2nd RobexDay

Stability analysis of n-dimensional nonlinear systems

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Introduction - Do you trust the pilots ?

Figure 1: Patrouille de France, a stable squadron

Introduction - Stability analysis can be difficult

Problematic: How to verify stability in an n-dimensional nonlinear system?

State of art:

- stability criterion
- BIBO stability
- linear stability
- Lyapunov stability

Problems:

- manual methods are time-consuming
- limited class of problems for numerical methods
- numerical methods may have exponential complexity

Summary

1. Case study
2. Numerical method for stability analysis
3. Applications

Case study - Platooning system

Robot model:

$$\begin{cases} \dot{a}_i = v_i \\ \dot{v}_i = u_i \end{cases} \quad (1)$$

System with two robots:

$$\begin{cases} x_1 = d_1 - d_d \\ x_2 = v_1 - v_d \\ x_3 = v_2 - v_d \\ \dot{x}_1 = x_3 - x_2 \\ \dot{x}_2 = u_1 \\ \dot{x}_3 = u_2 \end{cases} \quad (2)$$

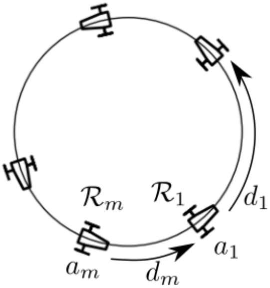

Figure 2: Platooning on the circle

Case study - Classical methods

Example of linear controller:

$$\begin{cases} u_1 & = x_1 + x_3 - 2x_2 \\ u_2 & = -x_1 + x_2 - 2x_3 \\ \dot{\boldsymbol{x}} & = \boldsymbol{A} \cdot \boldsymbol{x} \end{cases} \tag{3}$$

with $\boldsymbol{A} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$ Hurwitz

with the eigenvalues $\{-2, -1, -1\}$.

Lyapunov function:

$$V = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) \tag{4}$$

$$\begin{aligned} \dot{V} &= x_1\left(x_3 - x_2\right) \\ &\quad + x_2\left(x_1 + x_3 - 2x_2\right) \\ &\quad + x_3\left(-x_1 + x_2 - 2x_3\right) \\ &\quad = -x_2^2 - x_3^2 - \left(x_2 - x_3\right)^2 \\ &\quad \leq 0 \end{aligned} \tag{5}$$

Case study - disturbance

Adding disturbance $\boldsymbol{w} \in [\boldsymbol{w}]$:

$$\begin{cases} u_1 &= x_1 + x_3 - 2x_2 + \boldsymbol{w}_1 \\ u_2 &= -x_1 + x_2 - 2x_3 + \boldsymbol{w}_1 \\ \dot{\boldsymbol{x}} &= \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{w} \end{cases} \quad (6)$$

Assume:

$$|w_1| < |x_2| \quad (7)$$

$$|w_2| < |x_3| \quad (8)$$

Lyapunov function:

$$V = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) \quad (9)$$

$$\begin{aligned} \dot{V} &= -x_2\left(x_2 - w_1\right) - x_3\left(x_3 - w_2\right) \\ &\quad - \left(x_2 - x_3\right)^2 \\ &\leq 0 \end{aligned} \quad (10)$$

Case study - non-linearity with saturation

Adding saturation:

$$\begin{cases} u_1 &= \arctan\left(x_1 + x_3 - 2x_2 + \boldsymbol{w}_1\right) \\ u_2 &= \arctan\left(-x_1 + x_2 - 2x_3 + \boldsymbol{w}_2\right) \\ \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{w}\right) \end{cases} \quad (11)$$

Case study - with m-robots

$$\dot{x}_i = \begin{cases} x_{m+i} - x_{m+i-1} & \text{for } i < m \\ \arctan\left(x_{i-m+1} + x_{i+1} \right. \\ \left. -2x_i\right) + w_{i-m+1} & \text{for } m \leq i < n \\ \arctan\left(x_m - 2x_n \right. \\ \left. - \sum_{k=1}^{m-1} x_k\right) + w_m & \text{for } i = n \end{cases} \quad (12)$$

Numerical method - RPI Ellipsoid

Nonlinear system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w}) \qquad (13)$$

with the state $\boldsymbol{x} \in \mathbb{R}^n$ and the bounded disturbance $\boldsymbol{w} \in [\boldsymbol{w}] \subseteq \mathbb{R}^m$.

Centered ellipsoid:

$$\mathcal{E}(\boldsymbol{Q}) = \left\{\boldsymbol{x} \in \mathbb{R}^n | V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \leq 1\right\} \qquad (14)$$

Robust positive invariance (RPI):

$$\forall \boldsymbol{x}(0) \in \mathcal{E}(\boldsymbol{Q}), \forall \boldsymbol{w}(t) \in [\boldsymbol{w}], \forall t > 0, \boldsymbol{x}(t) \in \mathcal{E}(\boldsymbol{Q}) \qquad (15)$$

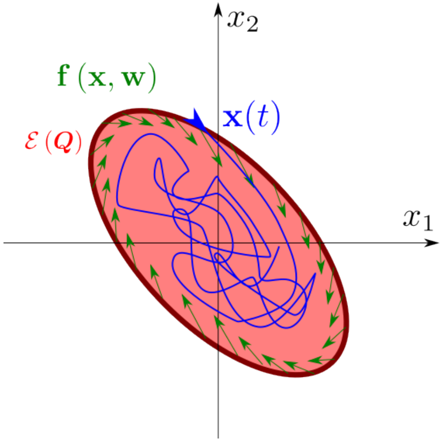

Figure 3: RPI 2D ellipsoid

Numerical method - Criteria for RPI

Euler prediction:

$$\boldsymbol{h}_\delta : (\boldsymbol{x}, \boldsymbol{w}) \mapsto \boldsymbol{x} + \delta \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w}) \qquad (16)$$

set of prediction:

$$\mathcal{Y}_\delta = \{\boldsymbol{y} \in \mathbb{R}^n | \exists (\boldsymbol{x}, \boldsymbol{w}) \in \partial\mathcal{E}(\boldsymbol{Q}) \times [\boldsymbol{w}], \quad \boldsymbol{y} = \boldsymbol{h}_\delta(\boldsymbol{x}, \boldsymbol{y})\}. \qquad (17)$$

Theorem

If $\mathcal{Y}_\delta \subseteq \mathcal{E}(\boldsymbol{Q})$ then $\mathcal{E}(\boldsymbol{Q})$ is RPI

Problem: $\mathcal{Y}_\delta$ hard to find

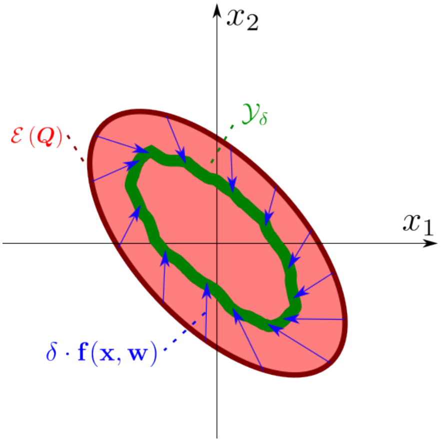

Figure 4: Euler set prediction

Numerical method - Practical method

Step 1: choose an ellipsoid $\mathcal{E}(\mathbf{Q})$ with the Lyapunov equation.

Step 2: compute an outer ellipsoidal approximation of $\mathcal{Y}_\delta$ such that $\mathcal{Y}_\delta \subseteq \mathcal{E}\left(\overline{\mathbf{Q}}\right)$.

Step 3: verify $\mathcal{E}\left(\overline{\mathbf{Q}}\right) \subseteq \mathcal{E}(\mathbf{Q})$ by showing that $\overline{\mathbf{Q}} - \mathbf{Q}$ is positive definite.

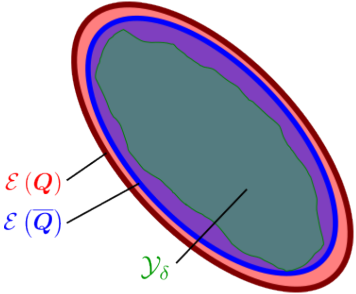

Figure 5: Illustration of the ellipsoidal approximation

Numerical Method - Algorithm

Algorithm 1 Algorithm of the enclosing method

Input $\boldsymbol{Q}$, $\boldsymbol{f}$,$[\boldsymbol{w}]$,δ | **Output** res

1: $\Gamma_{\boldsymbol{Q}} = \left(\sqrt{\boldsymbol{Q}}\right)^{-1}$

2: $[\boldsymbol{x}] = \text{enclose\_ellipse\_by\_box}\left(\boldsymbol{Q}\right)$

3: $\boldsymbol{A}_x = \boldsymbol{I}_n + \delta \cdot \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\left(0, w_m\right)$

4: $[\boldsymbol{b}] = \boldsymbol{b}\left(0, w_m\right) + \left(\Gamma_{\boldsymbol{Q}}^{-1}\boldsymbol{A}_x^{-1}\left(\boldsymbol{I}_n + \delta \cdot \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]\left([\boldsymbol{x}], [\boldsymbol{w}]\right)\right) - \Gamma_{\boldsymbol{Q}}^{-1}\right) \cdot [\boldsymbol{x}]$
$+\Gamma_{\boldsymbol{Q}}^{-1} \cdot \boldsymbol{A}_x^{-1} \cdot \delta \cdot \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{w}}\right]\left([\boldsymbol{x}], [\boldsymbol{w}]\right) \cdot \left([\boldsymbol{w}] - w_m\right)$

5: $[||\boldsymbol{b}||] = \text{norm2}\left([\boldsymbol{b}]\right)$

6: $\rho = \text{upper\_bound}\left([||\boldsymbol{b}||]\right)$

7: $\overline{\boldsymbol{Q}} = \frac{1}{(1+\rho)^2}\boldsymbol{A}_x^{-T}\boldsymbol{Q}\boldsymbol{A}_x^{-1}$

8: $\text{res} = \text{is\_definite\_positive}(\overline{\boldsymbol{Q}} - \boldsymbol{Q})$

Numerical Method - Advantages and Inconvenient

Advantages of the method:

- guaranteed numerical method
- polynomial complexity

Inconvenient:

- pessimist method
- good candidate ellipsoid can be hard to find

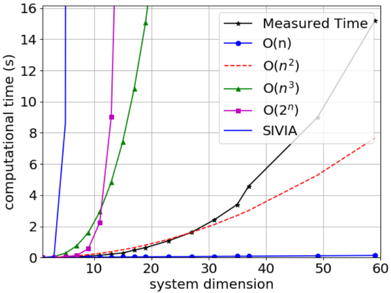

Figure 6: The computational time is smaller than with the SIVIA method

Application - illustration in 2D

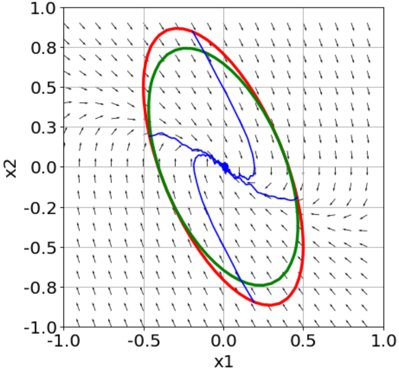

Figure 7: RPI ellipsoid

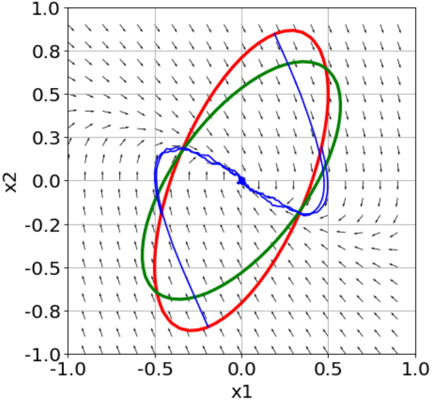

Figure 8: non-RPI ellipsoid

Application - Stability of the platooning system

With m robots, the system has a dimension $n = 2m - 1$ and the differential equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{w}) \tag{18}$$

Candidate ellipsoid $\mathcal{E}(\boldsymbol{Q}_n)$ chosen as solution of the Lyapunov equation

$$\boldsymbol{A}^T \boldsymbol{Q}_n + \boldsymbol{Q}_n \boldsymbol{A} = -10^j \boldsymbol{I}_n \tag{19}$$

with $\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(0, 0)$

Example:

$m = 5$

$[\boldsymbol{w}] = [-10^{-4}, 10^{-4}]^5$

$j = 4$

$\delta = 0.01$

Conclusion

Main result:

- Numerical method to study n-dimensional stability
- Pessimist but guaranteed

Future work:

- Guaranteed integration with positive invariance