Inner and Outer Set-membership State Estimation

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March 25, 2016

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Abstract

The main contribution of this paper is to provide a method (probably the first), based on separator algebra, which makes it possible to compute an inner and an outer approximations of the set $\mathbb{X}(t)$ of all states that are consistent with an initial set $\mathbb{X}(0)$ containing the initial state vector $\mathbf{x}(0)$ and a state equation of the form $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), t \in \mathbb{R}$. As an application, we consider the state estimation problem where feasible state vectors have to be consistent with some data intervals.

1 Introduction

Consider the following state estimation problem [1]

$$\begin{array}{ll} \text{(i)} & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), t \in \mathbb{R} \\ \text{(ii)} & \mathbf{g}(\mathbf{x}(t_k)) \in \mathbb{Y}(k), \, k \in \mathbb{N} \end{array} \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{g}(\mathbf{x}(t)) \in \mathbb{R}^m$. Equation (i) represents the evolution of the system and (ii) corresponds to the observation equation. The uncertainty on the measurements are represented by the subsets $\mathbb{Y}(k)$ of $\mathbb{R}^m$. Computing the the set of all state vectors $\mathbf{x}(t)$ consistent with the $\mathbb{Y}(k)$, for all k is known as a *set-membership estimation problem*. For this type of problem, interval analysis is often used, especialy in a nonlinear context (see, *e.g.*, [2], [3] or [4]). Our objective is to find an inner and an outer approximation of the set $\mathbb{X}(t)$ of all state vectors that are consistent with (1) at time t. Define by *flow map* φ_{t_1,t_2} as follows:

$$\left(\mathbf{x}(t_1) = \mathbf{x}_1 \text{ and } \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \Rightarrow \mathbf{x}_2 = \boldsymbol{\varphi}_{t_1,t_2}(\mathbf{x}_1)\right). \quad (2)$$

The set of all *causal feasible states* at time t is defined by [5]

$$\mathbb{X}(t) = \bigcap_{t_k \le t} \boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}(\mathbb{Y}(k)). \quad (3)$$

Remark. To understand this formula, it suffices to see that $\mathbf{g}^{-1}(\mathbb{Y}(k))$ corresponds to the set of all state vectors $\mathbf{x}$ at time t_k that are consistent with the measurement set $\mathbb{Y}(k)$. As a consequence, the set of all $\mathbf{x}$ at time t that are consistent with the measurement set $\mathbb{Y}(k)$ is $\boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}(\mathbb{Y}(k))$. To be consistent with all past measurements, we thus have to take the intersection of all sets associated with all measurements taken at time $t_k \leq t$.

In this paper, we show how it is possible to find both an inner and an outer approximations for $\mathbb{X}(t)$. Some existing methods are able to find an outer approximation [6], but, to my knowledge, none of them is able to get an inner approximation. The main idea is to copy a classical contractor approach [7] for state estimation, but to use separators [8] instead of contractors.

2 Separators

In this section, we present separators, recently introduced in [8], and show how they can be used by a paver in order to bracket the solution sets. An *interval* of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$. A box $[\mathbf{x}]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. A *contractor* $\mathcal{C}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that $\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$ and $[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$. A set $\mathbb{S}$ is *consistent* with the contractor $\mathcal{C}$ (we will write $\mathbb{S} \sim \mathcal{C}$) if for all $[\mathbf{x}]$, we have $\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}$. A *separator* $\mathcal{S}$ is a pair of contractors $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$, we have $\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}]$. A set $\mathbb{S}$ is *consistent* with the separator $\mathcal{S}$ (we write $\mathbb{S} \sim \mathcal{S}$), if $\mathbb{S} \sim \mathcal{S}^{\text{out}}$ and $\overline{\mathbb{S}} \sim \mathcal{S}^{\text{in}}$. where $\overline{\mathbb{S}} = \{\mathbf{x} \mid \mathbf{x} \notin \mathbb{S}\}$. By using a separator inside a *paver,* we can easily classify parts of the search space that are inside or outside a solution set $\mathbb{S}$ associated with the separator $\mathcal{S}$.

The algebra for separators is a direct extension of contractor algebra [7]. If $\mathcal{S}_i = \left\{\mathcal{S}_i^{\text{in}}, \mathcal{S}_i^{\text{out}}\right\}, i \in \{1,2\}$ are separators, we define

$$\begin{array}{rcll} \mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\right\} & \text{(intersection)} \\ \mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\right\} & \text{(union)} \\ \mathbf{f}^{-1}(\mathcal{S}_1) & = & \left\{\mathbf{f}^{-1}(\mathcal{S}_1^{\text{in}}), \mathbf{f}^{-1}(\mathcal{S}_1^{\text{out}})\right\} & \text{(inverse)} \end{array} \tag{4}$$

If $\mathbb{S}_i$ are sets of $\mathbb{R}^n$, we have [9] [8]

$$\begin{array}{cccc} \text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathcal{S}_1 \cap \mathcal{S}_2 \\ \text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathcal{S}_1 \cup \mathcal{S}_2 \\ \text{(iii)} & \mathbf{f}^{-1}(\mathbb{S}_1) & \sim & \mathbf{f}^{-1}(\mathcal{S}_1). \end{array} \tag{5}$$

Interval analysis [10] [11] combined with contractors [7] has been shown to be able to give an outer approximation for a large class of set defined by inequalities. The principle is to build a contractor for primitive sets and to build a contractor for more complex sets by intersections and unions.

Example. Consider the set

$$\mathbb{S} = \mathbb{S}_1 \cap (\mathbb{S}_2 \cup \mathbb{S}_3) \tag{6}$$

where

$$\mathbb{S}_i = \{\mathbf{x} \mid f_i(\mathbf{x}) \leq 0\} \tag{7}$$

If $\mathcal{C}_i$ are contractors for the sets $\mathbb{S}_i$, then a contractor for $\mathbb{S}$ is $\mathcal{C} = \mathcal{C}_1 \cap (\mathcal{C}_2 \cup \mathcal{C}_3)$.

For the inner approximation, we need to find a contractor $\overline{\mathcal{C}}$ for the set $\overline{\mathbb{S}}$, the complementary set $\mathbb{S}$. For this, we need to use the *De Morgan* rules:

$$\begin{array}{ccc} \overline{\mathbb{X}_1 \cap \mathbb{X}_2} & = & \overline{\mathbb{X}}_1 \cup \overline{\mathbb{X}}_2 \\ \overline{\mathbb{X}_1 \cup \mathbb{X}_2} & = & \overline{\mathbb{X}}_1 \cap \overline{\mathbb{X}}_2. \end{array} \tag{8}$$

Then, basic contractor techniques can be used to get an inner characterizations.

Example. Consider the previous example and let is now show how to buid a contractor for the complementary set of $\mathbb{S} = \mathbb{S}_1 \cap (\mathbb{S}_2 \cup \mathbb{S}_3)$. From the De Morgan rules, we have

$$\overline{\mathbb{S}} = \overline{\mathbb{S}}_1 \cup \left(\overline{\mathbb{S}}_2 \cap \overline{\mathbb{S}}_3\right) \tag{9}$$

with

$$\overline{\mathbb{S}}_i = \{\mathbf{x} \mid f_i(\mathbf{x}) > 0\} \tag{10}$$

If $\overline{\mathcal{C}}_i$ are contractors for the sets $\overline{\mathbb{S}}_i$, then a contractor for $\overline{\mathbb{S}}$ is $\overline{\mathcal{C}} = \overline{\mathcal{C}}_1 \cup \left(\overline{\mathcal{C}}_2 \cap \overline{\mathcal{C}}_3\right)$.

Now, the complementation task is tredious and the role of *separators* is to make it automatic.

Paver. Once, we have built the separator for a set $\mathbb{S}$, we can compute an approximation of $\mathbb{S}$ using a paver. A paver takes as an input an initial box which is sufficiently large to enclose $\mathbb{S}$ and then the paver partitions this box into smaller boxes. For each of these subboxes, the separator is called. Parts of the boxes that or proved to be inside or outside $\mathbb{S}$ are stored in a list in order to build an inner and an outer approximations of $\mathbb{S}$. Remaining boxes that are not too small are bisected and te operation is repeated until all remaining subboxes have a width smaller than a given value.

3 Transformation of separators

A transformation is an invertible function $\mathbf{f}$ such as an analytical expression is known for both $\mathbf{f}$ and $\mathbf{f}^{-1}$. The set of transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a group with respect to the composition $\circ$. Symmetries, translations, homotheties, rotations, ... are linear transformations.

Theorem *(Separator transformation)*. Consider a set $\mathbb{X}$ and a transformation $\mathbf{f}$. Denote by $[\mathbf{f}]$ and $\left[\mathbf{f}^{-1}\right]$ two inclusion functions for $\mathbf{f}$ and $\mathbf{f}^{-1}$. If $\mathcal{S}_{\mathbb{X}}$ is a separator for $\mathbb{X}$ then a separator $\mathcal{S}_{\mathbb{Y}}$ for $\mathbb{Y} = \mathbf{f}(\mathbb{X})$ is

$$[\mathbf{y}] \rightarrow \left\{\left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \left[\mathbf{f}^{-1}\right]\right)([\mathbf{y}]) \cap [\mathbf{y}], \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]\right)([\mathbf{y}]) \cap [\mathbf{y}]\right\} \tag{11}$$

or equivalently

$$\mathbf{f}(\mathbb{X}) \sim \left\{[\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \left[\mathbf{f}^{-1}\right] \cap \text{Id},\ [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right] \cap \text{Id}\right\} \tag{12}$$

where *Id* is the identity contractor.

Figure 3 illustrates this theorem. Figure 3.1 represents the box $[\mathbf{y}]$ to be separated. In Figure 3.2 we compute the box $\left[\mathbf{f}^{-1}\right]([\mathbf{y}])$. In Figure 3.3, this box is separated into two overlapping boxes using $\mathcal{S}_{\mathbb{X}}$. In Figure 3.4, the two contracted boxes are mapped into the $\mathbf{y}$-space via $[\mathbf{f}]$. In Figure 3.5 these boxes are intersected with $[\mathbf{y}]$.

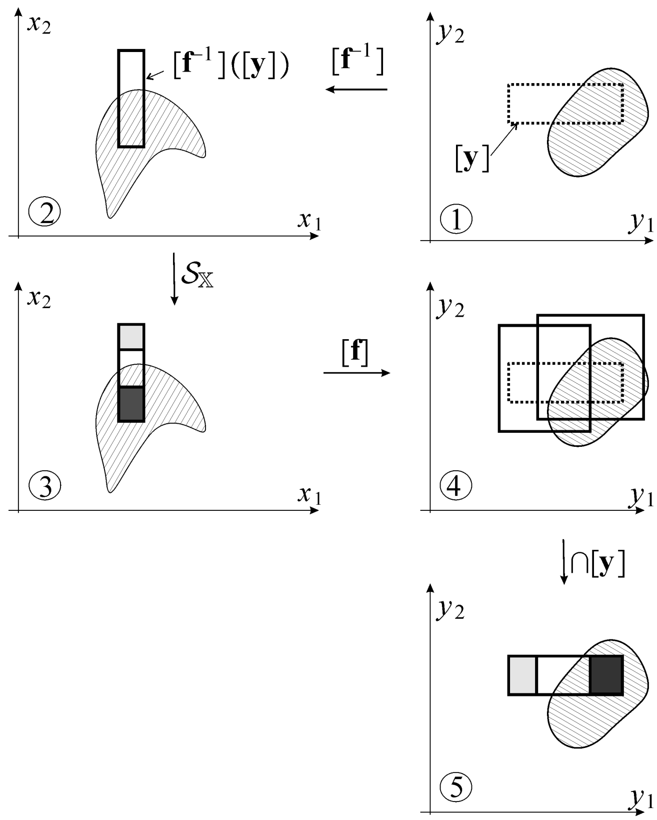

Figure 1: Illustration of the separator transformation Theorem

Remark. The separator defined by (11) corresponds to what we call the *transformation* of a separator by $\mathbf{f}$ and we write $\mathcal{S}_{\mathbb{Y}} = \mathbf{f}(\mathcal{S}_{\mathbb{X}})$. As a consequence, thanks to the theorem, we can add to (5) the property

$$\text{(iv)} \qquad \mathbf{f}(\mathbb{X}) \quad \sim \quad \mathbf{f}(\mathcal{S}_{\mathbb{X}}).$$

which will be used later for our state estimation problem.

Proof. The separator $\mathcal{S}_{\mathbb{Y}}$ is equivalent to $\mathbb{Y} = \mathbf{f}(\mathbb{X})$ if

$$\left\{ \begin{array}{lll} \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) \cap \mathbb{Y} & = & [\mathbf{y}] \cap \mathbb{Y} \\ \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}]) \cap \overline{\mathbb{Y}} & = & [\mathbf{y}] \cap \overline{\mathbb{Y}}. \end{array} \right. \tag{13}$$

Since $\mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}]) \subset [\mathbf{y}]$ and $\mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) \subset [\mathbf{y}]$, it suffices to prove that

$$\left\{ \begin{array}{lll} \text{(i)} & \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) & \supset & [\mathbf{y}] \cap \mathbb{Y} \\ \text{(ii)} & \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}]) & \supset & [\mathbf{y}] \cap \overline{\mathbb{Y}}. \end{array} \right. \tag{14}$$

Let us first prove (i). We have

$$\begin{array}{rcll} [\mathbf{y}] \cap \mathbb{Y} & = & \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \mathbf{f}^{-1}(\mathbb{Y})\right) & \mathbf{f} \text{ is bijective} \\ & = & \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \mathbb{X}\right) & \mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) \\ & \subset & \mathbf{f}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap \mathbb{X}\right) & \left[\mathbf{f}^{-1}\right] \text{ is an inclusion function for } \mathbf{f}^{-1} \\ & \subset & \mathbf{f}(\mathcal{S}_{\mathbb{X}}^{\text{out}}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}])\right)) & \mathcal{S}_{\mathbb{X}}^{\text{out}} \text{ is a contractor for } \mathbb{X} \\ & \subset & [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) & [\mathbf{f}] \text{ is an inclusion function for } \mathbf{f} \end{array} \tag{15}$$

Thus $[\mathbf{y}] \cap \mathbb{Y} \subset \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap [\mathbf{y}]\right) = \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}])$. Let us now prove (ii). We have

$$\begin{array}{rcll} [\mathbf{y}] \cap \overline{\mathbb{Y}} & = & \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \mathbf{f}^{-1}\left(\overline{\mathbb{Y}}\right)\right) & \mathbf{f} \text{ is bijective} \\ & = & \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \overline{\mathbb{X}}\right) & \overline{\mathbb{X}} = \mathbf{f}^{-1}\left(\overline{\mathbb{Y}}\right) \\ & \subset & \mathbf{f}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap \overline{\mathbb{X}}\right) & \left[\mathbf{f}^{-1}\right] \text{ is an inclusion function for } \mathbf{f}^{-1} \\ & \subset & \mathbf{f}(\mathcal{S}_{\mathbb{X}}^{\text{in}}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}])\right)) & \mathcal{S}_{\mathbb{X}}^{\text{in}} \text{ is a contractor for } \overline{\mathbb{X}} \\ & \subset & [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) & [\mathbf{f}] \text{ is an inclusion function for } \mathbf{f} \end{array} \tag{16}$$

Thus $[\mathbf{y}] \cap \mathbb{Y} \subset \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap [\mathbf{y}]\right) \cap \mathbb{Y} = \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}])$ which terminates the proof. ■

Example. Consider the constraint

$$\left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 - 1 \\ y_2 - 2 \end{pmatrix} \right\| \in [1, 3]. \tag{17}$$

If we apply an efficient forward-backward contractor in a paver, we get the contractions illustrated by the paving of Figure 3, left. Now, if we take

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 - 1 \\ y_2 - 2 \end{pmatrix} = \mathbf{f}^{-1}(\mathbf{y}) \tag{18}$$

or equivalently

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{f}(\mathbf{x}), \tag{19}$$

we get

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \text{ and } \|\mathbf{x}\| \in [1, 3]. \tag{20}$$

An optimal separator $\mathcal{S}_{\mathbb{X}}$ can be built for $\mathbf{x}$ and the separator transform provides us a separator $\mathcal{S}_{\mathbb{Y}}$ for $\mathbb{Y}$. As illustrated by Figure 3, right, the resulting separator $\mathcal{S}_{\mathbb{Y}}$ gets stronger contractions than the classical one based on forward-backward contractors.

Note that in case we are not able to have an inner approximation for $\mathbf{f}^{-1}$, the problem of finding an inner approximation of the image of a set $\mathbf{f}(\mathbb{X})$ becomes much more difficult. See, *e.g.*, [12] [13] [14].

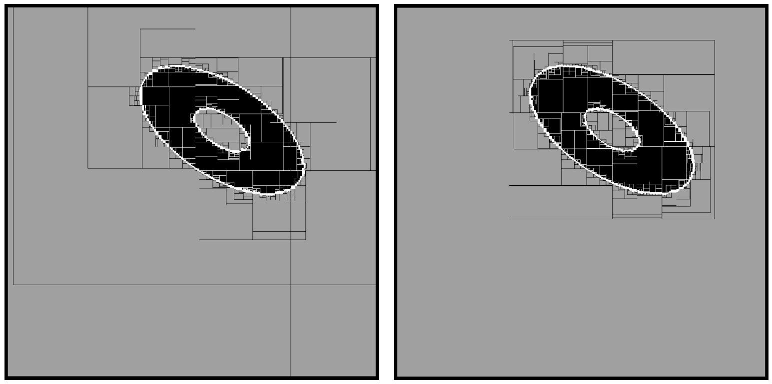

Figure 2: Left. Contractions obtained using a classical forward-backward propagation; Right. Contractions obtained using the separator transform. The frame corresponds to the box $[-6,6]^2$.

4 State estimation

In this section, we illustrate how the separators can be used to compute an inner and an outer approximations of the feasible set for the state vectors. For this, we will use the separators to implement the set-membership observer (3). If $\mathcal{S}_{\mathbb{Y}(k)}$ are separators for $\mathbb{Y}(k)$, then a separator for the set $\mathbb{X}(t)$ defined by (3) is

$$\mathcal{S}_{\mathbb{X}(t)} = \bigcap_{t_k \leq t} \boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right). \tag{21}$$

In this formula, $\mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right)$ is a separator. Due to the fact that $\boldsymbol{\varphi}_{t_k,t}$ is bijective and that we are able to find an inclusion function for $\boldsymbol{\varphi}_{t_k,t}$ and $\boldsymbol{\varphi}_{t_k,t}^{-1}$ [15] [16] [17], the separator $\boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right)$ is clearly defined using the separator transform.

The method we propose is thus the following. For each sampling time, $t = 0.1 \cdot k,\ k \in \mathbb{N}$, we call a paver with the separator given by (21). As a result, we get a guaranteed enclosure of $\mathbb{X}(t)$ with an inner and an outer approximations. Of course, the method could be made more efficient by taking into account the computations made before, in a recursive manner. Now, this method is sufficient to illustrate the main contribution of the paper which is the inner approximation of $\mathbb{X}(t)$.

To illustrate the method, let us consider a robot described by

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{pmatrix} v(t) \cdot \cos\theta(t) \\ v(t) \cdot \sin\theta(t) \end{pmatrix} & \text{(evolution)} \\ \|\mathbf{x}(t_k)\| \in y(t_k) + [-0.3, 0.3],\ t_k = 0.1 \cdot k,\ \ k \in \mathbb{N} & \text{(observation)} \end{cases} \tag{22}$$

where $v(t)$ and $\theta(t)$ are measured with an accuracy of ± 0.03. The observation equation is due to the fact that the robot measures every 0.1 sec its distance to the origin with an accuracy of ± 0.3. The actual (but unknown) trajectory for the robot is

$$\mathbf{x}(t) = \begin{pmatrix} 2 + 3\cos t \\ 2\sin t \end{pmatrix}. \tag{23}$$

For $t \in 0.2 \cdot k,\ k = 0, \ldots, 7$, the sets $\mathbb{X}(t)$ obtained by our observer are represented on Figure 4. Black boxes are inside $\mathbb{X}(t)$, grey boxes are outside and the white boxes cover the boundary. For $t = 0$, $\mathbb{X}(t)$ is a ring which becomes a small set for $t = 1.4$ once the robot has moved sufficiently. The fact that the white area covering the boundary becomes thick is mainly due to the state errors inside the evolution equation.

5 Conclusion

This paper has introduced the new concept of the transformation of a separator. Taking into account the property that the flow map is invertible for a deterministic system, we were able to build a method computing an inner and outer approximations of the set of all states that are consistent at time t with the initial set $\mathbb{X}(0)$ and a collection of data bars. One simple test-case has been presented in order to illustrate the efficiency of the method. In a near future, it would be interesting to compare/combine with the approach given in [18] to compute an inner approximation of the reachability set. An other improvement would be a recursive implementation of the state observer in order to factorize the computation at different steps.

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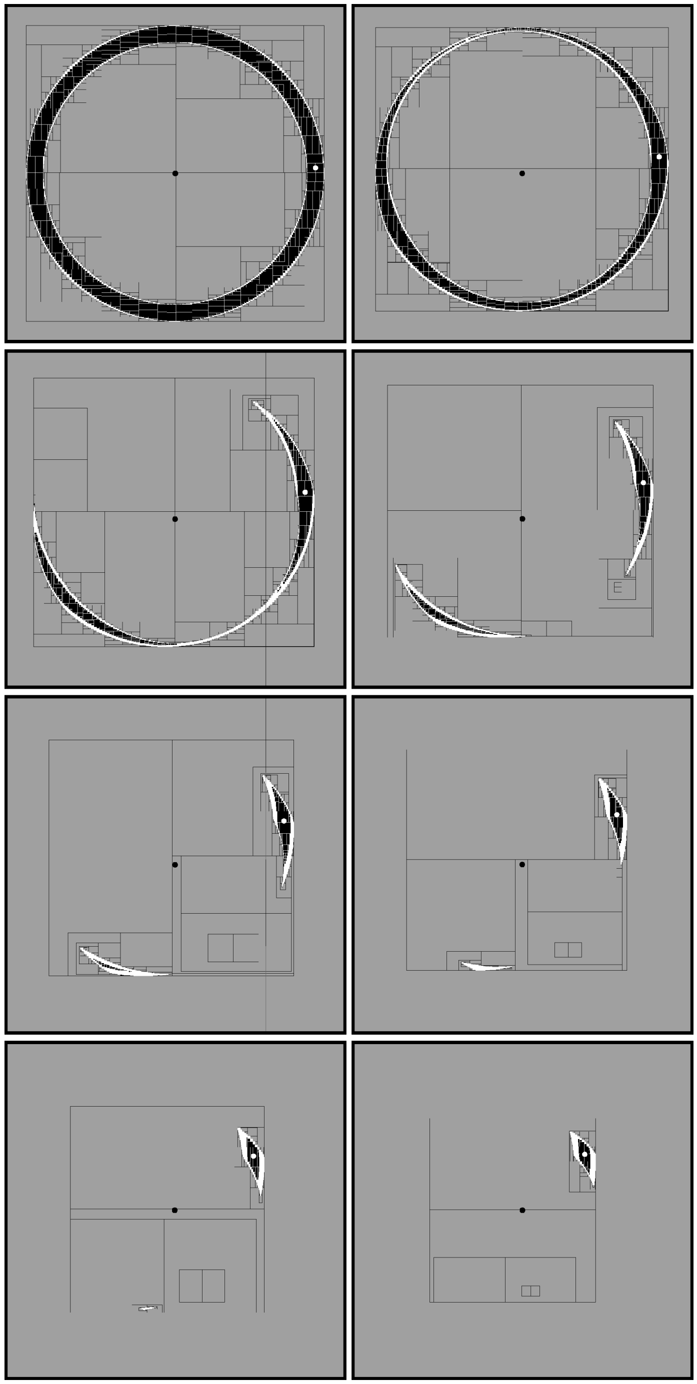

Figure 3: Inner and outer approximations of the set of all feasible state vectors $\mathbb{X}(t)$, for $t \in 0, 0.2, \ldots, 1.4$. The frame boxes are $[-6, 6]^2$.

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