Interval Integration of Triangular Systems

Luc Jaulin[1]

[1] ENSTA, Lab-STICC
2 rue François Verny, 29200, Brest
lucjaulin@ensta.fr

Keywords: Differential inclusion, Interval Methods, Reachability

Introduction

An *interval dynamical system* is a pair of two dynamical systems:

$$\begin{array}{rcl} \dot{\mathbf{x}}^-(t) & = & \mathbf{f}^-(\mathbf{x}^-(t), \mathbf{x}^+(t), \mathbf{u}^-(t), \mathbf{u}^+(t)) \\ \dot{\mathbf{x}}^+(t) & = & \mathbf{f}^+(\mathbf{x}^-(t), \mathbf{x}^+(t), \mathbf{u}^-(t), \mathbf{u}^+(t)) \end{array} \qquad (1)$$

Equivalently, the interval dynamical system we will be written as

$$[\dot{\mathbf{x}}](t) = [\![\mathbf{f}]\!]([\mathbf{x}](t), [\mathbf{u}](t)) \qquad (2)$$

where

$$\begin{array}{rcl} [\dot{\mathbf{x}}](t) & = & [\dot{\mathbf{x}}^-(t), \dot{\mathbf{x}}^+(t)] \\ [\mathbf{x}](t) & = & [\mathbf{x}^-(t), \mathbf{x}^+(t)] \\ [\mathbf{u}](t) & = & [\mathbf{u}^-(t), \mathbf{u}^+(t)] \\ [\![\mathbf{f}]\!] & = & [\mathbf{f}^-, \mathbf{f}^+] \end{array} \qquad (3)$$

The quantities $[\mathbf{u}](t), [\mathbf{x}](t), [\dot{\mathbf{x}}](t)$ correspond to tubes [3]. From an initial state $[\mathbf{x}](0)$, a tube $[\mathbf{x}](t)$ can easily be computed using a Runge-Kutta integration of Equation 6. An interval integration for ODE [2] can also be considered, if the guarantee is required [1]. The interval dynamical system (6) is *self consistent* if

$$\left.\begin{array}{c}\mathbf{x}^-(0) \leq \mathbf{x}^+(0) \\ \mathbf{u}^-(t) \leq \mathbf{u}^+(t), \forall t\end{array}\right\} \Rightarrow \mathbf{x}^-(t) \leq \mathbf{x}^+(t), \forall t$$

The interval dynamical system (2) is an *interval enclosure* of the system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ if

$$\left.\begin{array}{c}\mathbf{x}(0) = \mathbf{x}_0 \in [\mathbf{x}_0] \\ \mathbf{u}(t) \in [\mathbf{u}](t)\end{array}\right\} \Rightarrow \forall t, \mathbf{x}(t) \in [\mathbf{x}](t) \tag{4}$$

Interval enclosure of a triangular dynamical system

Consider the triangular system

$$\begin{array}{rcl}\dot{x}_1 & = & f_1(x_1, \mathbf{u}) \\ \dot{x}_2 & = & f_2(x_1, x_2, \mathbf{u}) \\ & \vdots & \\ \dot{x}_n & = & f_n(x_1, x_2, \ldots, x_n, \mathbf{u})\end{array} \tag{5}$$

Proposition 1. *Define*

$$[\![f_i]\!] = \left(\begin{array}{c} lb([f](x_i^-, [x_1] \times \cdots \times [x_{i-1}] \times [x_{i+1}] \times \cdots \times [x_n] \times [\mathbf{u}])) \\ ub([f](x_i^+, [x_1] \times \cdots \times [x_{i-1}] \times [x_{i+1}] \times \cdots \times [x_n] \times [\mathbf{u}]))\end{array}\right)$$

where $[f]$ is an inclusion function for f. An interval enclosure of (5) is

$$\begin{array}{rcl}[\dot{x}_1] & = & [\![f_1]\!]([x_1], [\mathbf{u}]) \\ [\dot{x}_2] & = & [\![f_2]\!]([x_2], [x_1] \times [\mathbf{u}]) \\ & \vdots & \\ [\dot{x}_n] & = & [\![f_m]\!]([x_n], [x_1] \times [x_2] \times \cdots \times [x_{n-1}] \times [\mathbf{u}])\end{array}$$

Ballast robot

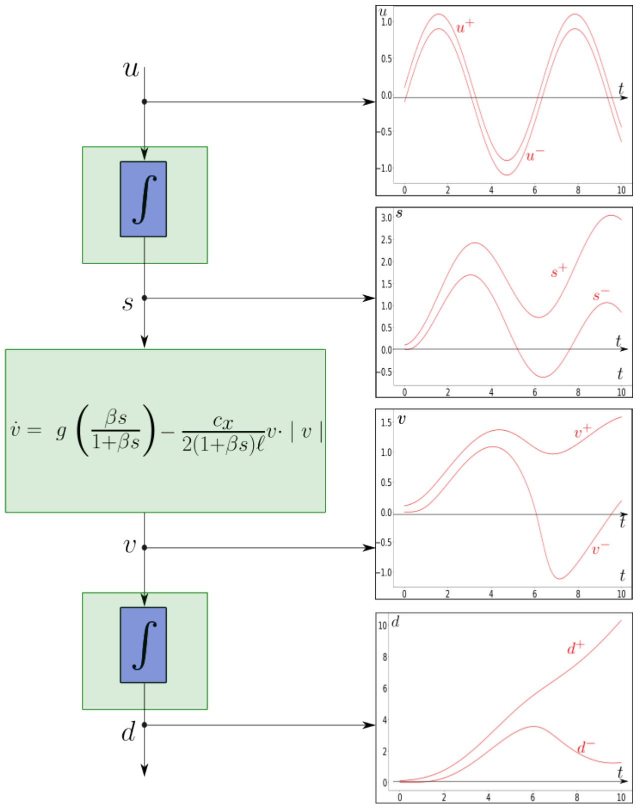
We consider an underwater float equipped with a ballast. It can only move upward to the surface and downward to the seafloor. The state equations are given by

$$\left\{\begin{array}{rcl} \dot{s} & = & u \\ \dot{v} & = & \frac{s}{1+s} - \frac{1}{1+s} v \cdot \mid v \mid \\ \dot{d} & = & v \end{array}\right. \tag{6}$$

where d is the depth, v is the vertical speed and s is the buoyancy. Since the system is triangular, Proposition 1 provides an interval enclosure of (6):

$$\begin{array}{rcl} \dot{s}^- & = & u^- \\ \dot{s}^+ & = & u^+ \\ \dot{v}^- & = & \text{lb}\left(\frac{[s]}{1+[s]} - \frac{1}{1+[s]} \cdot v^- \cdot \mid v^- \mid\right) \\ \dot{v}^+ & = & \text{ub}\left(\frac{[s]}{1+[s]} - \frac{1}{1+[s]} \cdot v^+ \cdot \mid v^+ \mid\right) \\ \dot{d}^- & = & d^- \\ \dot{d}^+ & = & d^+ \end{array}$$

The behavior of the corresponding interval simulator is illustrated by the figure below. We took $[s_0] = [v_0] = [d_0] = [0, 0.1]$ for the initial states. For the input, we have chosen $[u](t) = \sin(t) + [-0.1, 0.1]$.



References

[1] D. Efimov and T. Raïssi. Design of interval observers for uncertain dynamical systems. *Automation and Remote Control*, 77(2):191–225, 2016.

[2] T. Kapela, M. Mrozek, D. Wilczak, and P. Zgliczynski. CAPD: dynsys, A flexible C++ toolbox for rigorous numerical analysis of dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, 101:105578, 2021.

[3] S. Rohou, L. Jaulin, L. Mihaylova, F. Le Bars, and S. Veres. *Reliable Robot Localization*. Wiley, dec 2019.