Thick separators

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Abstract. If an interval of $\mathbb{R}$ is an uncertain real number, a *thick set* is an uncertain subset of $\mathbb{R}^n$. More precisely, a thick set is an interval of the powerset of $\mathbb{R}^n$ equipped with the inclusion $\subset$ as an order relation. It can generally be defined by parameters or functions which are not known exactly, but are known to belong to some intervals. In this paper, we show how to use constraint propagation methods in order to compute efficiently an inner and an outer approximations of a thick set. The resulting inner/outer contraction are made using an operator which is called a *thick separator*. Then, we show how thick separators can be combined in order to compute with thick sets.

1 Thick sets

A thin set is a subset of $\mathbb{R}^n$. It is qualified as *thin* because its boundary is thin. In this section, we present the definition of thick sets.

Thick set. Denote by $(\mathcal{P}(\mathbb{R}^n), \subset)$, the powerset of $\mathbb{R}^n$ equipped with the inclusion $\subset$ as an order relation. A *thick set* $[\![\mathbb{X}]\!]$ of $\mathbb{R}^n$ is an interval of $(\mathcal{P}(\mathbb{R}^n), \subset)$. If $[\![\mathbb{X}]\!]$ is a thick set of $\mathbb{R}^n$, there exist two subsets of $\mathbb{R}^n$, called the *subset bound* and the *supset bound* such that

$$[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!] = \{\mathbb{X} \in \mathcal{P}(\mathbb{R}^n) | \mathbb{X}^{\subset} \subset \mathbb{X} \subset \mathbb{X}^{\supset}\}. \quad (1)$$

Another representation for the thickset $[\![\mathbb{X}]\!]$ is the partition $\left\{\mathbb{X}^{in}, \mathbb{X}^{?}, \mathbb{X}^{out}\right\}$, where

$$\begin{array}{rcl} \mathbb{X}^{in} & = & \mathbb{X}^{\subset} \\ \mathbb{X}^{?} & = & \mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset} \\ \mathbb{X}^{out} & = & \overline{\mathbb{Z}^{\supset}}. \end{array} \quad (2)$$

The subset $\mathbb{X}^?$ is called the *penumbra* and plays an important role in the characterization of thick sets [3]. Thick sets can be used to represent uncertain sets (such as an uncertain map [4]) or a soft constraints [1].

2 Thick separators

To characterize a thin set using a paver, we may use a separator (which is a pair of two contractors [8]) inside a paver. Separators can be immediately generalized to thick sets. Now, the penumbra as a non-zero volume for thick sets. For efficiency reasons, it is important to avoid any accumulation of the paving deep inside the penumbra. This is the role of thick separators to avoid as much as possible bisections inside the penumbra.

Thick separators. A *thick separator* $[\![\mathcal{S}]\!]$ for the thick set $[\![\mathbb{X}]\!]$ is an extension of the concept of separator to thick sets. More precisely, a thick separator is a 3-uple of contractors $\left\{\mathcal{S}^{in}, \mathcal{S}^?, \mathcal{S}^{out}\right\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$

$$\begin{array}{c} \mathcal{S}^{in}([\mathbf{x}]) \cap \mathbb{X}^{in} = [\mathbf{x}] \cap \mathbb{X}^{in} \\ \mathcal{S}^?([\mathbf{x}]) \cap \mathbb{X}^? = [\mathbf{x}] \cap \mathbb{X}^? \\ \mathcal{S}^{out}([\mathbf{x}]) \cap \mathbb{X}^{out} = [\mathbf{x}] \cap \mathbb{X}^{out} \end{array} \tag{3}$$

In what follow, we define an algebra for thick separator in a similar manner than what as been done for contractors [2] or for separators [7].

3 Algebra

In this section, we show how we can define operations for thick sets (as a union, intersection, difference, etc.). The main motivation is to be able to compute with thick sets.

Intersection. Consider two thick sets $[\![\mathbb{X}]\!] = [\![\mathbb{X}^\subset, \mathbb{X}^\supset]\!]$ and $[\![\mathbb{Y}]\!] = [\![\mathbb{Y}^\subset, \mathbb{Y}^\supset]\!]$ with thick separators $[\![\mathcal{S}_\mathbb{X}]\!] = \left\{\mathcal{S}_\mathbb{X}^{in}, \mathcal{S}_\mathbb{X}^?, \mathcal{S}_\mathbb{X}^{out}\right\}$ and $[\![\mathcal{S}_\mathbb{Y}]\!] =$

$\{\mathcal{S}_{\mathbb{Y}}^{in}, \mathcal{S}_{\mathbb{Y}}^{?}, \mathcal{S}_{\mathbb{Y}}^{out}\}$. A thick separator for the thick set

$$[\![\mathbb{Z}]\!] = [\![\mathbb{Z}^{\subset}, \mathbb{Z}^{\supset}]\!] = [\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!] = [\![\mathbb{X}^{\subset} \cap \mathbb{Y}^{\subset}, \mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset}]\!] \tag{4}$$

is

$$\begin{array}{rcc} [\![\mathcal{S}_{\mathbb{Z}}]\!] & = & \{\mathcal{S}_{\mathbb{Z}}^{in}, \mathcal{S}_{\mathbb{Z}}^{?}, \mathcal{S}_{\mathbb{Z}}^{out}\} \\ & = & \{\mathcal{S}_{\mathbb{X}}^{in} \cap \mathcal{S}_{\mathbb{Y}}^{in}, (\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{in}) \sqcup (\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{?}) \sqcup (\mathcal{S}_{\mathbb{X}}^{in} \cap \mathcal{S}_{\mathbb{Y}}^{?}), \mathcal{S}_{\mathbb{X}}^{out} \sqcup \mathcal{S}_{\mathbb{Y}}^{out}\}. \end{array} \tag{5}$$

Proof. We have

$$\begin{array}{ccccc} \mathbb{Z}^{in} & = & \mathbb{Z}^{\subset} & = & \mathbb{X}^{\subset} \cap \mathbb{Y}^{\subset} \\ & & & = & \mathbb{X}^{in} \cap \mathbb{Y}^{in} \\ \mathbb{Z}^{?} & = & \mathbb{Z}^{\supset} \backslash \mathbb{Z}^{\subset} & = & \mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset} \backslash (\mathbb{X}^{\subset} \cap \mathbb{Y}^{\subset}) \\ & & & = & \mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset} \cap \overline{\mathbb{X}^{\subset} \cap \mathbb{Y}^{\subset}} \\ & & & = & \mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset} \cap (\overline{\mathbb{X}^{\subset}} \cup \overline{\mathbb{Y}^{\subset}}) \\ & & & = & (\mathbb{X}^{in} \cup \mathbb{X}^{?}) \cap (\mathbb{Y}^{in} \cup \mathbb{Y}^{?}) \cap ((\mathbb{X}^{out} \cup \mathbb{X}^{?}) \cup (\mathbb{Y}^{out} \cup \mathbb{Y}^{?})) \\ & & & = & (\mathbb{X}^{?} \cap \mathbb{Y}^{in}) \cup (\mathbb{X}^{?} \cap \mathbb{Y}^{?}) \cup (\mathbb{X}^{in} \cap \mathbb{Y}^{?}) \\ \mathbb{Z}^{out} & = & \overline{\mathbb{Z}^{\supset}} & = & \overline{\mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset}} \\ & & & = & \overline{\mathbb{X}^{\supset}} \cup \overline{\mathbb{Y}^{\supset}} \\ & & & = & \mathbb{X}^{out} \cup \mathbb{Y}^{out}. \end{array}$$

From the separator algebra, we get that a contractor for $\mathbb{Z}^{in}$ is $\mathcal{S}_{\mathbb{Z}}^{in} = \mathcal{S}_{\mathbb{X}}^{in} \cap \mathcal{S}_{\mathbb{Y}}^{in}$, a contractor for $\mathbb{Z}^{out}$ is $\mathcal{S}_{\mathbb{Z}}^{out} = \mathcal{S}_{\mathbb{X}}^{out} \sqcup \mathcal{S}_{\mathbb{Y}}^{out}$ and a contractor for $\mathbb{Z}^{?}$ is

$$\mathcal{S}_{\mathbb{Z}}^{?} = (\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{in}) \sqcup (\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{?}) \sqcup (\mathcal{S}_{\mathbb{X}}^{in} \cap \mathcal{S}_{\mathbb{Y}}^{?}). \tag{6}$$

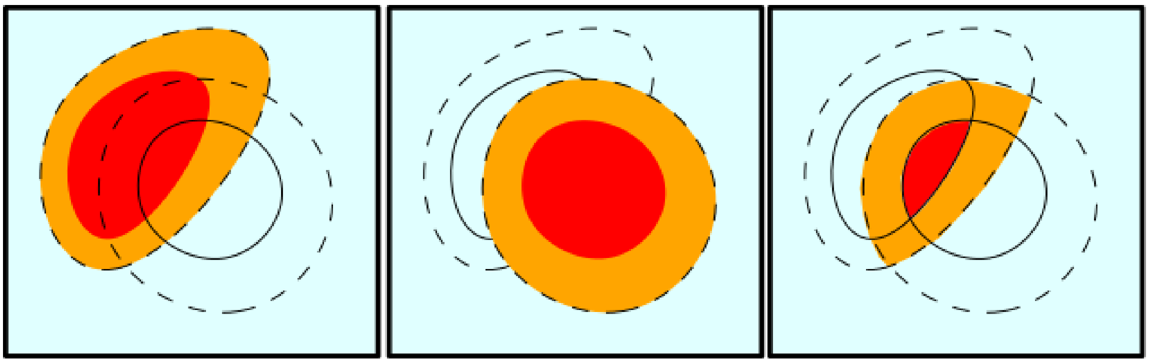
4 Using Karnaugh map

This expression could have been obtained using Figure 1.

Karnaugh map, as illustrated by Figure 2, can also can be used to get the expression for thick separators a more clear manner. For instance, if

$$[\![\mathbb{Z}]\!] = [\![\mathbb{X}]\!] \cup [\![\mathbb{Y}]\!], \tag{7}$$

we read from the Karnaugh map



$[\![\mathbb{X}]\!]$ $\quad$ $[\![\mathbb{Y}]\!]$ $\quad$ $[\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!]$

Figure 1: Intersection of two thick sets. Red means *inside*, Blue means *outside* and Orange means *uncertain*

$$\begin{array}{rcl} \mathbb{Z}^{in} & = & \mathbb{X}^{in} \cup \mathbb{Y}^{in} \\ \mathbb{Z}^{?} & = & \left(\mathbb{X}^{?} \cap \mathbb{Y}^{out}\right) \cup \left(\mathbb{X}^{?} \cap \mathbb{Y}^{?}\right) \cup \left(\mathbb{X}^{out} \cap \mathbb{Y}^{?}\right) \\ \mathbb{Z}^{out} & = & \mathbb{X}^{out} \cap \mathbb{Y}^{out}. \end{array} \tag{8}$$

Therefore a thick separator for the thick set $[\![\mathbb{Z}]\!]$ is

$$\begin{array}{rcl} [\![\mathcal{S}_{\mathbb{Z}}]\!] & = & \left\{\mathcal{S}_{\mathbb{Z}}^{in}, \mathcal{S}_{\mathbb{Z}}^{?}, \mathcal{S}_{\mathbb{Z}}^{out}\right\} \\ & = & \left\{\mathcal{S}_{\mathbb{X}}^{in} \sqcup \mathcal{S}_{\mathbb{Y}}^{in}, \left(\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{out}\right) \sqcup \left(\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{?}\right) \sqcup \left(\mathcal{S}_{\mathbb{X}}^{out} \cap \mathcal{S}_{\mathbb{Y}}^{?}\right), \mathcal{S}_{\mathbb{X}}^{out} \cap \mathcal{S}_{\mathbb{Y}}^{out}\right\} \end{array} \tag{9}$$

Now, if

$$[\![\mathbb{Z}]\!] = [\![\mathbb{X}]\!] \backslash [\![\mathbb{Y}]\!] \cup [\![\mathbb{Y}]\!] \backslash [\![\mathbb{X}]\!], \tag{10}$$

we read

$$\begin{array}{rcl} \mathbb{Z}^{in} & = & \left(\mathbb{X}^{in} \cap \mathbb{Y}^{out}\right) \cup \left(\mathbb{X}^{out} \cap \mathbb{Y}^{in}\right) \\ \mathbb{Z}^{?} & = & \mathbb{X}^{?} \cup \mathbb{Y}^{?} \\ \mathbb{Z}^{out} & = & \left(\mathbb{X}^{in} \cap \mathbb{Y}^{in}\right) \cup \left(\mathbb{X}^{out} \cap \mathbb{Y}^{out}\right). \end{array} \tag{11}$$

Therefore a thick separator for the thick set $[\![\mathbb{Z}]\!]$ is

$$\begin{array}{rcl} [\![\mathcal{S}_{\mathbb{Z}}]\!] & = & \left\{\mathcal{S}_{\mathbb{Z}}^{in}, \mathcal{S}_{\mathbb{Z}}^{?}, \mathcal{S}_{\mathbb{Z}}^{out}\right\} \\ & = & \left\{\mathcal{S}_{\mathbb{X}}^{in} \sqcup \mathcal{S}_{\mathbb{Y}}^{in}, \left(\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{out}\right) \sqcup \left(\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{?}\right) \sqcup \left(\mathcal{S}_{\mathbb{X}}^{out} \cap \mathcal{S}_{\mathbb{Y}}^{?}\right), \mathcal{S}_{\mathbb{X}}^{out} \cap \mathcal{S}_{\mathbb{Y}}^{out}\right\}. \end{array} \tag{12}$$

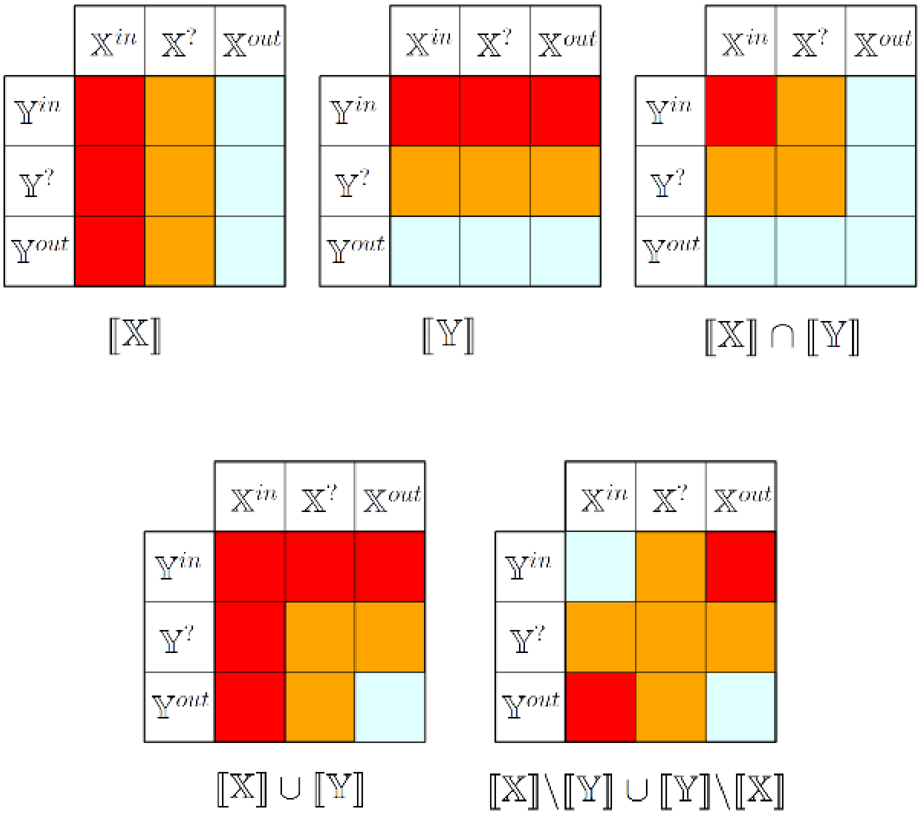


Figure 2: Karnaugh map

Note that when we build such an expression from a Karnaugh map, fake boundaries may appear. They could be avoided using the method proposed in [5].

Example. Take one box $[\mathbf{x}]$ as in Figure 3. We get

$$[\![\mathcal{S}_{\mathbb{X}}]\!]\left([\mathbf{x}]\right) = \left\{\mathcal{S}_{\mathbb{X}}^{in}, \mathcal{S}_{\mathbb{X}}^{?}, \mathcal{S}_{\mathbb{X}}^{out}\right\}\left([\mathbf{x}]\right) = \{[\mathbf{a}], [\mathbf{x}], \emptyset\} \tag{13}$$

where $[\mathbf{a}]$ the white box. Moreover,

$$[\![\mathcal{S}_{\mathbb{Y}}]\!]\left([\mathbf{x}]\right) = \left\{\mathcal{S}_{\mathbb{Y}}^{in}, \mathcal{S}_{\mathbb{Y}}^{?}, \mathcal{S}_{\mathbb{Y}}^{out}\right\}\left([\mathbf{x}]\right) = \{\emptyset, [\mathbf{x}], \emptyset\}. \tag{14}$$

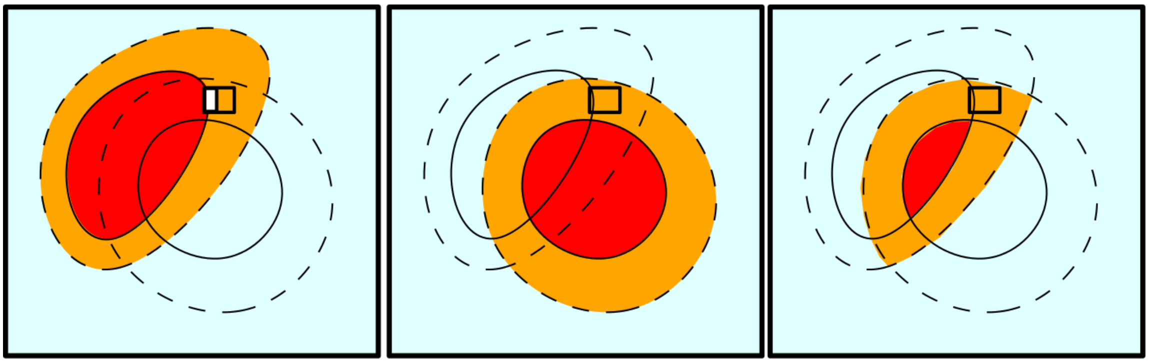


Figure 3: Illustration of the intersection of two separators

Thus

$$\begin{array}{rcl} [\![\mathcal{S}_{\mathbb{Z}}]\!] & = & \left\{\mathcal{S}_{\mathbb{Z}}^{in}, \mathcal{S}_{\mathbb{Z}}^{?}, \mathcal{S}_{\mathbb{Z}}^{out}\right\}([\mathbf{x}]) \\ & = \{ & \mathcal{S}_{\mathbb{X}}^{in} \cap \mathcal{S}_{\mathbb{Y}}^{in}([\mathbf{x}]), \\ & = & \left(\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{in}\right) \sqcup \left(\mathcal{S}_{\mathbb{X}}^{?} \cap \mathcal{S}_{\mathbb{Y}}^{?}\right) \sqcup \left(\mathcal{S}_{\mathbb{X}}^{in} \cap \mathcal{S}_{\mathbb{Y}}^{?}\right)([\mathbf{x}]), \\ & = & \mathcal{S}_{\mathbb{X}}^{out} \sqcup \mathcal{S}_{\mathbb{Y}}^{out}([\mathbf{x}]) \qquad \} \\ & = \{ & [\mathbf{a}] \cap \emptyset, ([\mathbf{x}] \cap \emptyset) \sqcup ([\mathbf{x}] \cap [\mathbf{x}]) \sqcup ([\mathbf{a}] \cap [\mathbf{x}]), \emptyset \sqcup \emptyset \quad \} \\ & = & \{\emptyset, [\mathbf{x}], \emptyset\} \end{array}$$

We conclude that $[\mathbf{x}] \subset \mathbb{Z}^{in}$.

5 Test case

Interval linear system [10] [11] are linear systems of equations the coefficients of which are uncertain and belong to some intervals. Consider for instance the following interval linear system [9]:

$$\left\{ \begin{array}{c} [2,4] \cdot x_1 + [-2,0] \cdot x_2 \in [-1,1] \\ [-1,1] \cdot x_1 + [2,4] \cdot x_2 \in [0,2] \end{array} \right. \tag{15}$$

For each constraint, a thick separator can be build and then combined using Equation 5. A thick set inversion algorithm provides the paving Figure 4. The solution set $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$ has for supset bound the

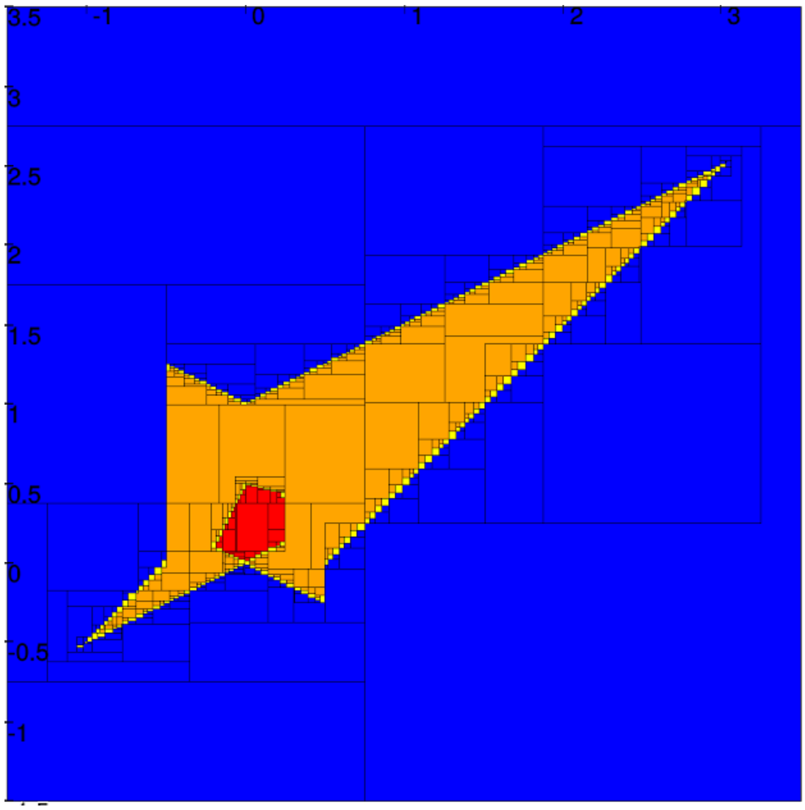


Figure 4: Thick set corresponding to the test-case. Red boxes are inside the thick solution set and the blue boxes are outside. The penumbra corresponds to the orange boxes.

tolerable solution set $\mathbb{X}^{\supset}$(red+orange) and for subset bound $\mathbb{X}^{\subset}$ the *united solution set* (red) [6]. Note that inside the penumbra, no accumulation can be observed.

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