Interval and fuzzy numbers. Applications to optimization

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Abstract

In this talk, we present the concepts of intervals and fuzzy numbers, as well as their arithmetics and partial orders. We discuss their applications to modeling and optimization problems. To the latter, some formulations and computational algorithms are presented for the case of linear problems. And for the case of nonlinear problems, some tools, such as differentials and optimiality conditions are commented.

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Basic Operations

Intervals

K_c is the class of all closed and bounded intervals in R.
$K_c = \{[a^L, a^U], a^L, a^U \in R, a^L \leq a^U\}$.

Let $A = [a^L, a^U]$, $B = [b^L, b^U]$, $A, B \in K_c$. (equivalently, $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, or $A = [a^-, a^+]$, $B = [b^-, b^+]$)

Sum of intervals

$A + B = \{a + b : a \in A, b \in B\} = [a^L + b^L, a^U + b^U]$.

Example. $A = [-3, 2]$, $B = [-8, -1]$. $A + B = [-3 + (-8), 2 + (-1)] = [-11, 1]$.

An opposite value

$-A = \{-a : a \in A\} = [-a^U, -a^L]$.

Example. $A = [7, 9]$, $-A = [-9, -7]$.

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Basic Operations

Multiplication by number

$$\lambda A = \{\lambda a : a \in A\} = \begin{cases} [\lambda a^L, \lambda a^U], & \text{if } \lambda \geq 0, \\ |\lambda|[-a^U, -a^L], & \text{if } \lambda < 0; \end{cases}$$

where $\lambda \in R$ is a real number.

Example. If $A = [2, 5]$, $\lambda = 4$, then $\lambda A = [4 \cdot 2, 4 \cdot 5] = [8, 20]$.
If $\lambda = -4$, then $\lambda A = |-4|[-5, -2] = [-20, -8]$.

Multiplication of intervals

$A \times B = ab : a \in A, b \in B = [\text{mín}_{ab}, \text{máx}_{ab}]$, where
$\text{mín}_{ab} = \text{mín}\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$.

Example. Consider $A = [-1, 3], B = [2, 5] \in \mathcal{K}_C$,

$$a^L b^L = -1 \cdot 2 = -2, \qquad a^L b^U = -1 \cdot 5 = -5,$$
$$a^U b^L = 3 \cdot 2 = 6, \qquad a^U b^U = 3 \cdot 5 = 15,$$

then $A \times B = [-5, 15]$.

Basic Operations

Multiplication by number

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Example. Consider $A = [-1, 3], B = [2, 5] \in \mathcal{K}_C$,

$$a^L b^L = -1 \cdot 2 = -2, \qquad a^L b^U = -1 \cdot 5 = -5,$$
$$a^U b^L = 3 \cdot 2 = 6, \qquad a^U b^U = 3 \cdot 5 = 15,$$

then $A \times B = [-5, 15]$.

Difference

Minkowski difference

The difference can be definded via a sum: $A - B = A + (-B) = [a^L - b^U, a^U - b^L]$.

Example. $A = [-6, 7]$. It is expected that $A - A = 0$. But:
$A - A = A + (-A) = [-6, 7] + (-1)[-6, 7] = [-6, 7] + [-7, 6] = [-13, 13] \neq 0$.

Definition (Hukuhara difference)

Let $A = [a^L, a^U]$, $B = [b^L, b^U]$ be two closed intervals in K_c. If $a^L - b^L \leq a^U - b^U$, then the Hukuhara difference $C = A \ominus_H B$ exists and $C = [a^L - b^L, a^U - b^U]$.

Example. $[1, 3] \ominus_H [2, 8] = C$ but C does not exist because the width of $[2, 8]$ is bigger than $[1, 3]$, so the condition $a^L - b^L \leq a^U - b^U$ is not satisfied.

Difference

Definition (Generalized Hukuhara difference or gH-difference)

Let A, B be two closed intervals in K_c. This differense is defined as follows

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) A = B + C, \\ \text{or } (b) B = A + (-1)C. \end{cases}$$

Example. $A = [-2, 3]$, $A \ominus_{gH} A = [-2, 3] \ominus_{gH} [-2, 3] = [0, 0] = \{0\}$.

Proposition (gH-difference)

The gH-difference $C = A \ominus_{gH} B$ *of two intervals* $A = [a^L, a^U]$ *and* $B = [b^L, b^U]$ *always exists and*

$$[a^L, a^U] \ominus_{gH} [b^L, b^U] = [c^L, c^U],$$

where $c^L = \text{mín}\{a^L - b^L, a^U - b^U\}$, $c^U = \text{máx}\{a^L - b^L, a^U - b^U\}$.

Example. $A = [2, 6]$, $B = [1, 8]$, $A \ominus_{gH} B = C$, $C = [2, 6] \ominus_{gH} [1, 8] = [-2, 1]$.
$B = A + (-1)C = [2, 6] + (-1)[-2, 1] = [2, 6] + [-1, 2] = [1, 8]$.

orders and metric

Hausdorff metric

Given $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$:

$$H(A, B) = \text{máx}\left\{\left|\underline{a} - \underline{b}\right|, \left|\overline{a} - \overline{b}\right|\right\}.$$

$(\mathcal{K}_C, H)$ is a complete metric space

Ordering relation

Let $A = [a^L, a^U]$, $B = [b^L, b^U]$, $A \in K_c$, $B \in K_c$.

- $A \leqq_{LU} B \Leftrightarrow a^L \leq b^L$ and $a^U \leq b^U$,
- $A \leq_{LU} B \Leftrightarrow A \leqq_{LU} B$ and $A \neq B$,
- $A <_{LU} B \Leftrightarrow a^L < b^L$ and $a^U < b^U$.

Differentiation of interval-valued functions

Definition (H-derivative)

Let X be an open set in R. An interval-valued function $f : X \to K_c$ is called H-differentiable (or strongly differentiable) at x_0 if there exists a closed interval $A(x_0)$ (note that this interval depends on x_0) in R such that the limits

$$\lim_{h \to 0+} \frac{f(x_0 + h) \ominus f(x_0)}{h} \text{ and } \lim_{h \to 0+} \frac{f(x_0) \ominus f(x_0 - h)}{h}$$

both exists and are equal to $A(x_0)$. In this case, $A(x_0)$ is called the H-derivative of f at x_0.

Definition (gH-derivative)

Let $x_0 \in T = (t_1, t_2)$ and let $f : T \to K_c$ be an interval-valued function, then the generalized Hukuhara derivative (gH-derivative, for short) of f at x_0 is defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h}.$$

Differentiation of interval-valued functions

Theorem (gH-differetiable function)

Let $f : (a, b) \to K_c$ be such that $f(x) = [f^L(x), f^U(x)]$. If $f^L(x)$ and $f^U(x)$ are differentiable functions at $x \in (a, b)$, then $f(x)$ is gH-differetiable at x, and

$$f'(x) = [\text{mín}\{(f^L)'(x_0), (f^U)'(x_0)\}, \text{máx}\{(f^L)'(x_0), (f^U)'(x_0)\}].$$

Definition (Gradient of the interval-valued function)

Let $f(x)$ be an interval-valued function defined on Ω, where Ω is an open subset of R^n, $x_0 = \left(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}\right)$. The partial differentiation with respect to the ith variable $x_i, i = 1, \ldots, n$ has the following form

$$\frac{\partial f}{\partial x_i}(x_0) = \left[\text{mín}\left\{\frac{\partial f^L}{\partial x_i}(x_0), \frac{\partial f^U}{\partial x_i}(x_0)\right\}, \text{máx}\left\{\frac{\partial f^L}{\partial x_i}(x_0), \frac{\partial f^U}{\partial x_i}(x_0)\right\}\right],$$

the gradient of the interval-valued function $f(x)$ at x_0 has the form:

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0)\right)^t.$$

Differentiation of interval-valued functions

Definition

Let $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$, $F(x) = (\widehat{F}(x); \widetilde{F}(x)) = [\widehat{F}(x) - \widetilde{F}(x), \widehat{F}(x) + \widetilde{F}(x)]$ and let $x^{(0)} \in K$ such that $x^{(0)} + h \in K$, for all $h \in \mathbb{R}^n$ with $||h|| < \delta$ for a given $\delta > 0$. We say that F is gH-differentiable at $x^{(0)}$ if and only if there exist two vectors $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$, $\widehat{w} = (\widehat{w}_1, ..., \widehat{w}_n)$, $\widetilde{w} = (\widetilde{w}_1, ..., \widetilde{w}_n)$ and two functions $\widehat{\varepsilon}(h)$, $\widetilde{\varepsilon}(h)$ with $\lim_{h\to 0} \widehat{\varepsilon}(h) = \lim_{h\to 0} \widetilde{\varepsilon}(h) = 0$, such that, for all $h \neq 0$,

$$\widehat{F}(x^{(0)} + h) - \widehat{F}(x^{(0)}) = \sum_{j=1}^{n} h_j \widehat{w}_j + ||h|| \widehat{\varepsilon}(h), \quad \left|\widetilde{F}(x^{(0)} + h) - \widetilde{F}(x^{(0)})\right| = \left|\sum_{j=1}^{n} h_j \widetilde{w}_j + ||h|| \widetilde{\varepsilon}(h)\right|.$$

The gH-differential is the interval-valued function $D_{gH}F(x^{(0)}) : \mathbb{R}^n \to \mathcal{K}_C$

$$W(h) = D_{gH}F(x^{(0)})(h) = \left(\sum_{j=1}^{n} h_j \widehat{w}_j ; \left|\sum_{j=1}^{n} h_j \widetilde{w}_j\right|\right). \tag{1}$$

$$\lim_{h\to 0} \frac{\left(F(x^{(0)} + h) \ominus_{gH} F(x^{(0)})\right) \ominus_{gH} W(h)}{||h||} = 0 \tag{2}$$

$$\nabla_{gH} F(x^{(0)}) = (w_1, w_2, ..., w_n) = \left(\frac{\partial F(x^{(0)})}{\partial x_1}, ..., \frac{\partial F(x^{(0)})}{\partial x_n}\right) \tag{3}$$

Contents

Fuzzy numbers

Fuzzy set on $\mathbb{R}^n$

- $u : \mathbb{R}^n \to [0, 1]$
- $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$
- Support: $supp(u) = \{x \in \mathbb{R}^n \mid u(x) > 0\}$
- Closure: $[u]^0 = cl(supp(u))$

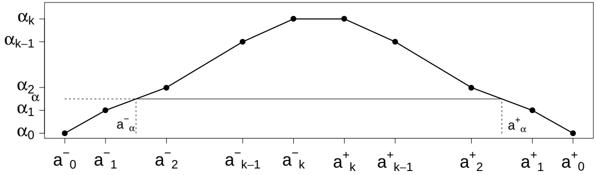
Fuzzy number

A fuzzy set u on $\mathbb{R}$ is said to be a fuzzy interval if

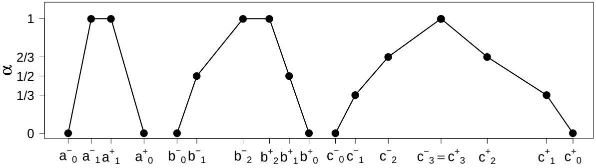
- u is normal, i.e. there exists $x_0 \in R$ such that $u(x_0) = 1$,
- u is an upper semi-continuous function,
- $u(\lambda x + (1 - \lambda)y) \geq \text{mín}\{u(x), y(x)\}, x, y \in R, \lambda \in [0, 1]$,
- $[u]^0$ is compact.

Fuzzy numbers

k−polygonal Fuzzy number



k−polygonal regular Fuzzy number



Operations and orders in with fuzzy numbers

Arithmetic operations

Consider the fuzzy intervals and $\lambda \in \mathbb{R}$. Then $u + v$ and λu, in terms of α-levels:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha = \left[(\underline{u+v})_\alpha, (\overline{u+v})_\alpha\right] = \left[\underline{u}_\alpha + \underline{v}_\alpha\ ,\ \overline{u}_\alpha + \overline{v}_\alpha\right]$$

$$[\lambda u]^\alpha = \lambda[u]^\alpha = \left[(\underline{\lambda u})_\alpha, (\overline{\lambda u})_\alpha\right] = \left[\text{mín}\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}, \text{máx}\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}\right].$$

Definition

Given two fuzzy intervals u, v, the generalized Hukuhara difference (gH-difference for short) is the fuzzy interval w, if it exists, such that

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} & (i)\ u = v + w, \\ \text{or} & (ii)\ v = u + (-1)w. \end{cases}$$

If $u \ominus_{gH} v$ exists then, in terms of α-levels, we have $[u \ominus_{gH} v]^\alpha = [u]^\alpha \ominus_{gH} [v]^\alpha$ for all $\alpha \in [0, 1]$.

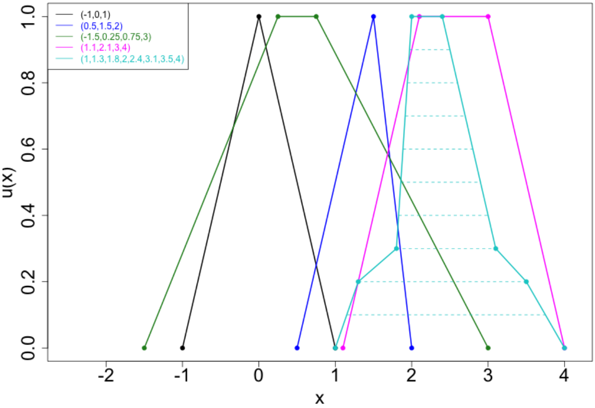
Orders

Given $u, v \in \mathcal{F}_C$, we say that

(i) $u \leqq v$ if and only if $\underline{u}_\alpha \leq \underline{v}_\alpha$ and $\overline{u}_\alpha \leq \overline{v}_\alpha$, for all $\alpha \in [0, 1]$,

(ii) $u \prec v$ if and only if $\underline{u}_\alpha < \underline{v}_\alpha$ and $\overline{u}_\alpha < \overline{v}_\alpha$, for all $\alpha \in [0, 1]$.

Fuzzy numbers



Definition

Let $K \subset \mathbb{R}$ with $F : K \to \mathcal{F}_C$ a fuzzy function and $x_0 \in K$ and h be such that $x_0 + h \in K$. Then the generalized Hukuhara derivative (gH-derivative, for short) of F at x_0 is defined as

$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0 + h) \ominus_{gH} F(x_0)}{h}. \tag{4}$$

If $F'(x_0) \in \mathcal{F}_C$ satisfying (4) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

Example

We consider the fuzzy mapping $F : \mathbb{R} \to \mathcal{F}_C$ defined by $F(x) = C \cdot x$, where C is a fuzzy interval where $[C]^\alpha = [\underline{C}_\alpha, \overline{C}_\alpha]$ with $\underline{C}_\alpha < \overline{C}_\alpha$. Note that in this case F is a generalization of a linear function. Then

$$F_\alpha(x) = \begin{cases} \left[\underline{C}_\alpha x, \overline{C}_\alpha x\right] & \text{if} \quad x \geq 0; \\ \left[\overline{C}_\alpha x, \underline{C}_\alpha x\right] & \text{if} \quad x < 0. \end{cases}$$

We can see that the endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are not differentiable at $x = 0$. However F is gH-differentiable on $\mathbb{R}$ and $F^{'}(x) = C$ for all $x \in \mathbb{R}$.

Contents

Fuzzy Problem

$$
\begin{array}{lll}
(FP) & \text{Minimize} & F(x) \\
 & \text{subject to:} & x \in K
\end{array}
$$

where $K \subseteq \mathbb{R}^n$ is an nonempty open set, and $F : K \subseteq \mathbb{R}^n \to \mathcal{F}_C$ a fuzzy mapping. K is said to be the feasible set.

$F_\alpha : K \to \mathcal{K}_C$, with $F_\alpha(x) = [\underline{f_\alpha}(x), \overline{f_\alpha}(x)] = [\underline{f}(\alpha, x), \overline{f}(\alpha, x)] \qquad \forall \alpha \in [0, 1]$

Methods for ranking fuzzy numbers

Definition (Ranking Value Function τ)

Let $\tau : \mathcal{F}_C \to \mathbb{R}$ be a function defined by

$$\tau(u) = \int_0^1 \alpha \left[\underline{u}(\alpha) + \overline{u}(\alpha)\right] d\alpha, \quad u \in \mathcal{F}_C. \tag{5}$$

Definition (Order relation $\preceq$)

u precedes v ($u \preceq v$) if and only if $\tau(u) \leq \tau(v)$.
u strictly precedes v ($u \prec v$) if $\tau(u) < \tau(v)$.

Definition

For each fuzzy mapping $F : K \to \mathcal{F}_C$, the ranking function $T_F : K \to \mathbb{R}$ associated to F is defined by

$$T_F(x) = \int_0^1 \alpha \left[\underline{f}(\alpha, x) + \overline{f}(\alpha, x)\right]$$

Note that the real-valued function T_F can be rewritten as being $T_F(x) = \tau(F(x))$.

Definition (Optimal Solution)

Let $x^* \in K$. It is said that x^* is a minimum or optimal solution for $F : K \to \mathfrak{F}_C$ if there exists no $x \in K$ such that $T_F(x) < T_F(x^*)$.

Definition (Stationary Point)

We say that $x^* \in K$ is a stationary point for a gH-differentiable fuzzy mapping $F : K \to \mathfrak{F}_C$ if $\nabla T_F(x^*) = 0$.

Theorem (Optimal Solution → Stationary Point)

If $x^* \in K$ is an optimal solution for F then x^* is an stationary point.

Related works

- Consider several fuzzy objectives (Arana et al. (2015) Inf Sci).
- KKT optimality conditions without ranking functions (Stefanini and Arana (2019) FSS).
- Newton method using ranking functions (Arana and Burgos (2020) Cambridge Scholars P.)

Contents

Triangular fuzzy numbers

Triangular fuzzy numbers

A triangular fuzzy number is a special type of fuzzy number which is well determined by three real numbers $a \leq b \leq c$. It is written $u = (a, b, c)$, and

$$[u]^\alpha = [a + (b - a)\alpha, c - (c - b)\alpha], \quad \text{for all } \alpha \in [0, 1].$$

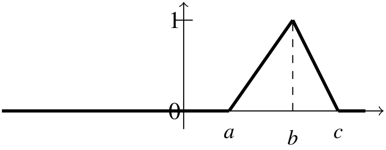


Figura: Graph of the triangular fuzzy number (a, b, c).

Arithmetic operations on the set of triangular fuzzy numbers (see [?, ?])

(i) $\tilde{b} + \tilde{e} = (a, b, c) + (d, e, f) = (a + d, b + e, c + f)$.

(ii) $\lambda\tilde{b} = (\lambda a, \lambda b, \lambda c)$, if $\lambda \geq 0$; and, $\lambda\tilde{b} = (\lambda c, \lambda b, \lambda a)$, if $\lambda < 0$.

(iii) If $\tilde{e}$ is a nonnegative triangular fuzzy number, then

$$\tilde{b}\tilde{e} = \begin{cases} (ad, be, cf), & \text{if } a \geq 0, \\ (af, be, cf), & \text{if } a < 0,\ c \geq 0, \\ (af, be, cd), & \text{if } c < 0. \end{cases}$$

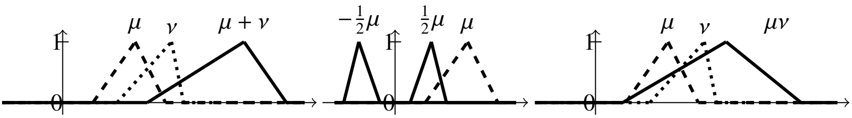


Figura: Operations of triangular fuzzy numbers.

Formulation of a fully fuzzy linear programming problem

(FFLP) Min/Max $\tilde{z} = \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j$

subject to $\sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j \leqq \tilde{b}_i, \quad i = 1, \ldots, m,$

$\tilde{x}_j \geqq 0, \quad j = 1, \ldots, n.$

$\tilde{x}_j$ is a nonnegative fuzzy triangular number, $\quad j = 1, \ldots, n,$

where $\tilde{z}$ is the fuzzy objective function, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c_n})$ is the fuzzy vector with the fuzzy objective function coefficients, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ is the vector with the fuzzy decision variables, and $\tilde{a}_{ij}$ and $\tilde{b}_i$ are the thechnical coeficients.

From now on, and for convenience, triangular fuzzy numbers is noted as $\tilde{u} = (u^-, \hat{u}, u^+)$.

Formulation of a fully fuzzy linear programming problem

$$
\begin{array}{lll}
\text{(FFLP)} & \text{Min/Max} & \tilde{z} = \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j \\
& \text{subject to} & \sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \leqq \tilde{b}_i, \quad i = 1, \ldots, m, \\
& & \tilde{x}_j \geqq 0, \quad j = 1, \ldots, n. \\
& & \tilde{x}_j \text{ is a nonnegative fuzzy triangular number}, \quad j = 1, \ldots, n,
\end{array}
$$

- Lofti et al.(2009) solve the problem under symmetric triangular fuzzy numbers and equaility constraints.
- Kumar er al.(2011) propose a new method for finding the fuzzy optimal solution of (FFLP) problems with equality constraints, with triangular fuzzy numbers involved, although they use ranking function. This type of order relationship has got the advantage that any two triangular fuzzy numbers $\tilde{u}$ and $\tilde{v}$ can be compared, that is,

$$\tilde{u} \leqq_{\mathcal{R}} (\geqq_{\mathcal{R}}) \tilde{v} \quad \text{iff} \quad \mathcal{R}(\tilde{u}) \leqq (\geqq) \mathcal{R}(\tilde{v}).$$

  They propose $\mathcal{R}_1(u) = \frac{u^- + 2\hat{u} + u^+}{4}$.
- Khan et al. (2013, 2017) deal with FFLP with inequalities, and they also compare the objective function values via ranking functions. They propose $\mathcal{R}_2(u) = \frac{7u^- + 5u^+}{6}$.

Formulation of a fully fuzzy linear programming problem

$$
\text{(FFLP)} \quad \text{Min/Max} \quad \tilde{z} = \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j
$$

$$
\begin{aligned}
\text{subject to} \quad & \sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \leqq \tilde{b}_i, \quad i = 1, \ldots, m, \\
& \tilde{x}_j \geqq 0, \quad j = 1, \ldots, n. \\
& \tilde{x}_j \text{ is a nonnegative fuzzy triangular number}, \quad j = 1, \ldots, n,
\end{aligned}
$$

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$$

  They propose $\mathcal{R}_1(u) = \frac{u^- + 2\hat{u} + u^+}{4}$.
- Khan et al. (2013, 2017) deal with FFLP with inequalities, and they also compare the objective function values via ranking functions. They propose $\mathcal{R}_2(u) = \frac{7u^- + 5u^+}{6}$.

Formulation of a fully fuzzy linear programming problem

$$\begin{array}{lll} \text{(FFLP)} \quad \text{Min/Max} & \tilde{z} = \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j & \\ \text{subject to} & \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j \leqq \tilde{b}_i, \quad i = 1, \ldots, m, & \\ & \tilde{x}_j \geqq 0, \quad j = 1, \ldots, n. & \\ & \tilde{x}_j \text{ is a nonnegative fuzzy triangular number}, \quad j = 1, \ldots, n, & \end{array}$$

- Lofti et al.(2009) solve the problem under symmetric triangular fuzzy numbers and equaility constraints.
- Kumar er al.(2011) propose a new method for finding the fuzzy optimal solution of (FFLP) problems with equality constraints, with triangular fuzzy numbers involved, although they use ranking function. This type of order relationship has got the advantage that any two triangular fuzzy numbers $\tilde{u}$ and $\tilde{v}$ can be compared, that is,

$$\tilde{u} \leqq_{\mathcal{R}} (\geqq_{\mathcal{R}}) \tilde{v} \quad \text{iff} \quad \mathcal{R}(\tilde{u}) \leqq (\geqq) \mathcal{R}(\tilde{v}).$$

  They propose $\mathcal{R}_1(u) = \frac{u^- + 2\hat{u} + u^+}{4}$.
- Khan et al. (2013, 2017) deal with FFLP with inequalities, and they also compare the objective function values via ranking functions. They propose $\mathcal{R}_2(u) = \frac{7u^- + 5u^+}{6}$.

Partial order relationship

The relationship between two triangular fuzzy numbers $\tilde{u}$ and $\tilde{v}$, under ranking function, depends on the definition of $\mathcal{R}$. In order to ovoid this dependence, we propose the following.

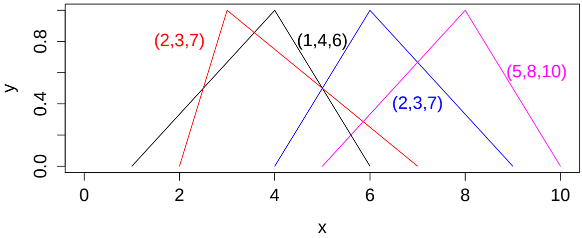
Definition

Given $u, v \in \mathcal{F}_C$, we say that

(i) $u \leqq v$ if and only if $\underline{u}_\alpha \leq \underline{v}_\alpha$ and $\overline{u}_\alpha \leq \overline{v}_\alpha$, for all $\alpha \in [0,1]$,

(ii) $u \prec v$ if and only if $\underline{u}_\alpha < \underline{v}_\alpha$ and $\overline{u}_\alpha < \overline{v}_\alpha$, for all $\alpha \in [0,1]$.

However, and with respect to the partial orders introduced by the previous definition, we have that if $u \prec (\leqq) v$, then $\mathcal{R}_i(\tilde{u}) < (\leqq) \mathcal{R}_i(\tilde{v})$, for $i = 1, 2$; and the same for $u \succ (\geqq) v$.

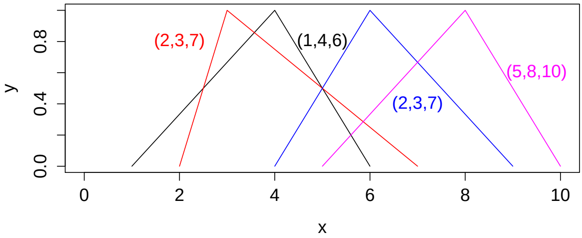


Theorem

Given two triangular fuzzy numbers $\tilde{u} = (u^-, \hat{u}, u^+)$ and $\tilde{v} = (v^-, \hat{v}, v^+)$, it follows that

(i) $\tilde{u} < \tilde{v}$ *if and only if* $u^- < v^-$, $\hat{u} < \hat{v}$ *and* $u^+ < v^+$.

(ii) $\tilde{u} \leqq \tilde{v}$ *if and only if* $u^- \leq v^-$, $\hat{u} \leq \hat{v}$ *and* $u^+ \leq v^+$.



Theorem

Given two triangular fuzzy numbers $\tilde{u} = (u^-, \hat{u}, u^+)$ *and* $\tilde{v} = (v^-, \hat{v}, v^+)$*, it follows that*

(i) $\tilde{u} < \tilde{v}$ *if and only if* $u^- < v^-$, $\hat{u} < \hat{v}$ *and* $u^+ < v^+$.

(ii) $\tilde{u} \leqq \tilde{v}$ *if and only if* $u^- \leq v^-$, $\hat{u} \leq \hat{v}$ *and* $u^+ \leq v^+$.

Nondominated solution

$$
\begin{array}{lll}
\text{(FFLP)} & \text{Min/Max} & \tilde{z} = \displaystyle\sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j \\
 & \text{subject to} & \displaystyle\sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j \leqq \tilde{b}_i, \quad i = 1, \ldots, m, \\
 & & \tilde{x}_j \geqq 0, \quad j = 1, \ldots, n. \\
 & & \tilde{x}_j \text{ is a nonnegative fuzzy triangular number}, \quad j = 1, \ldots, n,
\end{array}
$$

Definition

Let $\tilde{\bar{x}}$ be a feasible solution for (FFLP). In case of Minimization, $\tilde{\bar{x}}$ is said to be a nondominated solution of (FFLP) if there does not exist a feasible solution $\tilde{x}$ for (FFLP) such that $\displaystyle\sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j \prec \sum_{j=1}^{n} \tilde{c}_j \tilde{\bar{x}}_j$.

Multiobjective formulation

$$
\begin{aligned}
\text{(MLP)} \quad \text{Min/Max} \quad & f(x) = (f_1(x), f_2(x), f_3(x)) = \Big(\sum_{j=1}^{n}(\tilde{c}_j\tilde{x}_j)^-, \sum_{j=1}^{n}(\widehat{\tilde{c}_j\tilde{x}_j}), \sum_{j=1}^{n}(\tilde{c}_j\tilde{x}_j)^+\Big) \\
\text{subject to} \quad & \sum_{j=1}^{n}(\tilde{a}_{ij}\tilde{x}_j)^- \leq b_i^-, \quad i = 1,\ldots,m, \\
& \sum_{j=1}^{n}(\widehat{\tilde{a}_{ij}\tilde{x}_j}) \leq \hat{b}_i, \quad i = 1,\ldots,m, \\
& \sum_{j=1}^{n}(\tilde{a}_{ij}\tilde{x}_j)^+ \leq b_i^+, \quad i = 1,\ldots,m, \\
& x_j^- - \hat{x}_j \leq 0, \quad j = 1,\ldots,n, \\
& \hat{x}_j - x_j^+ \leq 0, \quad j = 1,\ldots,n, \\
& x_j^- \geq 0, \hat{x}_j \geq 0, x_j^+ \geq 0, \quad j = 1,\ldots,n.
\end{aligned}
$$

Theorem

$\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ with $\tilde{x}_j = (x_j^-, \hat{x}_j, x_j^+) \in \mathcal{F}_C$, $j = 1,\ldots,n$, is a nondominated solution of (FFLP) if and only if $x = (x_1^-, \hat{x}_1, x_1^+,\ldots,x_n^-, \hat{x}_n, x_n^+) \in \mathbb{R}^{3n}$ is a weakly efficient solution of (MLP).

Algorithm to generalte nondominated solutions

Algorithm	
Step 1	Define $r \in \mathbb{N}$ and a set of weights $S_W = \{(w_{s1}, w_{s2}, \ldots, w_{sn}) : s = 1, \ldots r\}$. ($w_s = (w_{s1}, w_{s2}, \ldots, w_{sn}) \in S_W$ means that $w_{s1}, w_{s2}, \ldots, w_{sn} \geq 0$ and $\sum_{i=1}^{n} w_{si} = 1$, for all $s = 1, \ldots r$) $D \leftarrow \emptyset$ $s \leftarrow 1$
Step 2	Solve (MLP)$_{w_s}$ $\rightarrow x_s = (x^-_{s,1}, \hat{x}_{s,1}, x^+_{s,1}, \ldots, x^-_{s,n}, \hat{x}_{s,n}, x^+_{s,n}) \in \mathbb{R}^{3n}$ If no solution, then go to Step 4
Step 3	$\tilde{x}_{s,j} \leftarrow (x^-_{s,j}, \hat{x}_{s,j}, x^+_{s,j}),\ j = 1, \ldots n$ $\tilde{x}_s \leftarrow (\tilde{x}_{s,1}, \tilde{x}_{s,2}, \ldots \tilde{x}_{s,n})$ $D \leftarrow D \cup \{\tilde{x}_s\}$
Step 4	$s \leftarrow s + 1$ If $s \leq r$, then go to Step 2
Step 5	End

Example

Let us consider the following example proposed by Khan et al. (2013, 2017), but whose inequality relations are now defined under no ranking functions.

(FFLP1) Max $\tilde{z} = (2,4,8)\tilde{x}_1 + (3, \frac{37}{6}, 10)\tilde{x}_2 + (5, \frac{34}{3}, 15)\tilde{x}_3 \in \mathcal{F}_C$

subject to

$$(2,5,8)\tilde{x}_1 + (3, \tfrac{41}{6}, 10)\tilde{x}_2 + (5, \tfrac{31}{3}, 18)\tilde{x}_3 \leqq (6, \tfrac{50}{3}, 30)$$

$$(4\tfrac{32}{3}, 12)\tilde{x}_1 + (5, \tfrac{73}{6}, 20)\tilde{x}_2 + (7, \tfrac{105}{6}, 30)\tilde{x}_3 \leqq (10, 30, 50)$$

$$(3,5,7)\tilde{x}_1 + (5,15,20)\tilde{x}_2 + (5,10,15)\tilde{x}_3 \leqq (2, \tfrac{145}{6}, 30)$$

$$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \geqq 0$$

Example

We apply the algorithm to (FFLP1), and then we generate nondominated solutions of (FFLP1).

s	$w_{s,1}$	$w_{s,2}$	w_{s3}	$\tilde{x}_{s,1}$	$\tilde{x}_{s,2}$	$\tilde{x}_{s,3}$	$\tilde{z}_s$
1	0,8	0,1	0,1	(0, 0,715, 1)	(0, 0,4, 0,4)	(0,4, 1, 1)	(2, 17,378, 27)
2	0,5	0,3	0,2	(0, 0,806, 0,999)	(0, 0, 0)	(0,4, 1,222, 1,222)	(2, 17,889, 26,332)
3	0,4	0	0,6	(0,666, 0,666, 3,75)	(0, 0, 0)	(0, 0, 0)	(1,333, 3,333, 30)
4	0,1	0,7	0,2	(0, 0, 0,080)	(0, 0, 0,0322)	(0,4, 1,612, 1,612)	(2, 18,279, 25,161)
5	0	0,4	0,6	(0, 2,432, 3,333)	(0, 0,333, 0,333)	(0, 0, 0)	(0, 14,217, 30)
6	0	0,9	0,1	(0, 0, 0,120)	(0, 0, 0)	(0, 1,612, 1,612)	(0, 18,279, 25,161)

Contents

Applications of interval and fuzzy optimization

- Extension to several fuzzy objectives (Arana (2023) RAIRO).
- Application to fuzzy discrete optimization: p-center location problem (Arana and Blanco (2019) CAIE).
- Application to the fuzzy maximal covering location problem (Arana, Fernandez and Blanco (2020) EJOR).
- Application to Volterra interval-valued integral equation, with gH difference (Arana, Berenguer, Gamez, Guillen, Ruiz (2020) CNSNS)
- Application to nonparametric dual control algorithm of multidimensional objects with interval-valued observations (Arana, Medvedev, and Ekaterina Chzhan (2023) Axioms)
- Application to interval and fuzzy Data Envelopment Analysis (Arana, Younesi, Sanchez and Lozano (2020-2023) FSS, JCAM, FODM, etc).

Thanks for your attention!