Inner and outer approximation of the polar diagram of a sailboat

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Abstract : A sailboat can be described by a system of nonlinear differential equations. A polar speed diagram shows the set of all feasible speed that can be reached for different values of the orientation angle of the boat, in a permanent regime. This paper shows how interval analysis can be used to characterize the polar speed diagram in a reliable way.

1 Introduction

Many problem from control theory have a solution set that can be written into the form

$$\mathbb{S} \triangleq \{\mathbf{p} \in \mathbf{P} \mid \exists \mathbf{q} \in \mathbf{Q}, \mathbf{f}(\mathbf{p}, \mathbf{q}) = \mathbf{0}\}, \tag{1}$$

where $\mathbf{P}$ is a box of $\mathbb{R}^{n_p}$, $\mathbf{Q}$ is a box of $\mathbb{R}^{n_q}$ and $\mathbf{f}$ is a continuous function from $\mathbb{R}^{n_p+n_q}$ to $\mathbb{R}^{n_f}$. A challenging problem is to find an algorithm which is able to find an inner and an outer approximation of $\mathbb{S}$ with arbitrary accuracy. When the dimension n_f of $\mathbf{f}(\mathbf{p},\mathbf{q})$ is greater or equal than 2, to our knowledge, no general algorithm is available to provide an inner approximation of $\mathbb{S}$. When $n_f = 1$, this problem can be solved using existence theorems (e.g. Miranda theorem [4]) or by taking advantage of the theory of Modal Interval Analysis [3]. The aim of this paper is to provide an algorithm to compute an inner and an outer approximation for $\mathbb{S}$ when $n_f = 1$ and to illustrate the efficiency of the algorithm on an application related to the control of a sailboat.

Section 2 shows that the polar speed diagram of a sailboat can be expressed into the form given by (1). Then, using some formal transformations in Section 3, we shall eliminate all but one of the variables $q_1, \ldots, q_{n_q}$ in order to be able to find an inner approximation for $\mathbb{S}$ in Section 4. On Section 5, the approach will be used to characterize the polar speed diagram of the sailboat.

2 Sailboat problem

The sailboat represented on Figure 1 is described by the following state equations [6]

$$\left\{\begin{array}{lclr}
\dot{x} & = & v\cos\theta, & \text{(i)}\\
\dot{y} & = & v\sin\theta - \beta V, & \text{(ii)}\\
\dot{\theta} & = & \omega, & \text{(iii)}\\
\dot{\delta}_s & = & u_1, & \text{(iv)}\\
\dot{\delta}_r & = & u_2, & \text{(v)}\\
\dot{v} & = & \frac{f_s\sin\delta_s - f_r\sin\delta_r - \alpha_f v}{m}, & \text{(vi)}\\
\dot{\omega} & = & \frac{(\ell - r_s\cos\delta_s)f_s - r_r\cos\delta_r f_r - \alpha_\theta\omega}{J}, & \text{(vii)}\\
f_s & = & \alpha_s\left(V\cos\left(\theta+\delta_s\right) - v\sin\delta_s\right), & \text{(viii)}\\
f_r & = & \alpha_r v\sin\delta_r. & \text{(ix)}
\end{array}\right. \tag{2}$$

where $\dot{x}, \dot{y}, \ldots$ represents the derivatives of $x, y, \ldots$ with respect to the time t. The state vector $\mathbf{x} = (x, y, \theta, \delta_s, \delta_r, v, \omega)^{\mathrm{T}} \in \mathbb{R}^7$ is composed with

- the coordinates x, y of the inertial center G of the boat
- the orientation θ,
- the sail angle δ_s
- the rudder angle δ_r

- the tangential speed of G
- the angular velocity ω of the boat around G.

The intermediate variables are

- the thrust force f_s of the wind on the sail,
- the force f_r of the water on the rudder.

The parameters (that are assumed to be known) are

- the speed V of the wind,
- the distance r_r between the rudder and G,
- the distance r_s between the mast and G,
- the rudder lift α_r,
- the sail lift α_s,
- the tangential friction α_f of the boat with respect to the water,
- the angular friction α_θ of the boat with respect to the water,
- the angular inertia J of the boat,
- the distance ℓ between the mast and the thrust center of the sail,
- and the drift coefficient β.

These parameters will be chosen as

$$\begin{aligned}\beta = 0.05, r_s = 1, r_r = 2, \ell = 1, V = 10,\\ m = 1000, J = 2000, \alpha_f = 60,\\ \alpha_\theta \in 500, \alpha_s = 500, \alpha_r = 300.\end{aligned}$$

The inputs u_1 and u_2 of the system are the derivatives of the angles δ_s and δ_r.

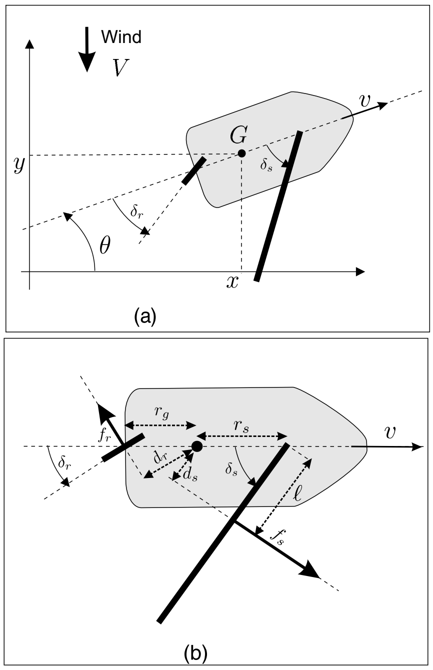

Figure 1: Sailing boat

The *Polar Diagram* of the sailboat is the set $\mathbb{S}$ of all pairs (θ, v) that can be reached by the boat, in a cruising behavior. During a cruising behavior of the boat, the speed of the boat, its course, its angular velocity, ... are constant, i.e.,

$$\dot{\theta} = 0, \dot{\delta}_s = 0, \dot{\delta}_r = 0, \dot{v} = 0, \dot{\omega} = 0. \tag{3}$$

From Equation (2), we get

$$\left\{ \begin{array}{ccc} 0 & = & \frac{f_s \sin \delta_s - f_r \sin \delta_r - \alpha_f v}{m}, \\ 0 & = & \frac{(\ell - r_s \cos \delta_s) f_s - r_r \cos \delta_r f_r}{J}, \\ f_s & = & \alpha_s \left(V \cos \left(\theta + \delta_s \right) - v \sin \delta_s \right), \\ f_r & = & \alpha_r v \sin \delta_r. \end{array} \right. \tag{4}$$

which is equivalent to

$$\left\{ \begin{array}{ccc} \alpha_s \left(V \cos \left(\theta + \delta_s \right) - v \sin \delta_s \right) \sin \delta_s - \alpha_r v \sin^2 \delta_r - \alpha_f v & = & 0 \\ \left(\ell - r_s \cos \delta_s \right) \alpha_s \left(V \cos \left(\theta + \delta_s \right) - v \sin \delta_s \right) - r_r \alpha_r v \sin \delta_r \cos \delta_r & = & 0 \end{array} \right. \tag{5}$$

The polar speed diagram is the set $\mathbb{S}$ of all feasible vectors (v, θ) in a cruising regime, i.e.,

$$\mathbb{S} = \left\{ (v, \theta) \mid \exists \delta_r, \exists \delta_s, \mathbf{f} \left(v, \theta, \delta_r, \delta_s \right) = \mathbf{0} \right\}, \tag{6}$$

where

$$\mathbf{f}(v,\theta,\delta_r,\delta_s)=\begin{pmatrix}\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\sin\delta_s-\alpha_r v\sin^2\delta_r-\alpha_f v\\ \left(\ell-r_s\cos\delta_s\right)\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)-r_r\alpha_r v\sin\delta_r\cos\delta_r\end{pmatrix} \quad (7)$$

3 Transformation of the problem

Characterizing an inner approximation for the set $\mathbb{S}$ given by equation (6) cannot be done using existing interval algorithm. The raison for that is that the dimension n_f of $\mathbf{f}$ is equal to 2. In this section, we shall eliminate the variable δ_r using symbolic calculus in order to cast into the case where $n_f=1$.

Since

$$\sin^2\delta_r=\frac{1-\cos\left(2\delta_r\right)}{2}\text{ and }\sin\delta_r\cos\delta_r=\frac{\sin(2\delta_r)}{2}, \quad (8)$$

Equation (7) transforms into

$$\left\{\begin{array}{rcl}\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\sin\delta_s-\alpha_r v\frac{1-\cos(2\delta_r)}{2}-\alpha_f v&=&0\\ \left(\ell-r_s\cos\delta_s\right)\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)-r_r\alpha_r v\frac{\sin(2\delta_r)}{2}&=&0\end{array}\right. \quad (9)$$

i.e.,

$$\left\{\begin{array}{rcl}\alpha_r v\frac{1-\cos(2\delta_r)}{2}&=&\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\sin\delta_s-\alpha_f v\\ r_r\alpha_r v\frac{\sin(2\delta_r)}{2}&=&\left(\ell-r_s\cos\delta_s\right)\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\end{array}\right. \quad (10)$$

or equivalently

$$\left\{\begin{array}{rcl}1-\cos(2\delta_r)&=&\frac{2}{\alpha_r v}\left(\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\sin\delta_s-\alpha_f v\right)\\ \sin(2\delta_r)&=&\frac{2}{r_r\alpha_r v}\left(\ell-r_s\cos\delta_s\right)\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\end{array}\right.$$

i.e.

$$\left\{\begin{array}{rcl}-\cos(2\delta_r)&=&-1+\frac{2}{\alpha_r v}\left(\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\sin\delta_s-\alpha_f v\right)\\ \sin(2\delta_r)&=&\frac{2\alpha_s}{r_r\alpha_r v}\left(\ell-r_s\cos\delta_s\right)\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\end{array}\right. \quad (11)$$

Since $\sin^2(2\delta_r)+\cos^2(2\delta_r)-1=0$, we get that

$$\begin{array}{l}\left(-1+\frac{2}{\alpha_r v}\left(\alpha_s\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\sin\delta_s-\alpha_f v\right)\right)^2\\ +\left(\frac{2\alpha_s}{r_r\alpha_r v}\left(\ell-r_s\cos\delta_s\right)\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\right)^2-1\;=\;0\end{array} \quad (12)$$

i.e.,

$$\begin{array}{l}\left(\left(\alpha_r+2\alpha_f\right)v-2\alpha_s V\cos\left(\theta+\delta_s\right)\sin\delta_s+2\alpha_s v\sin^2\delta_s\right)^2\\ +\left(\frac{2\alpha_s}{r_r}\left(\ell-r_s\cos\delta_s\right)\left(V\cos\left(\theta+\delta_s\right)-v\sin\delta_s\right)\right)^2-\alpha_r^2v^2\;=\;0\end{array} \quad (13)$$

The polar speed diagram can thus be written as

$$\mathbb{S}=\left\{(\theta,v)|\exists\delta_s\in[-\frac{\pi}{2},\frac{\pi}{2}]\mid f_1(\theta,v,\delta_s)=0\right\}. \tag{14}$$

where $f_1(\theta,v,\delta_s)$ is given by

$$\begin{aligned} f_1(\theta,v,\delta_s) &= \left((\alpha_r+2\alpha_f)\,v-2\alpha_s V\cos(\theta+\delta_s)\sin\delta_s+2\alpha_s v\sin^2\delta_s\right)^2 \\ &\quad+\left(\tfrac{2\alpha_s}{r_r}\,(\ell-r_s\cos\delta_s)\,(V\cos(\theta+\delta_s)-v\sin\delta_s)\right)^2-\alpha_r^2 v^2 \end{aligned}$$

4 Algorithm

This section provides an algorithm to compute an inner and an outer approximation of the set

$$\mathbb{S}\triangleq\{\mathbf{p}\in\mathbf{P}\mid\exists\mathbf{q}\in\mathbf{Q},f(\mathbf{p},\mathbf{q})=0\}, \tag{15}$$

which is a particular case of the problem expressed by Equation 1, where $n_f=1$. Then, this algorithm will be used to characterize the set described by (14).

Let us first recall some definitions needed to understand our algorithm. An interval is a closed and connected subset of $\mathbb{R}$. For instance, $[1,3]$, $\{1\}$, $]-\infty,6]$, $\mathbb{R}$ and $\emptyset$ are considered as intervals. A *box*, or *interval vector,* $\mathbf{X}$ of $\mathbb{R}^n$ is a vector with interval components. The *width* $w(\mathbf{X})$ of a box $\mathbf{X}$ is the length of its largest side. If $\mathbf{X}$ and $\mathbf{Y}$ are two boxes of $\mathbb{R}^n$, $\mathbf{X}\sqcup\mathbf{Y}$ denotes the smallest box which contains $\mathbf{X}\cup\mathbf{Y}$ and $\mathbf{X}\backslash\mathbf{Y}$ denotes the set of all $\mathbf{x}\in\mathbf{X}$ such that $\mathbf{x}\notin\mathbf{Y}$. The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$. Consider a closed subset $\mathbb{X}$ of $\mathbb{R}^n$. The operator $\mathcal{C}_{\mathbb{X}}:\mathbb{IR}^n\to\mathbb{IR}^n$ is a *contractor*[2] for the set $\mathbb{X}$ of $\mathbb{R}^n$ if it satisfies

$$\forall\mathbf{X}\in\mathbb{IR}^n,\begin{cases}\mathcal{C}_{\mathbb{X}}(\mathbf{X})\subset\mathbf{X} & \text{(contractance)},\\ \mathcal{C}_{\mathbb{X}}(\mathbf{X})\cap\mathbb{X}=\mathbf{X}\cap\mathbb{X} & \text{(completeness)}.\end{cases} \tag{16}$$

$\mathcal{C}_{\mathbb{X}}$ is *idempotent* if for all $\mathbf{X}$, $\mathcal{C}_{\mathbb{X}}(\mathcal{C}_{\mathbb{X}}(\mathbf{X}))=\mathcal{C}_{\mathbb{X}}(\mathbf{X})$. It is *thin* if for any singleton $\{\mathbf{x}\}$, $\mathcal{C}_{\mathbb{X}}(\{\mathbf{x}\})=\{\mathbf{x}\}\cap\mathbb{X}$. $\mathcal{C}_{\mathbb{X}}$ is said to be *convergent* if for almost any point $\mathbf{x}$, and for all sequences of nested boxes $\mathbf{X}(k)$,

$$\mathbf{X}(k)\to\mathbf{x}\Rightarrow\mathcal{C}_{\mathbb{X}}(\mathbf{X}(k))\to\{\mathbf{x}\}\cap\mathbb{X}. \tag{17}$$

It is said to be *minimal* if

$$\forall\mathbf{X}\in\mathbb{IR}^n,\mathcal{C}_{\mathbb{X}}(\mathbf{X})=[\mathbf{X}\cap\mathbb{X}], \tag{18}$$

where $[\mathbf{X}\cap\mathbb{X}]$ denotes the smallest box containing $\mathbf{X}\cap\mathbb{X}$. The contractor $\mathcal{C}_{\mathbb{X}}$ is said to be *binary* if $\forall\mathbf{X}\in\mathbb{IR}^n$, either $\mathcal{C}_{\mathbb{X}}(\mathbf{X})=\mathbf{X}$ or $\mathcal{C}_{\mathbb{X}}(\mathbf{X})=\emptyset$.

If $\mathbb{X}$ is a set defined by nonlinear inequalities, there exist different methods to build a contractor for $\mathbb{X}$ and for its complementary set $\neg\mathbb{X}$. Most of them are based on interval constraint propagation [1, 2]. For the set $\mathbb{S}$ defined by (15), these methods can still be extended to build a contractor for $\mathbb{S}$ [9, 7], as illustrated by Figure 2.

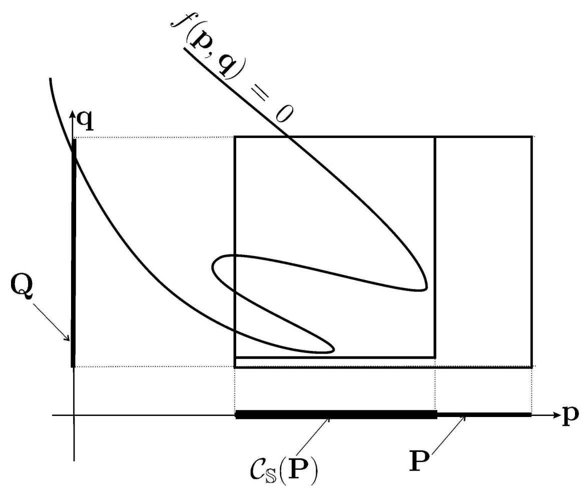

Figure 2: A contractor for $\mathbb{S}$.

Let us now show how classical contractors can be extended to build a contractor for $\neg\mathbb{S}$ in the case where f is continuous. We have

$$\begin{cases}\mathbf{p}\in\mathbb{S} & \Leftrightarrow \quad (\exists\mathbf{q}\in\mathbf{Q}, f(\mathbf{p},\mathbf{q})=0)\\ & \Leftrightarrow \quad \left(\max_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\geq 0\right)\wedge\left(\min_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\leq 0\right)\end{cases} \tag{19}$$

If $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ are two points of $\mathbf{Q}$ we have

$$f(\mathbf{p},\hat{\mathbf{q}}_1)\geq 0\Rightarrow \max_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\geq 0, \tag{20}$$

$$f(\mathbf{p},\hat{\mathbf{q}}_2)\leq 0\Rightarrow \min_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\leq 0. \tag{21}$$

Thus, if $\hat{\mathbf{q}}_1\in\mathbf{Q}$ and $\hat{\mathbf{q}}_2\in\mathbf{Q}$, from (19), we have

$$(f(\mathbf{p},\hat{\mathbf{q}}_1)\geq 0)\wedge(f(\mathbf{p},\hat{\mathbf{q}}_2)\leq 0)\Rightarrow\mathbf{p}\in\mathbb{S}. \tag{22}$$

or its contraposite

$$\mathbf{p}\in\neg\mathbb{S}\Rightarrow(f(\mathbf{p},\hat{\mathbf{q}}_1)<0)\vee(f(\mathbf{p},\hat{\mathbf{q}}_2)>0)$$

As a consequence, a contractor for the set

$$\mathbb{S}^{out} \stackrel{\text{def}}{=} \{\mathbf{p} \in \mathbf{P} \mid \ (f(\mathbf{p}, \hat{\mathbf{q}}_1) < 0) \vee (f(\mathbf{p}, \hat{\mathbf{q}}_2) > 0)\} \tag{23}$$

is also a contractor for $\neg\mathbb{S}$. Figure 3 gives an illustration of how the previous development can be used to build a contractor for $\neg\mathbb{S}$.

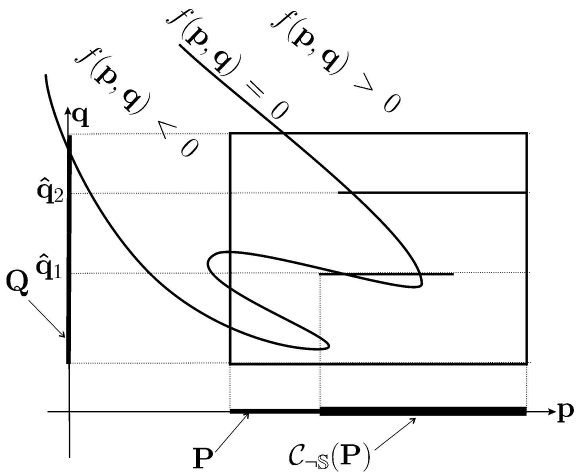

Figure 3: A contractor for $\neg\mathbb{S}$.

The following algorithm provides an inner and an outer approximation for $\mathbb{S}$. All points inside the resulting white area have been proved to belong to $\mathbb{S}$ whereas all points inside the grey area have been proved to be outside $\mathbb{S}$.

Algorithm ENCLOSE(in: $\mathbf{P}, \mathbf{Q}$)	
1	$\mathcal{L} := \{\mathbf{P}\}$;
2	while $\mathcal{L} \neq \emptyset$,
3	pop a box $\mathbf{P}$ out of $\mathcal{L}$;
4	Using a local method, choose $\hat{\mathbf{q}}_1$ such that $f(\text{center}(\mathbf{P}),\hat{\mathbf{q}}_1)$ is maximal;
5	Using a local method, choose $\hat{\mathbf{q}}_2$ such that $f(\text{center}(\mathbf{P}),\hat{\mathbf{q}}_2)$ is minimal;
6	$\mathbf{P}_1 := \mathcal{C}_{\neg\mathbb{S}}(\mathbf{P}) = \mathcal{C}_{\{\mathbf{p} \mid f(\mathbf{p},\hat{\mathbf{q}}_1)<0\}}(\mathbf{P}) \cup \mathcal{C}_{\{\mathbf{p} \mid f(\mathbf{p},\hat{\mathbf{q}}_2)>0\}}(\mathbf{P})$;
7	Paint $\mathbf{P}/\mathbf{P}_1$ white; $\mathbf{P} := \mathbf{P}_1$
8	$\mathbf{P}_2 \times \mathbf{Q}_2 := \mathcal{C}_{\mathbb{S}}(\mathbf{P} \times \mathbf{Q}) = \mathcal{C}_{\{(\mathbf{p},\mathbf{q}) \mid f(\mathbf{p},\mathbf{q})=0\}}(\mathbf{P} \times \mathbf{Q})$;
9	Paint $\mathbf{P}/\mathbf{P}_2$ grey; $\mathbf{P} := \mathbf{P}_2$
10	if $w(\mathbf{P}) < \varepsilon$, paint $\mathbf{P}$ black and go to 2;
11	bisect $\mathbf{P}$ and store the two resulting boxes into $\mathcal{L}$;
12	end while.

Remark: In the case where the available contractors are binary, the algorithm ENCLOSE can be viewed as a special instance of the Quantified Set Inversion algorithm ($\mathcal{QSI}$) [5], which implementation is based on Modal Interval Analysis [3] and for which an online web version exists at [8].

5 Results

By using $\mathcal{QSI}$ solver with an epsilon of $\epsilon = 0.02$, in less than 60 seconds on a Pentium IV, the result expressed in polar coordinates and showed in Figure 4 is obtained.

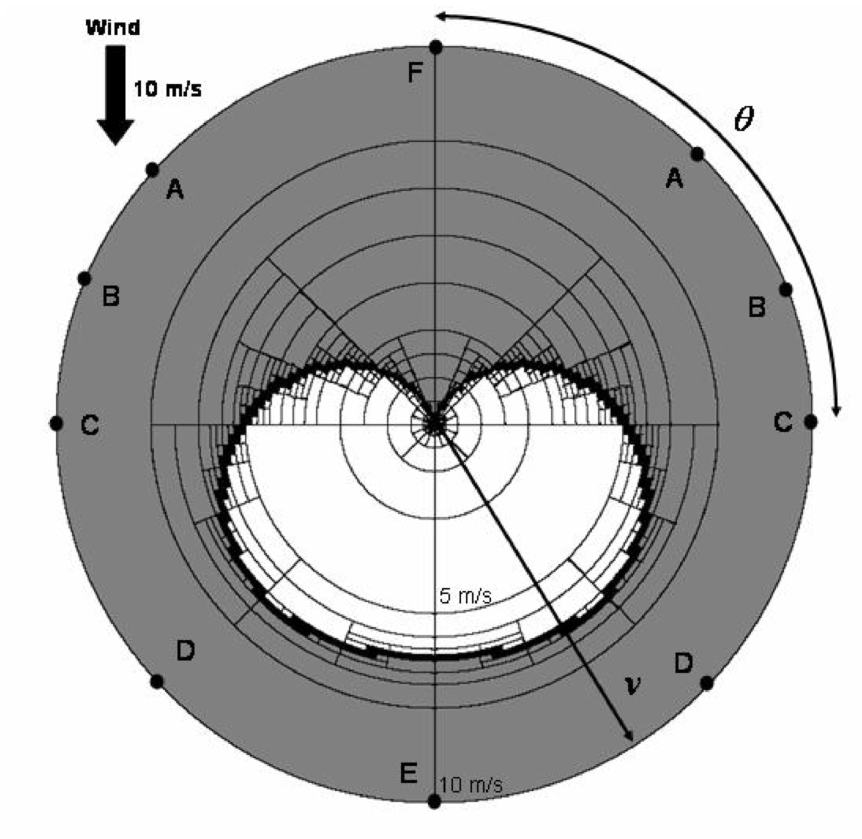

Figure 4: Polar diagram obtained by $\mathcal{QSI}$ solver.

Where the white area corresponds to the set of points (θ, v) which can be potentially reached by the sailboat, the grey area corresponds to the set of non feasible points and the black one is undefined.

The different points of sail, which refer to the position of the sailboat in relation to the wind, are also outlined in Figure 4. These points are called: *close hauled* (A), *close reach* (B), *beam reach* (C), *broad reach* (D), *run* (E) and *in irons* (F). From Figure 4, some observation can be done which could not be obvious for non-expert sailors. For instance, it can be observed that a sailboat cannot sails directly into the wind (F). In order to sailing up wind it is necessary to use the maneuver of *tacking*, which consists on sailing switching periodically from the two existing *close hauled* sail points (A). Another interesting information which can be obtained from Figure 4 is that contrarily to what would seem logical, the point of sail *run* (E), when the wind is blowing from behind the sailboat, is not the one for which the sailboat runs faster. The faster sail point corresponds to the *broad reach* position (D).

6 Conclusion

A reliable methodology, based on formal calculus and interval analysis, for finding inner and outer approximations of the *Polar Diagram* of a sailboat has been presented. For its numerical resolution, the $\mathcal{QSI}$ solver, which implementation is based on Modal Interval Analysis, has been used.

7 Future work

In practice, some parameters (speed of the wind, friction coefficients, ...) are not known exactly, but they are known to belong to some given intervals. The next step of the presented work consists on finding the *Robust Polar Diagram* of a sailboat by taking into account these uncertainties.

8 Acknowledgements

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