array}{l} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{array} \right. \quad \text{associated to } y = \sqrt{x}$$




Contractor graph



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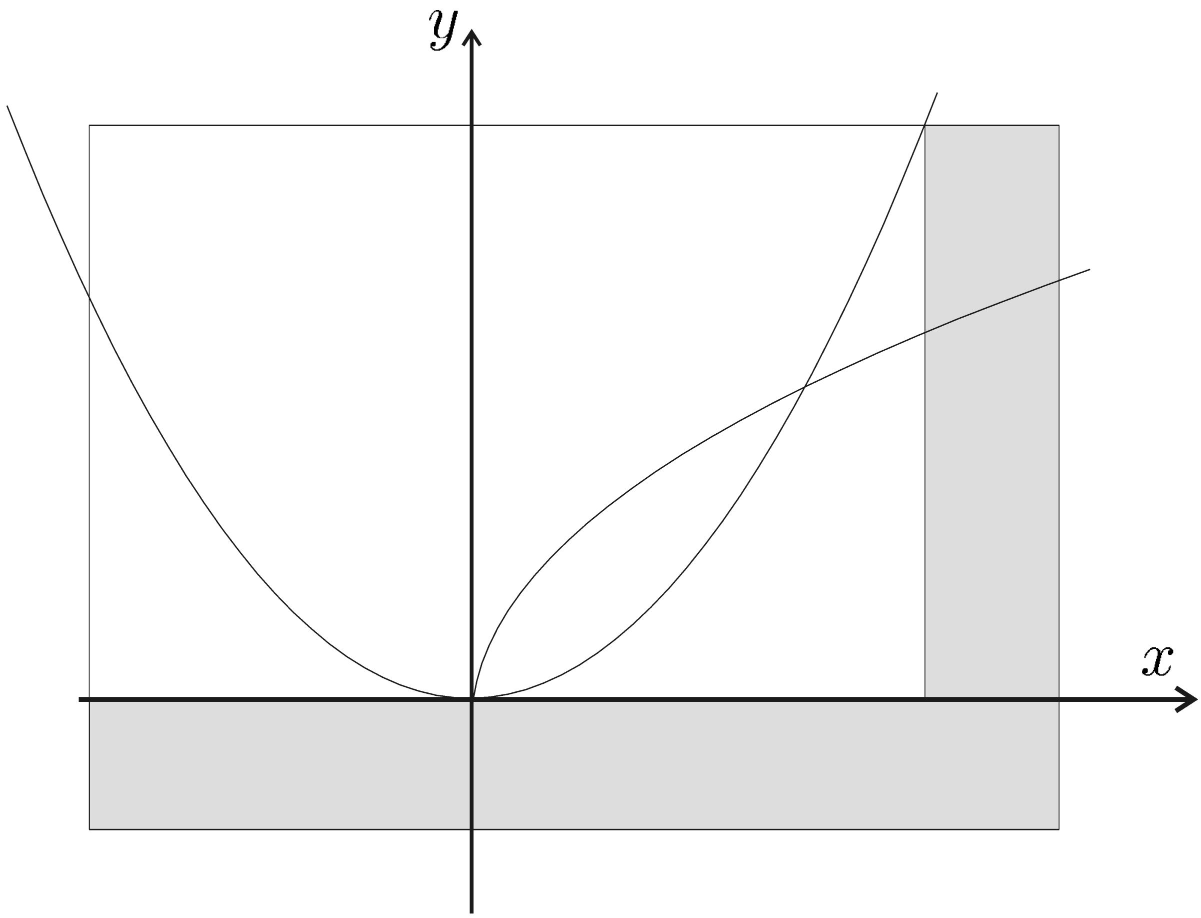

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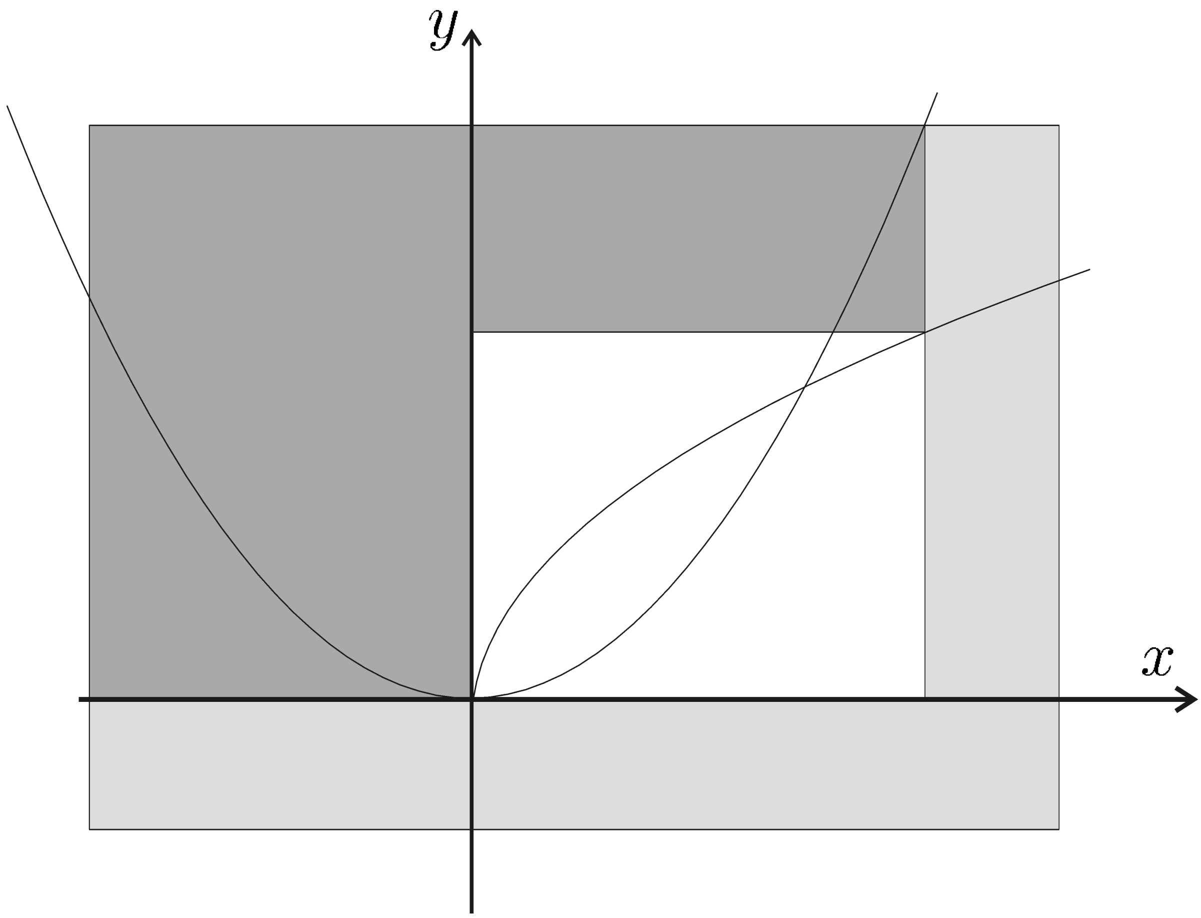

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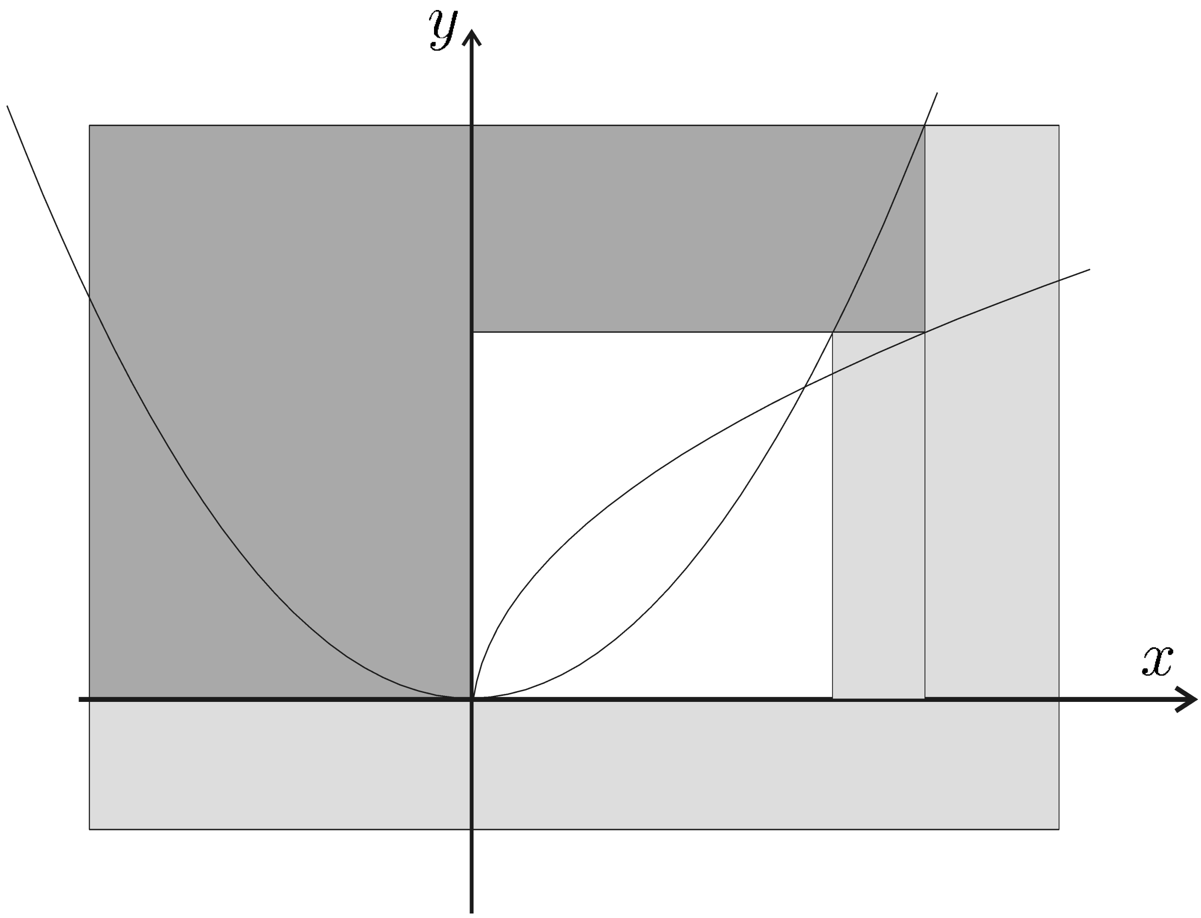

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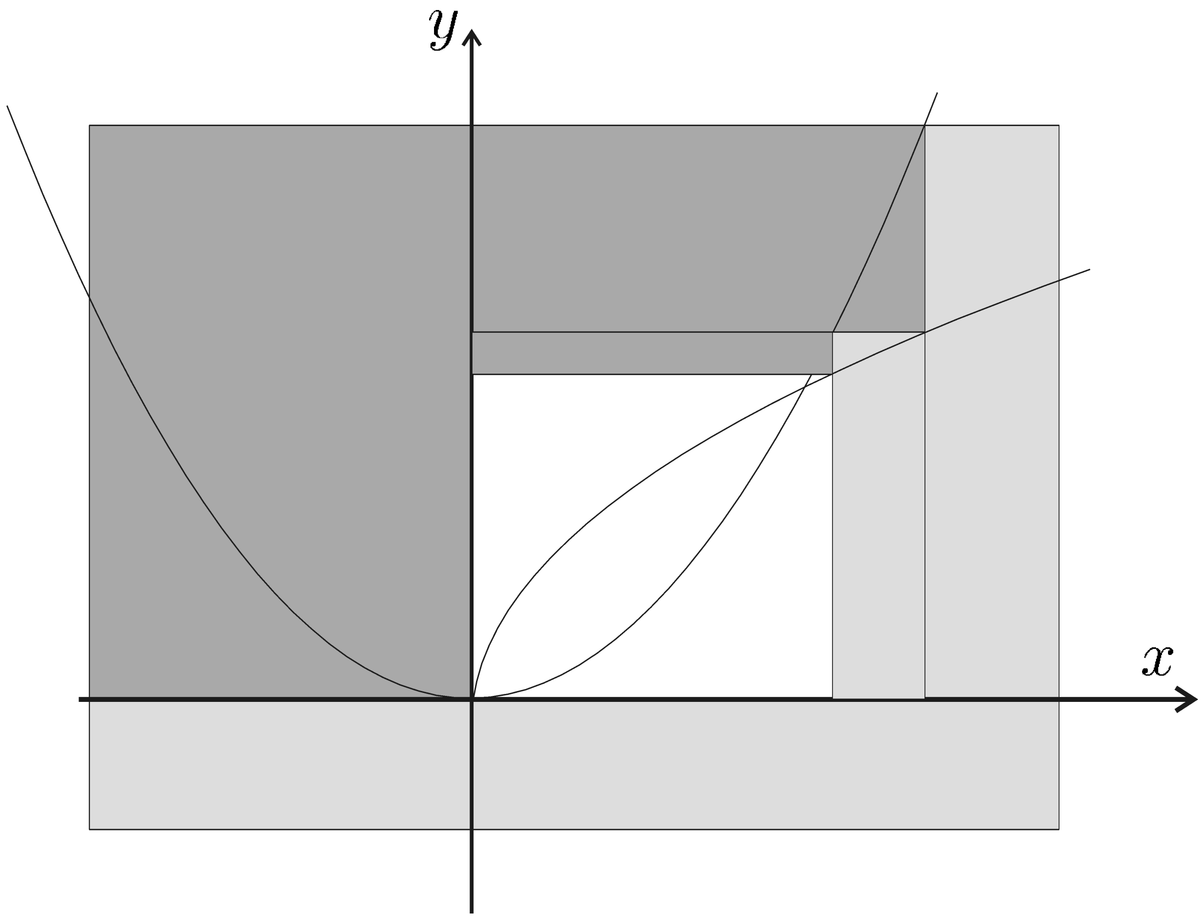

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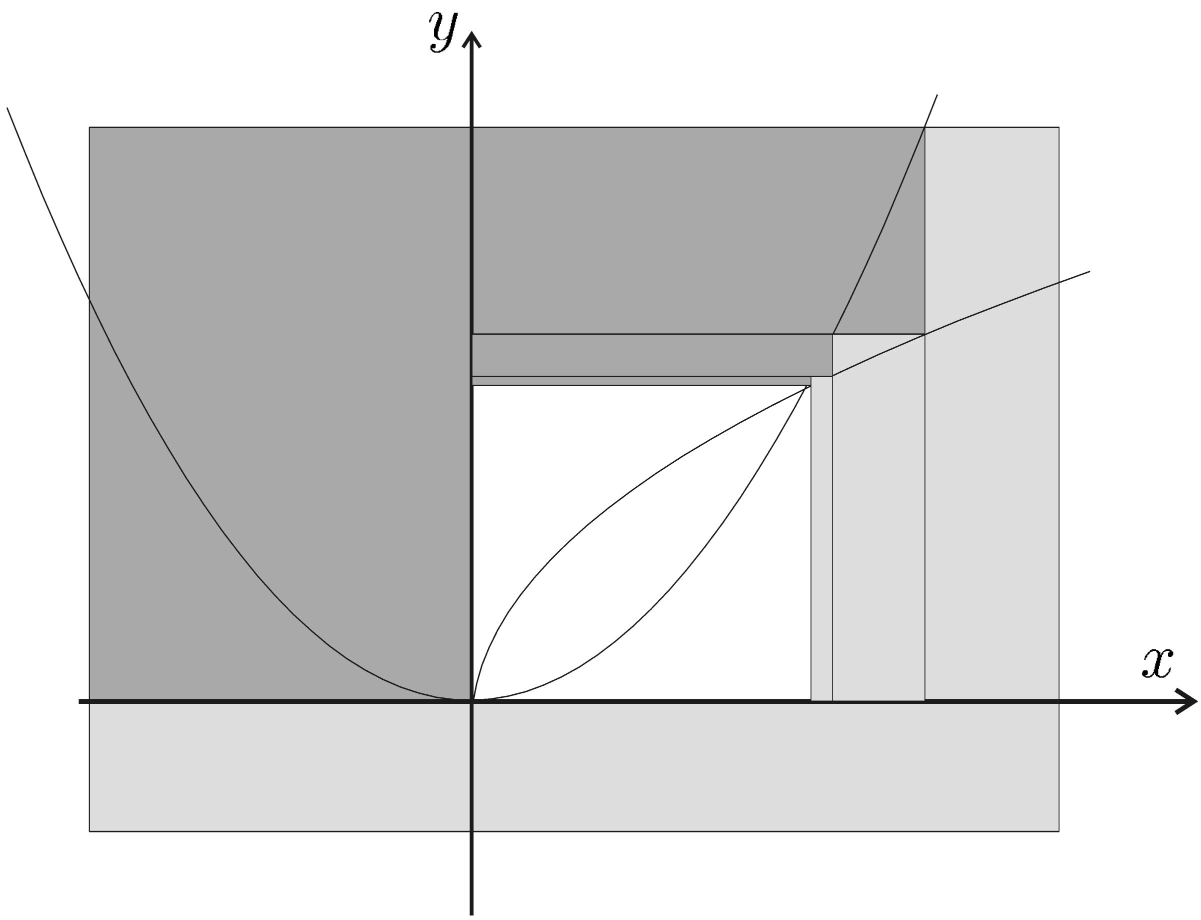
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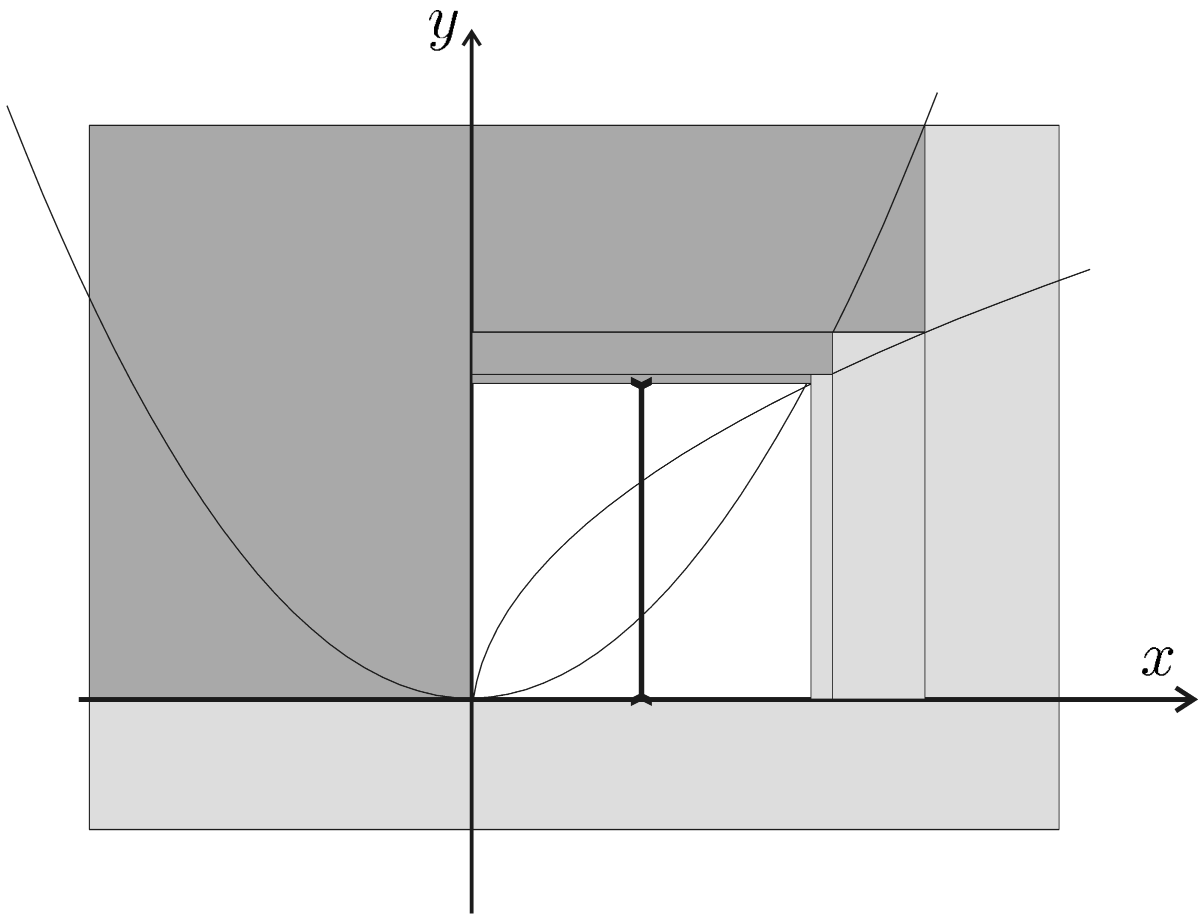

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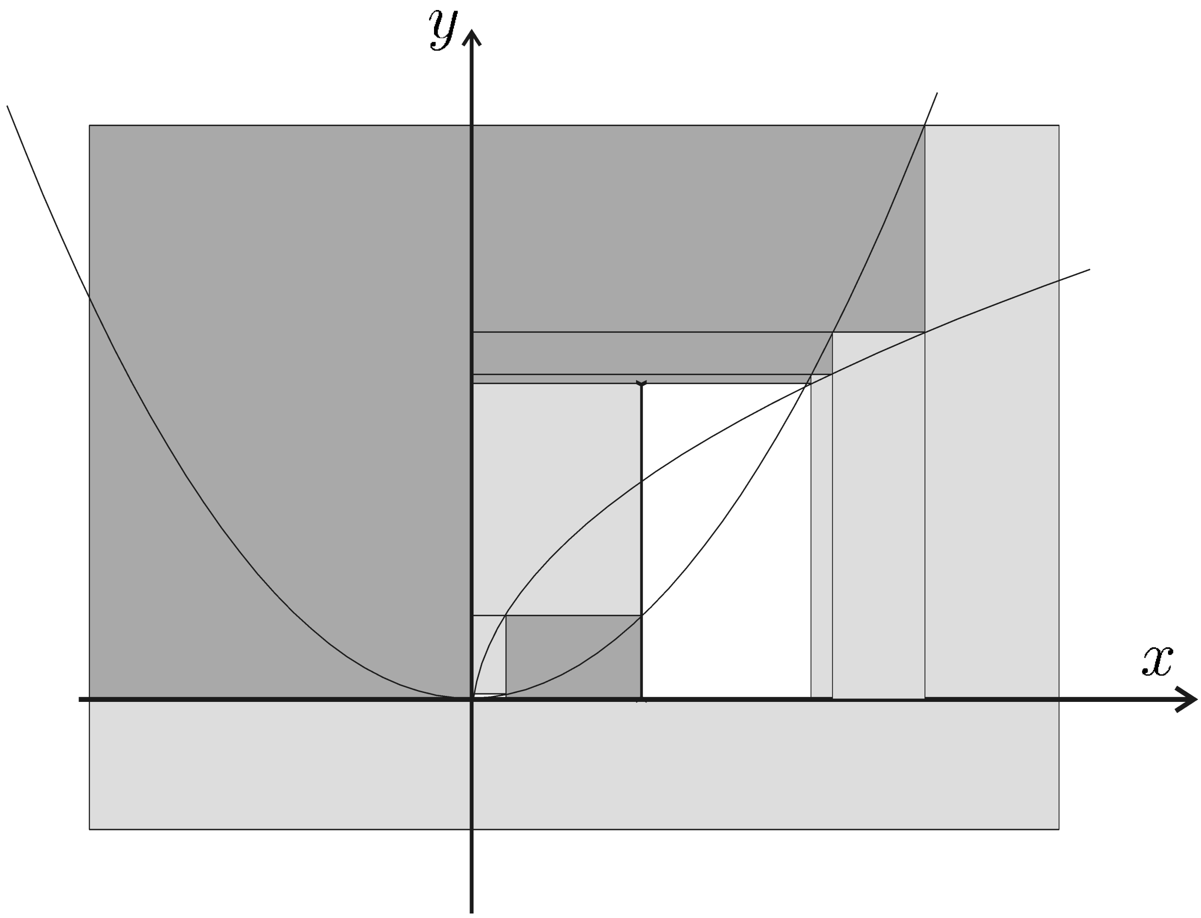

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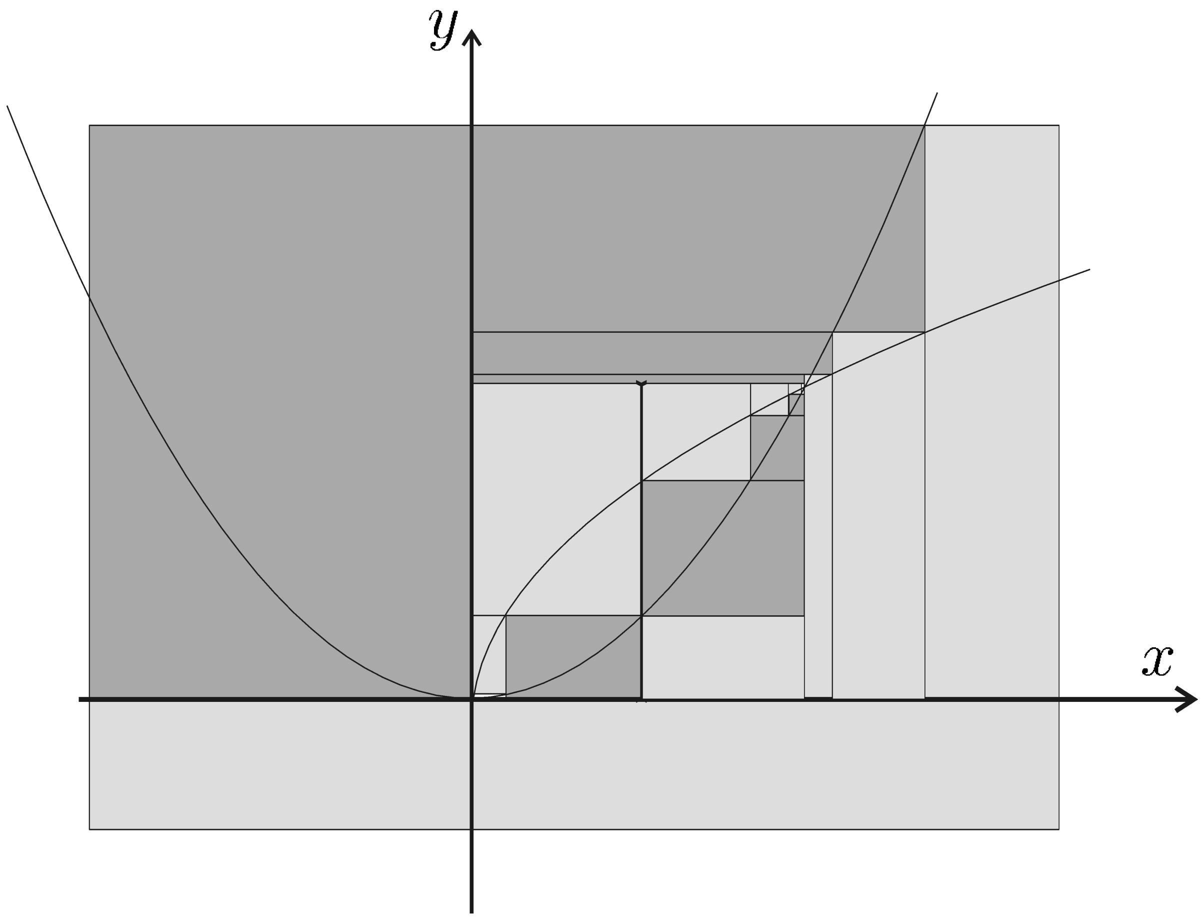


$y$
$x$

## 4.2 Example 2 (local consistency)

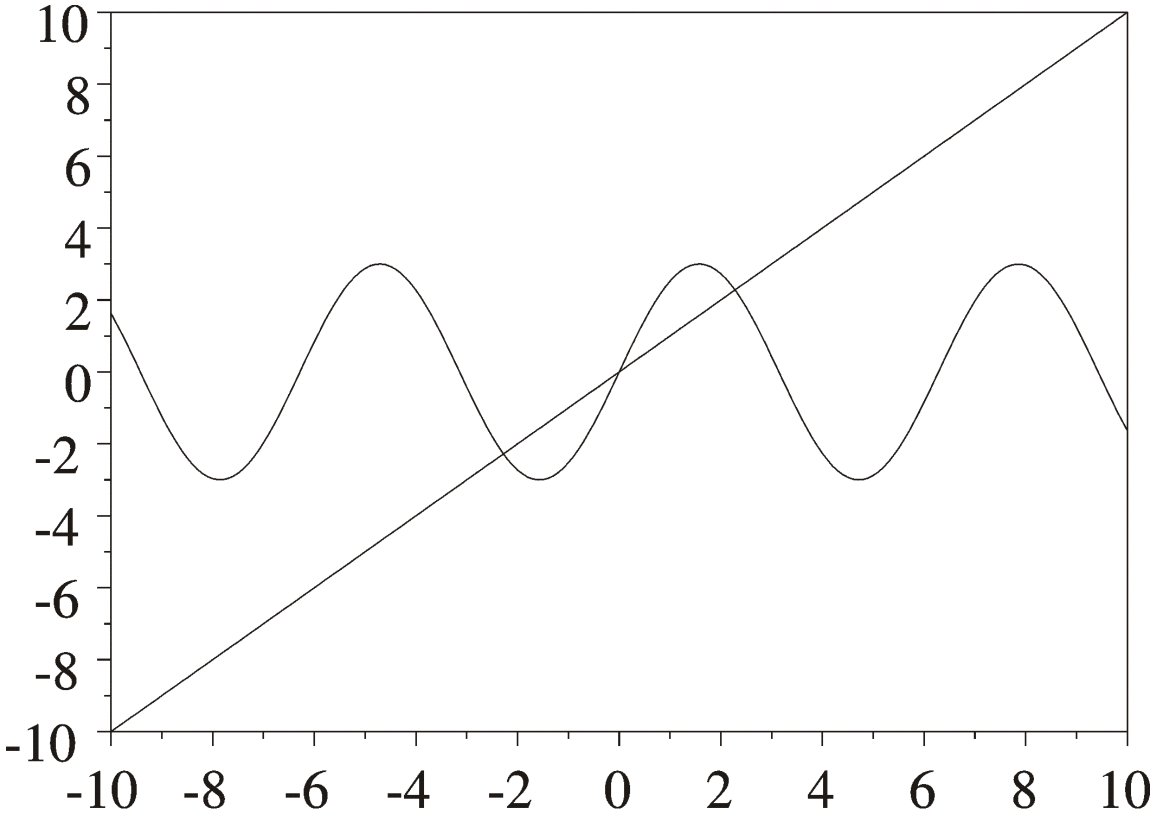
If $\mathcal{C}_{\mathbb{S}_1}^*$ and $\mathcal{C}_{\mathbb{S}_2}^*$ are two minimal contractors for $\mathbb{S}_1$ and $\mathbb{S}_2$ then

$$\mathcal{C}_{\mathbb{S}} = \mathcal{C}_{\mathbb{S}_1}^* \circ \mathcal{C}_{\mathbb{S}_2}^* \circ \mathcal{C}_{\mathbb{S}_1}^* \circ \mathcal{C}_{\mathbb{S}_2}^* \circ \dots$$

is a contractor for $\mathbb{S} = \mathbb{S}_1 \cap \mathbb{S}_2$, but it is not always optimal. This is the *local consistency effect.*

**Exemple**. Consider the system

$$\left\{ \begin{array}{rcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$



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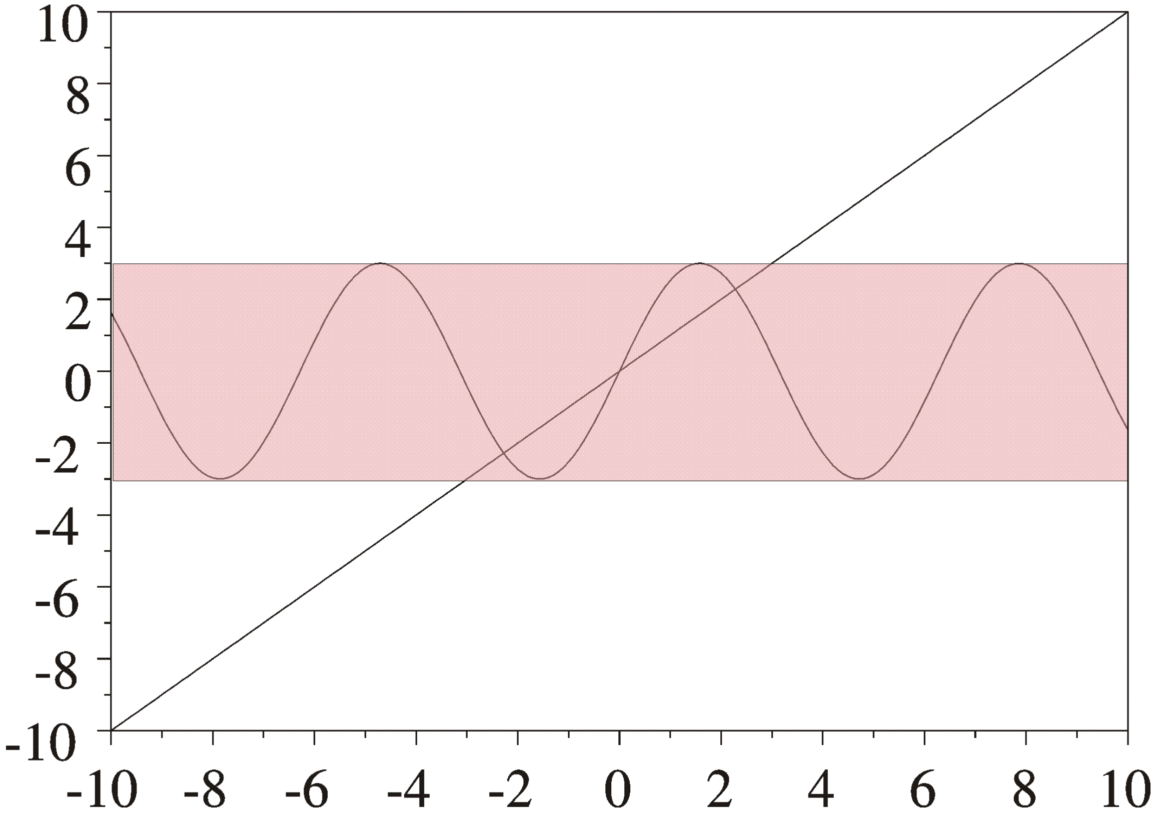

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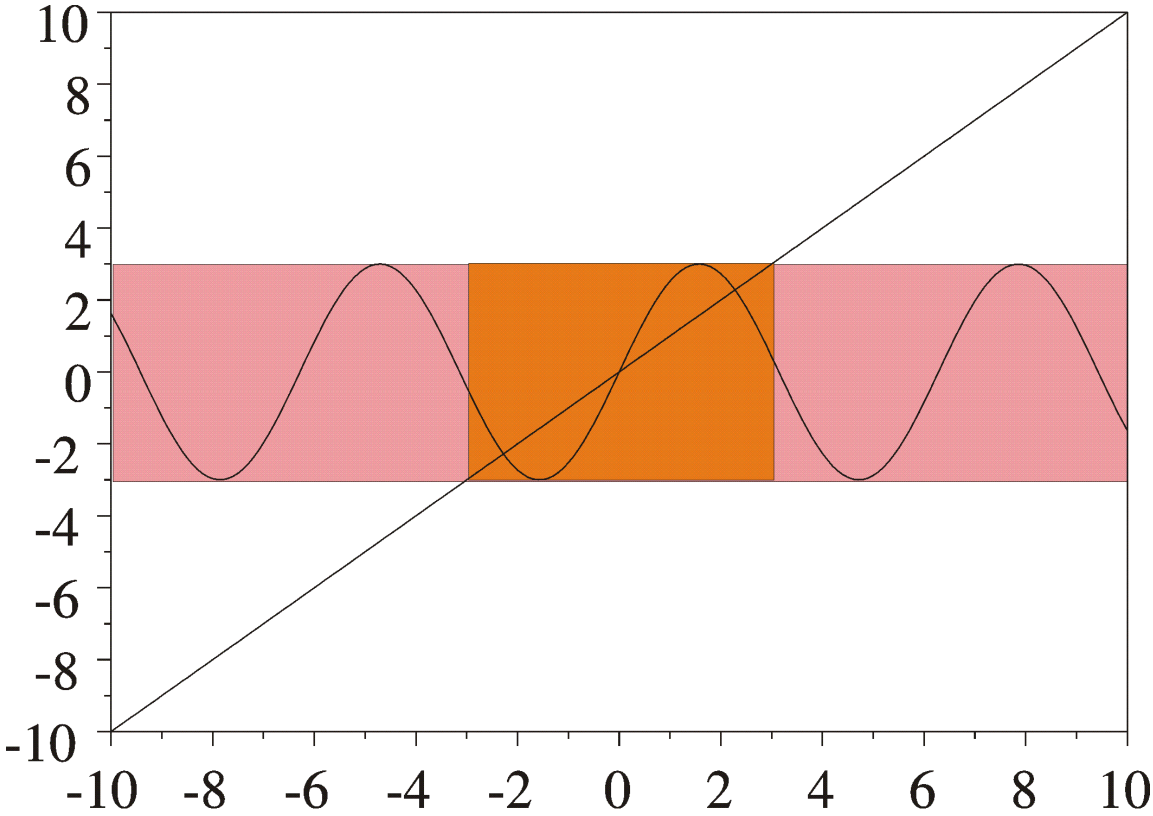

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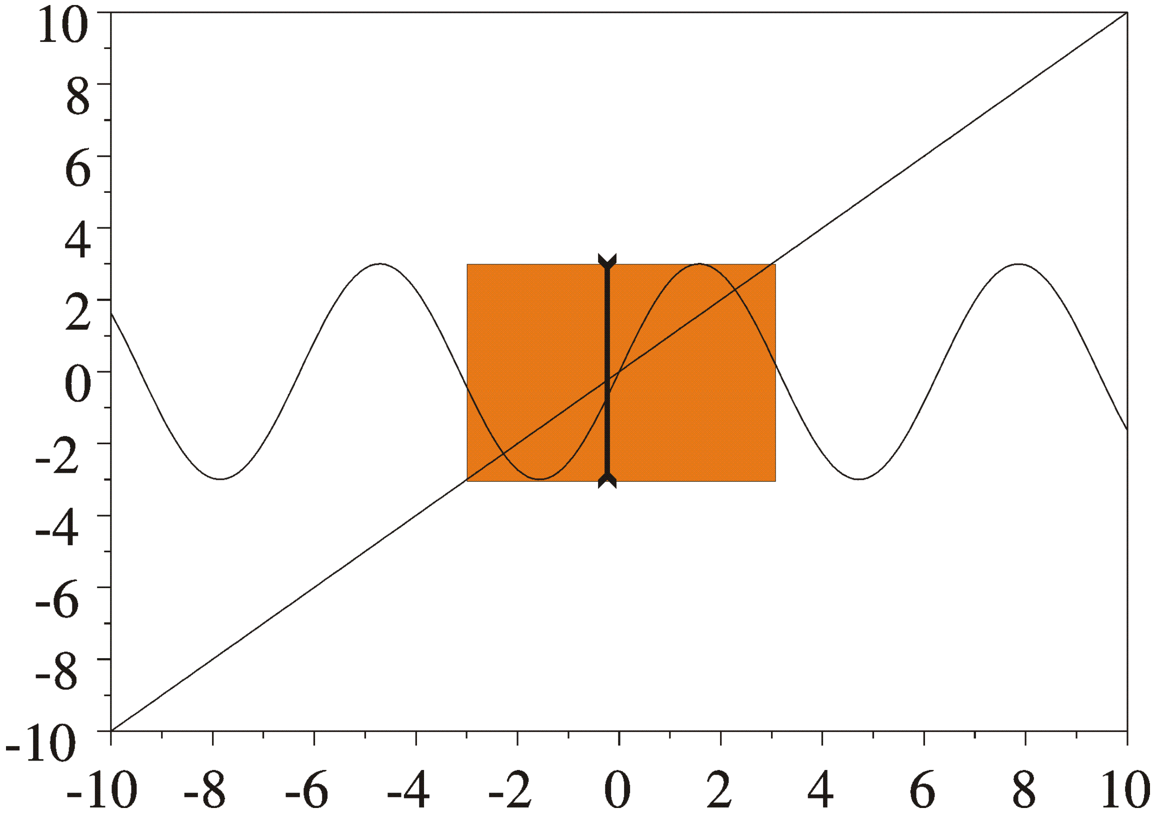
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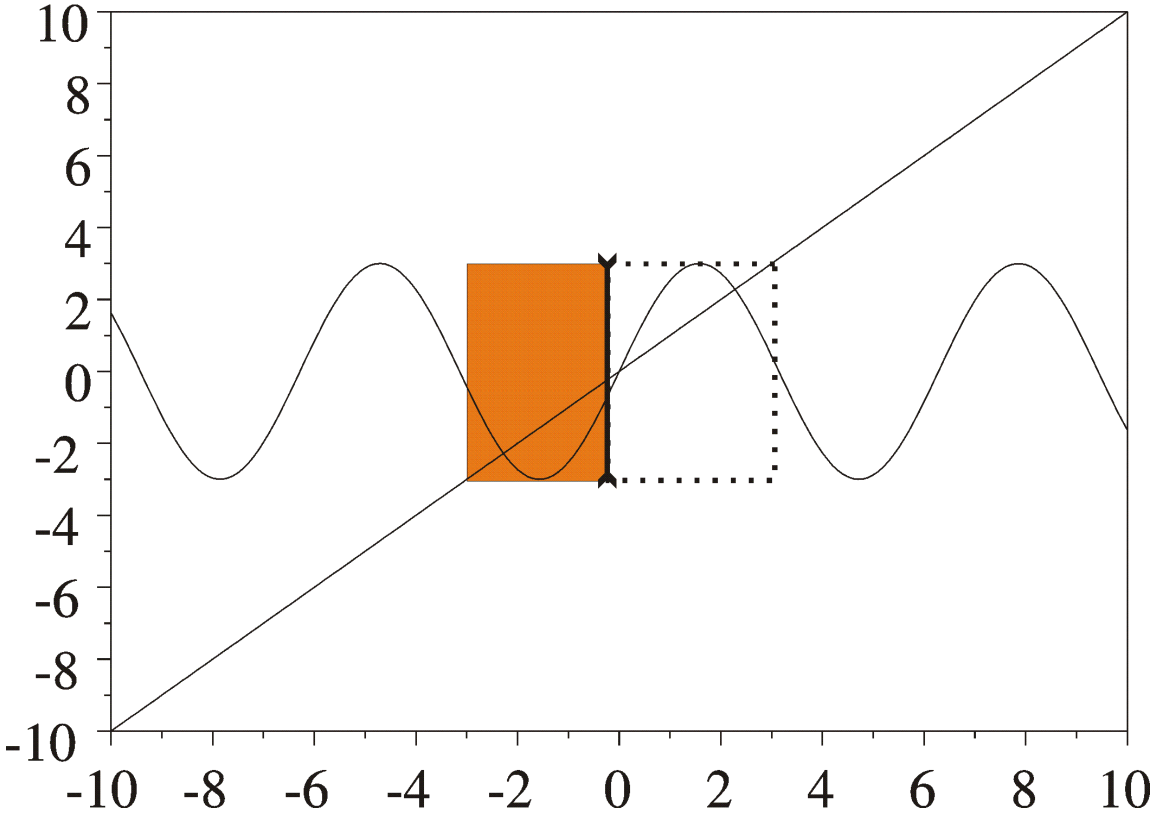

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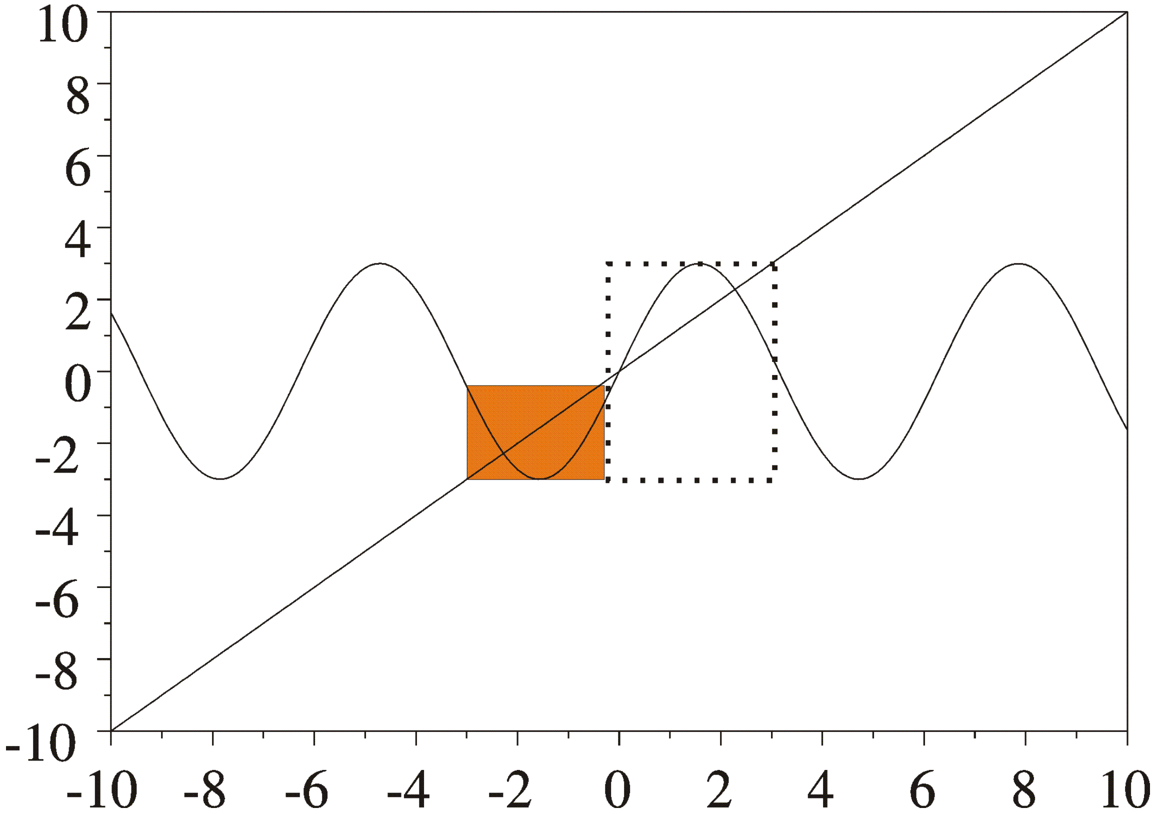

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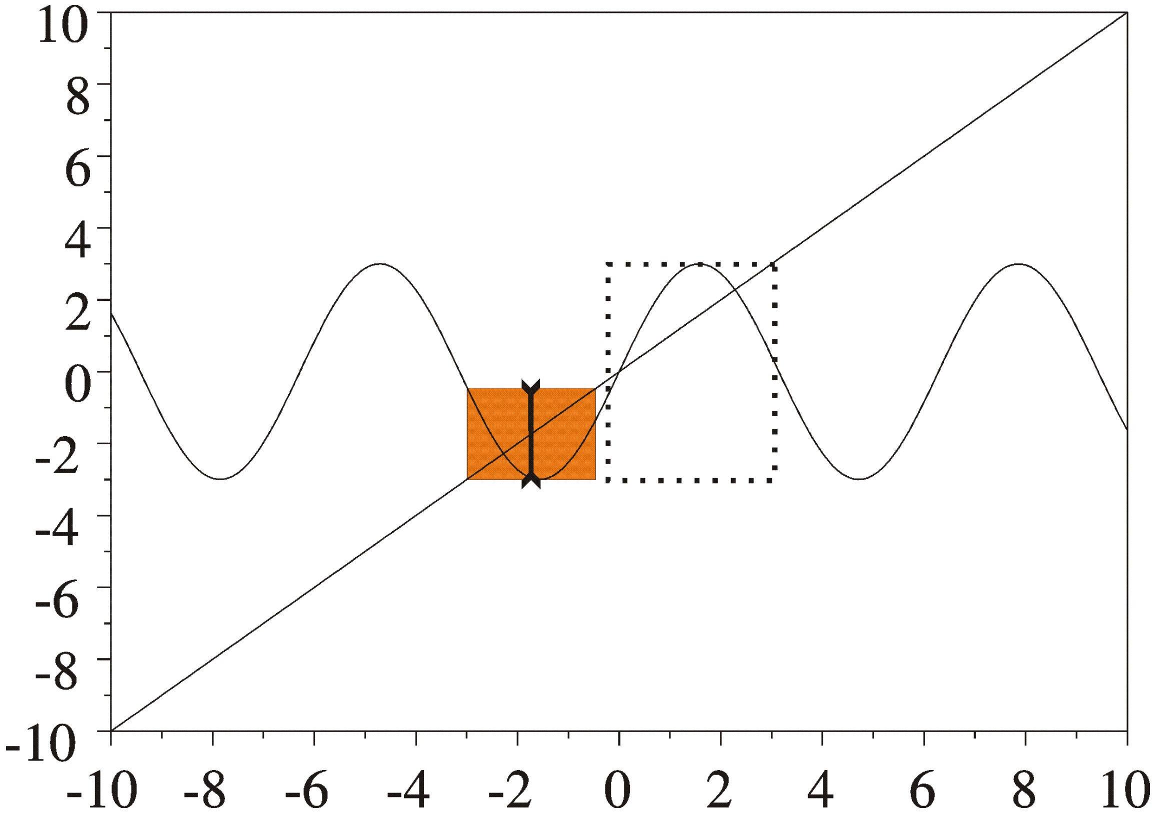

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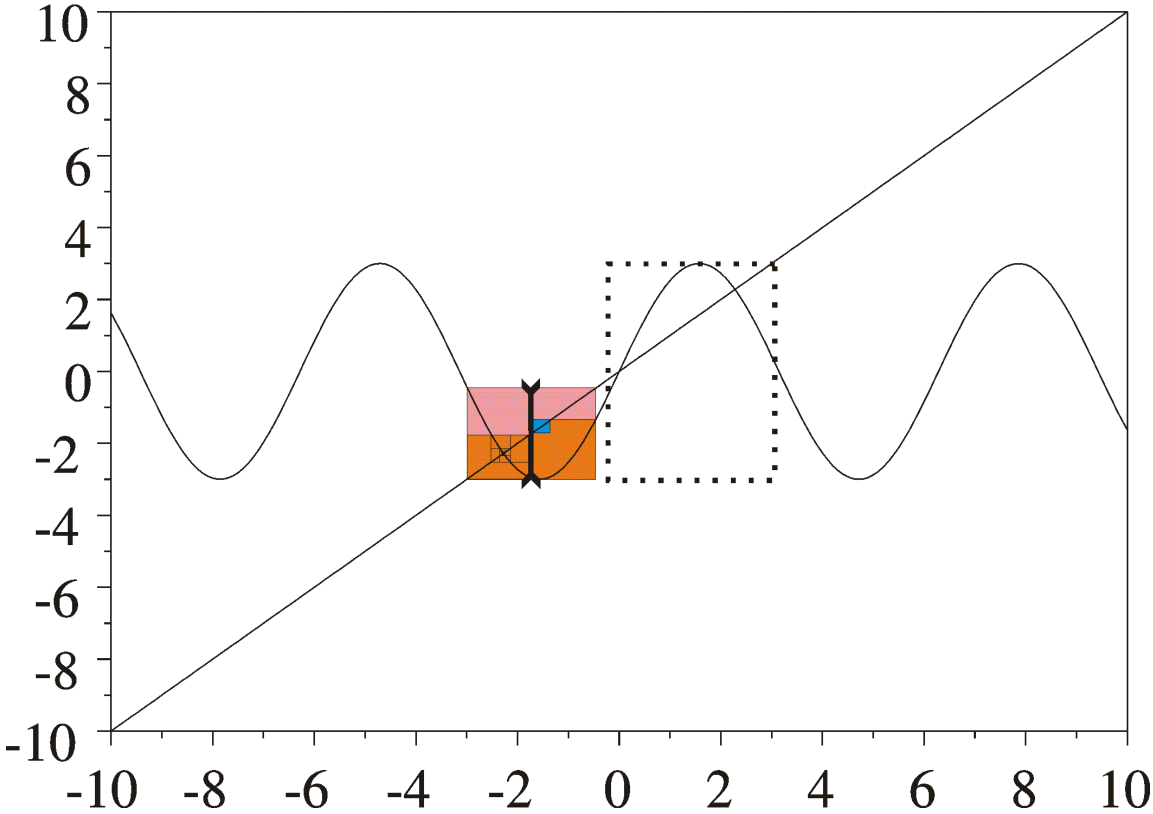
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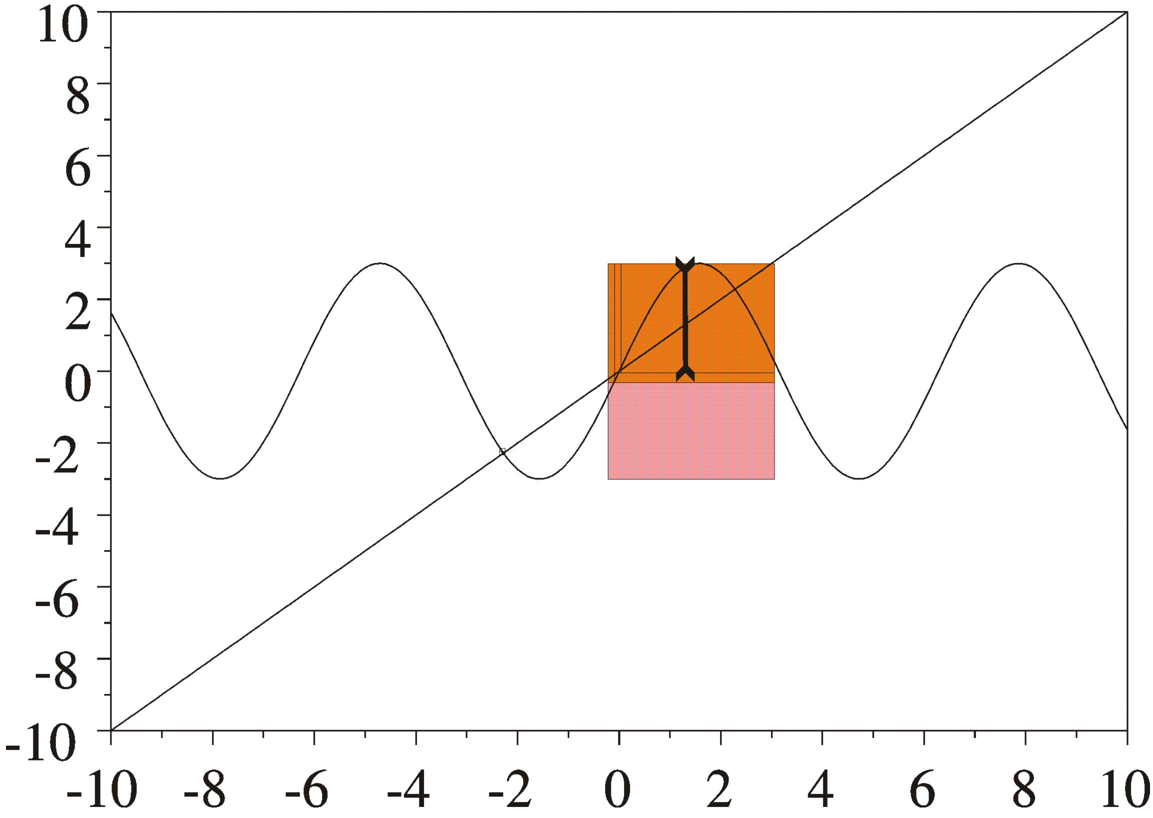
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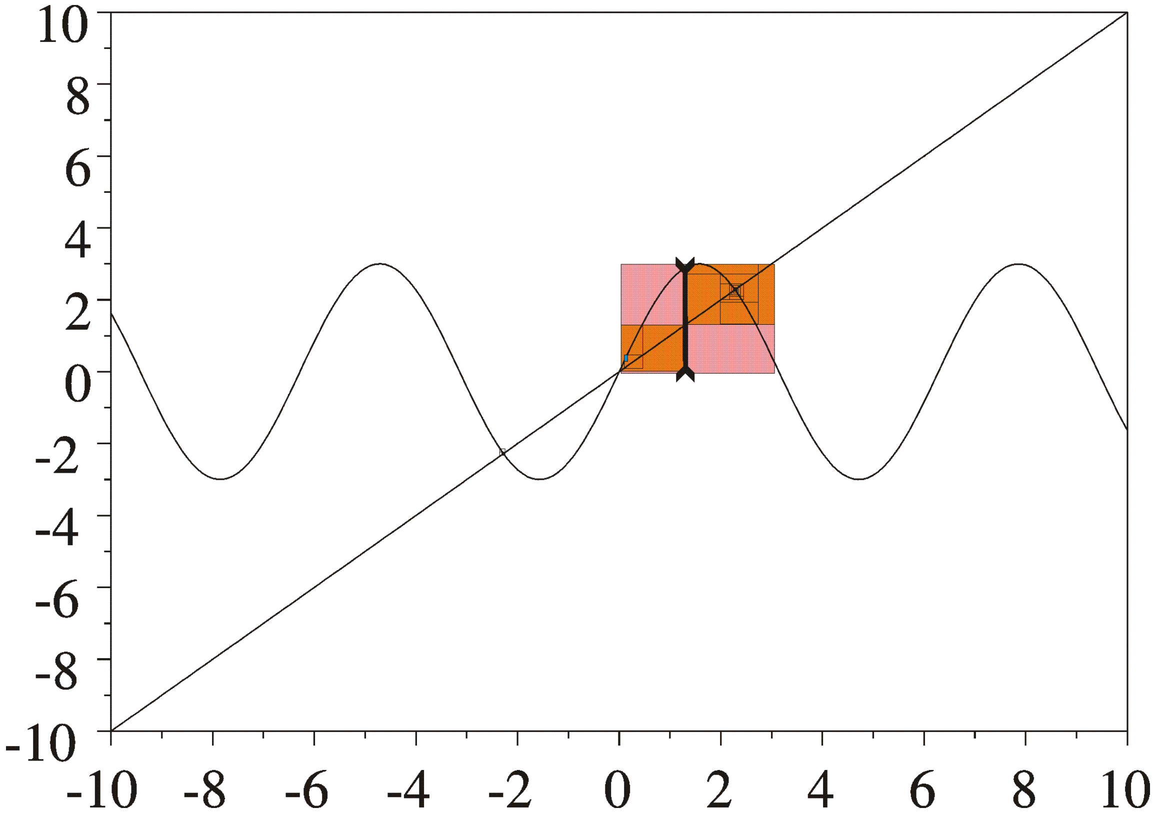
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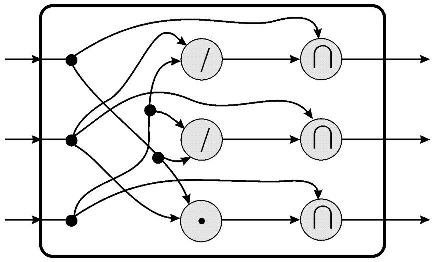
# 5 Contractor algebra

| | |
|---|---|
| intersection | $(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$ |
| union | $(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$ |
| composition | $(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$ |
| repetition | $\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$ |
| repeat intersection | $\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$ |
| repeat union | $\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^\infty$ |

# 6 A link between matrices and contractors

$$\begin{array}{ccc} \text{linear application} & \longrightarrow & \text{matrices} \\ \mathcal{L}:\left\{\begin{array}{lcl} \alpha & = & 2a+3h \\ \gamma & = & h-5a \end{array}\right. & \longrightarrow & \mathbf{A}=\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \end{array}$$

We have a matrix algebra and Matlab.
We have: var$(\mathcal{L}) = \{a, h\}$, covar$(\mathcal{L}) = \{\alpha, \gamma\}$.
But we cannot write: var$(\mathbf{A}) = \{a, h\}$, covar$(\mathbf{A}) = \{\alpha, \gamma\}$.

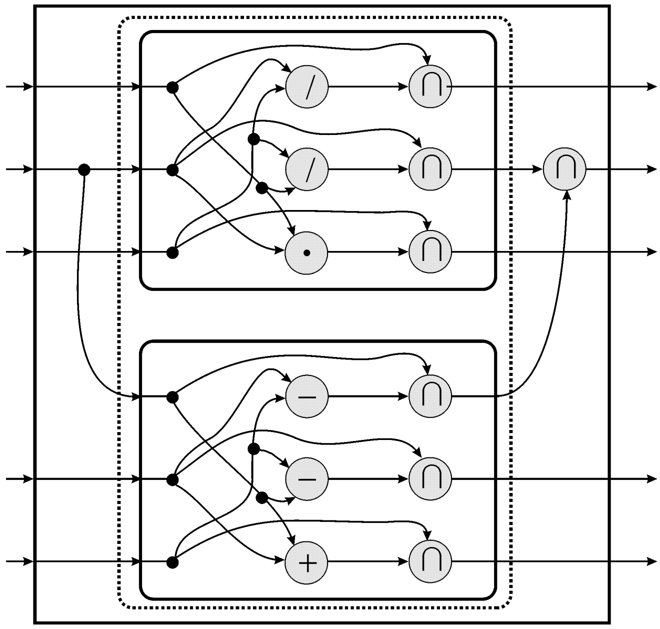
| constraint | $\longrightarrow$ | contractor |
|---|---|---|
| $a \cdot b = z$ | $\longrightarrow$ |  |

## Contractor fusion

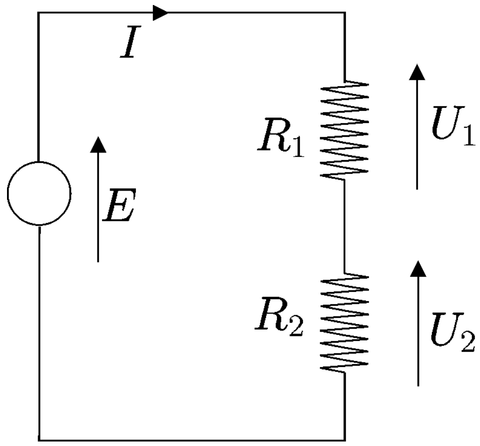
$$\begin{cases} a \cdot b = z & \rightarrow & \mathcal{C}_1 \\ b + c = d & \rightarrow & \mathcal{C}_2 \end{cases}$$

Since $b$ occurs in both constraints, we fuse the two contractors as:

$$\begin{aligned} \mathcal{C} &= \mathcal{C}_1 \times \mathcal{C}_2 \rfloor_{(2,1)} \\ &= \mathcal{C}_1 | \mathcal{C}_2 \text{ (for short)} \end{aligned}$$



# 7 Circuits



$I$
$R_1$
$U_1$
$E$
$R_2$
$U_2$

Domains

$$
\begin{aligned}
E &\in [23V, 26V]; I \in [4A, 8A]; \\
U_1 &\in [10V, 11V]; U_2 \in [14V, 17V]; \\
P &\in [124W, 130W]; R_1 \in [0, \infty[ \text{ and } R_2 \in [0, \infty[.
\end{aligned}
$$

Constraints

(i) $P = EI$, (ii) $E = (R_1 + R_2)\, I$, (iii) $U_1 = R_1 I$, (iv) $U_2 = R_2 I$, (v) $E = U_1 + U_2$.

Solution set

$$\mathbb{S} = \left\{ \begin{pmatrix} E \\ R_1 \\ R_2 \\ I \\ U_1 \\ U_2 \\ P \end{pmatrix} \in \begin{pmatrix} [23, 26] \\ [0, \infty[ \\ [0, \infty[ \\ [4, 8] \\ [10, 11] \\ [14, 17] \\ [124, 130]; \end{pmatrix}, \left\{ \begin{array}{l} P = EI \\ E = (R_1 + R_2)\, I \\ U_1 = R_1 I \\ U_2 = R_2 I \\ E = U_1 + U_2 \end{array} \right. \right\}$$

```
variables
 E in [23 ,26];
 I in [4,8];
 U1 in [10,11];
 U2 in [14 ,17];
 P in [124,130];
 R1 in [0 ,1e08 ];
 R2 in [0 ,1e08 ];
contractor_list L
 P=E*I;
 E=(R1+R2)*I;
 U1=R1*I;
 U2=R2*I;
 E=U1+U2;
end
```

```
contractor C
   compose(L);
end
contractor epsilon
   precision(1);
end
```

Quimper returns

$$[24;26] \times [1.846;2.307] \times [2.584;3.355]$$
$$\times [4.769;5.417] \times [10;11] \times [14;16] \times [124;130]\,,$$

i.e.,

$$\begin{array}{ll} E \in [24;26]\,, & R_1 \in [1.846;2.307]\,, \\ R_2 \in [2.584;3.355], & I \in [4.769;5.417]\,, \\ U_1 \in [10;11]\,, & U_2 \in [14;16]\,, \\ P \in [124;130]\,. & \end{array}$$