Modelisation of a rolling disk with Sympy

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Abstract. This paper proposes a Lagrangian approach to find the state equations of a disk rolling on a plane without friction. The approach takes advantage of a symbolic computation to simplify the reasoning.

1 Introduction

Consider a disk rolling on a plane without friction not sliding, as shown on Figure 1.

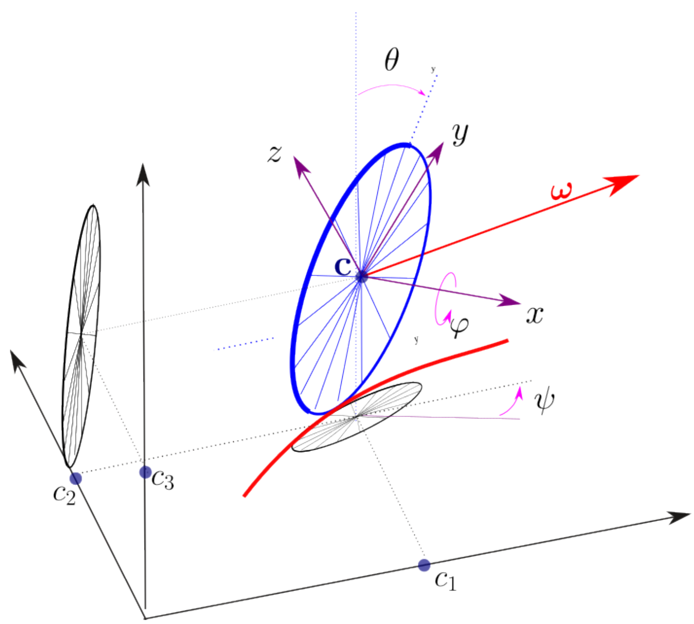

Fig. 1: Disk (blue) rolling on a plane. The vertical and horizontal projections are painted black

We assume that the disk mass is $m = 5$kg and its radius is $r = 1m$. The gravity is taken as $g = 9.81ms^{-2}$. In this paper, we want to find the state equation describing the motion of the disk. This problem has already been solved for over a century (see *e.g.* [1]) and even highly studied since (see *e.g.* [5]). Extension to more general wheel based vehicle have also been proposed (see *e.g.* [2] for the bicycle).

Computing the state equation for a rolling disk is a tedious task. This paper takes advantage of symbolic computing (here the `sympy` package of `Python`) in order to derive these state equations. The Lagrangian approach [6], often applied to model robots [3] will be chosen.

2 Modelisation

2.1 State vector

We take the state vector $\mathbf{x} = (c_1, c_2, \varphi, \theta, \psi, \dot{\varphi}, \dot{\theta}, \dot{\psi})$ where (c_1, c_2) is the vertical projection of center $\mathbf{c} = (c_1, c_2, c_3)$ of the disk and φ, θ, ψ are the three Euler angle. As illustrated by Figure 1,

- φ is the spin angle
- θ is the stand angle, *i.e.*, when $\theta = 0$, the disk is vertical
- ψ is the heading, *i.e.*, the horizontal orientation of the disk.

To use sympy, we first declare the symbolic variables and functions to be used:

```
t=symbols('t')
m,g,r= symbols('m g r')
c1,c2 = Function('c1')('t'),Function('c2')('t')
dc1,dc2 = Function('dc1')('t'),Function('dc2')('t')
ddc1,ddc2 = Function('ddc1')('t'),Function('ddc2')('t'),
ψ,θ,φ = Function('ψ')('t'),Function('θ')('t'),Function('φ')('t')
dψ,dθ,dφ = Function('dψ')('t'),Function('dθ')('t'),Function('dφ')('t')
ddψ,ddθ,ddφ = Function('ddψ')('t'),Function('ddθ')('t'),Function('ddφ')('t')
φ,θ,ψ = Function('φ')('t'),Function('θ')('t'),Function('ψ')('t')
dφ,dθ,dψ = Function('dφ')('t'),Function('dθ')('t'),Function('dψ')('t')
ddφ,ddθ,ddψ = Function('ddφ')('t'),Function('ddθ')('t'),Function('ddψ')('t')
λ1,λ2 = Function('λ1')('t'),Function('λ2')('t')
```

2.2 Orientation

The orientation of the disk is fixed by the three Euler angles φ, θ, ψ. The corresponding orientation matrix is

$$\mathbf{R} = \begin{pmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\theta\cos\psi\sin\varphi & \sin\psi\sin\varphi + \sin\theta\cos\psi\cos\varphi \\ \cos\theta\sin\psi & \cos\psi\cos\varphi + \sin\theta\sin\psi\sin\varphi & -\cos\psi\sin\varphi + \sin\theta\cos\varphi\sin\psi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix} \tag{1}$$

It is built by the following sympy function

```
def Reuler(φ,θ,ψ):
   Rφ = Matrix([[1,0,0],[0,cos(φ),-sin(φ)],[0,sin(φ),cos(φ)]])
   Rθ = Matrix([[cos(θ),0,sin(θ)],[0,1,0],[-sin(θ),0,cos(θ)]])
   Rψ = Matrix([[cos(ψ),-sin(ψ),0],[sin(ψ),cos(ψ),0],[0,0,1]])
   Return  Rψ*Rθ*Rφ
```

The rotation vector depends on the Euler angles and their derivatives. Its expression [4] can be obtained using the relation

$$\mathbf{R}^{\mathrm{T}}\dot{\mathbf{R}} = \begin{pmatrix} 0 & -\omega_{r3} & \omega_{r2} \\ \omega_{r3} & 0 & -\omega_{r1} \\ -\omega_{r2} & \omega_{r1} & 0 \end{pmatrix} \tag{2}$$

which gives us the following sympy function

```
def wr(R):
   W=Transpose(R)*diff(R,t)
   return Matrix([[-W[1,2]],[W[0,2]],[-W[0,1]]])
```

We get

$$\boldsymbol{\omega}_r = \begin{pmatrix} \dot{\varphi} - \dot{\psi}\sin\theta \\ \dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\cos\theta \\ -\dot{\theta}\sin\varphi + \dot{\psi}\cos\theta\cos\varphi \end{pmatrix}. \tag{3}$$

2.3 Lagrangian

In order, to get the state equation of the rolling disk, we use a Lagrangian approach. For this, we need to express the Lagrangian $\mathcal{L}$ with respect to the state variables. Recall that

$$\mathcal{L} = E_K - E_p \tag{4}$$

where E_K is the kinetic energy and E_p is the potential energy. We have

$$E_K = \frac{1}{2}\boldsymbol{\omega}_r^{\mathrm{T}}\mathbf{I}\boldsymbol{\omega}_r + \frac{1}{2}m\|\dot{\mathbf{c}}\|^2 \tag{5}$$

where $\mathbf{c} = (c_1, c_2, c_3)$ is the center of the disk and

$$\mathbf{I} = \begin{pmatrix} \frac{mr^2}{2} & 0 & 0 \\ 0 & \frac{mr^2}{4} & 0 \\ 0 & 0 & \frac{mr^2}{4} \end{pmatrix} \tag{6}$$

is the inertia matrix of the disk. Moreover

$$E_p = mgr\cos\theta. \tag{7}$$

Denote by

$$\mathbf{q} = (c_1, c_2, \varphi, \theta, \psi) \tag{8}$$

the generalized coordinates of the system, *i.e.*, the degrees of freedom of the system. The Lagrangian, which is a function of $(\mathbf{q}, \dot{\mathbf{q}})$, is computed with `sympy` :

```
c1,c2,φ,θ,ψ=list(q)
c=Matrix([[c1],[c2],[r*cos(θ)]])
dc=diff(c,t)
R=Reuler(φ,θ,ψ)
Ep=m*g*c[2]
I=Matrix([[1/2*m*r**2,0,0],[0,1/4*m*r**2,0],[0,0,1/4*m*r**2]])
Ek=1/2*m*(dc1**2+dc2**2+dc3**2)+(1/2)*wr(R).dot(I*wr(R))
L=Ek-Ep
```

which yields

$$\begin{aligned} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) \quad = \quad & \tfrac{1}{2}m\left(\dot{c}_1^2 + \dot{c}_2^2 + r^2\sin^2\theta\dot{\theta}^2\right) \\ & + \tfrac{1}{8}mr^2\left(2\left(\dot{\varphi} - \sin\theta\dot{\psi}\right)^2 + \dot{\theta}^2 + \cos^2\theta\dot{\psi}^2\right) \\ & - mgr\cos\theta \end{aligned} \tag{9}$$

The evolution of $\mathbf{q}$ obeys to the Lagrange's equation for holonomic systems

$$\underbrace{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\right) - \frac{\partial\mathcal{L}}{\partial\mathbf{q}}}_{\mathcal{Q}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})} = \boldsymbol{\tau} \tag{10}$$

where $\boldsymbol{\tau}$ are the constraint forces. The ith component τ_i of $\boldsymbol{\tau}$ is associated to the ith component q_i of $\mathbf{q}$. Now

$$\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} = \begin{pmatrix} m\dot{c}_1 \\ m\dot{c}_2 \\ \frac{1}{2}mr^2(\dot{\varphi} - \dot{\psi}\sin\theta) \\ mr^2(\sin^2\theta + \frac{1}{4})\dot{\theta} \\ \frac{1}{4}mr^2(-2\dot{\varphi}\sin\theta + \dot{\psi}\sin^2\theta + \dot{\psi}) \end{pmatrix} \tag{11}$$

Thus

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} m\ddot{c}_1 \\ m\ddot{c}_2 \\ mr^2(\ddot{\varphi} - \ddot{\psi}\sin\theta - \dot{\theta}\dot{\psi}\cos\theta)/2 \\ \frac{1}{4}mr^2(4\ddot{\theta}\sin^2\theta + \ddot{\theta} + 4\dot{\theta}^2\sin(2\theta)) \\ \frac{1}{4}mr^2(-2\ddot{\varphi}\sin\theta + \ddot{\psi}\sin^2\theta + \ddot{\psi} - 2\dot{\theta}\dot{\varphi}\cos\theta + \dot{\theta}\dot{\psi}\sin(2\theta)) \end{pmatrix} \tag{12}$$

Moreover

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4}mr\left(4g\sin\theta + 2r\dot{\theta}^2\sin(2\theta) - 2r\dot{\varphi}\dot{\psi}\cos\theta + \frac{r\dot{\psi}^2\sin(2\theta)}{2}\right) \\ 0 \end{pmatrix} \tag{13}$$

The left hand side of the Euler Lagrange equation (10) is thus

$$\mathcal{Q}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) = \begin{pmatrix} m\ddot{c}_1 \\ m\ddot{c}_2 \\ \frac{1}{2}mr^2(\ddot{\varphi} - \ddot{\psi}\sin\theta - \dot{\theta}\dot{\psi}\cos\theta) \\ \frac{1}{8}mr\left(-8g\sin\theta + 8r\ddot{\theta}\sin^2\theta + 2r\ddot{\theta} + 4r\dot{\theta}^2\sin(2\theta) + 4r\dot{\varphi}\dot{\psi}\cos\theta - r\dot{\psi}^2\sin(2\theta)\right) \\ \frac{1}{4}mr^2\left(-2\ddot{\varphi}\sin\theta + \ddot{\psi}\sin^2\theta + \ddot{\psi} - 2\dot{\theta}\dot{\varphi}\cos\theta + \dot{\theta}\dot{\psi}\sin(2\theta)\right) \end{pmatrix} \tag{14}$$

This expression for $\mathcal{Q}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})$ is obtained directly using `sympy` by

```
q=Matrix([c1,c2,φ,θ,ψ])
dq=Matrix([dc1,dc2,dφ,dθ,dψ])
ddq=Matrix([ddc1,ddc2,ddφ,ddθ,ddψ])
Q=diff(L.jacobian(dq),t)-L.jacobian(q)
```

2.4 Non holonomic constraints

If the ground is a flat frozen lake where the disk can slide without any friction in both direction (horizontally and laterally), then the state vector is

$$(\mathbf{q},\dot{\mathbf{q}}) = (c_1, c_2, \varphi, \theta, \psi, \dot{c}_1, \dot{c}_2, \dot{\varphi}, \dot{\theta}, \dot{\psi}), \tag{15}$$

i.e., the state is composed of the degrees of freedom $\mathbf{q}$ and they derivatives $\dot{\mathbf{q}}$. Now, in our case, no sliding is possible and the disk can only roll. Due to this rolling constraint, $(\mathbf{q},\dot{\mathbf{q}})$ are linked by some differential constraints. These constraints are needed to derive the state equations with $\mathbf{x} = (c_1, c_2, \varphi, \theta, \psi, \dot{\varphi}, \dot{\theta}, \dot{\psi})$ as a state vector. Since we have two variables to eliminate (here $\dot{c}_1$ and $\dot{c}_2$), we need to find two more differential constraints which are generated by the ground forces. These constraints translate the fact that the point of the disk in contact with the ground has a zero velocity. It means that the disk neither slides tangentially (first equation) nor laterally (second equation). We understand that these two equations have the form

$$\begin{array}{rcl} \dot{c}_1 & = & \alpha_1 \cdot \dot{\varphi} + \alpha_2 \cdot \dot{\theta} + \alpha_3 \cdot \dot{\psi} \\ \dot{c}_2 & = & \beta_1 \cdot \dot{\varphi} + \beta_2 \cdot \dot{\theta} + \beta_3 \cdot \dot{\psi} \end{array} \tag{16}$$

where the α_i's and the $\beta_i's$ depend on $\mathbf{q}$. More precisely, this corresponds to the *non holonomic* constraints given by

$$\begin{array}{rcl} \dot{c}_1 & = & r\sin\psi \cdot \dot{\varphi} + r\cos\psi\cos\theta \cdot \dot{\theta} - r\sin\psi\sin\theta \cdot \dot{\psi} \\ \dot{c}_2 & = & -r\cos\psi \cdot \dot{\varphi} + r\sin\psi\cos\theta \cdot \dot{\theta} + r\cos\psi\sin\theta \cdot \dot{\psi} \end{array} \tag{17}$$

Figure 2 illustrates how this formula is obtained. The left subfigure shown that, when $\dot{\theta} = 0$ and $\dot{\psi} = 0$, we have

$$\begin{array}{rcl} \dot{c}_1 & = & r \sin \psi \cdot \dot{\varphi} \\ \dot{c}_2 & = & -r \cos \psi \cdot \dot{\varphi} \end{array} \tag{18}$$

The subfigure in the center illustrates that if $\dot{\varphi} = 0$ and $\dot{\psi} = 0$,

$$\begin{array}{rcl} \dot{c}_1 & = & r \cos \psi \cos \theta \cdot \dot{\theta} \\ \dot{c}_2 & = & r \sin \psi \cos \theta \cdot \dot{\theta} \end{array} \tag{19}$$

The right subfigure illustrates that if $\dot{\theta} = 0$ and $\dot{\varphi} = 0$,

$$\begin{array}{rcl} \dot{c}_1 & = & -r \sin \psi \sin \theta \cdot \dot{\psi} \\ \dot{c}_2 & = & r \cos \psi \sin \theta \cdot \dot{\psi} \end{array} \tag{20}$$

By superposition, we get Equation (17). These constraints are said to be non holonomic since they will not allow us to express our system with a state composed of some degrees of freedom $\mathbf{q}$ and their derivatives $\dot{\mathbf{q}}$.

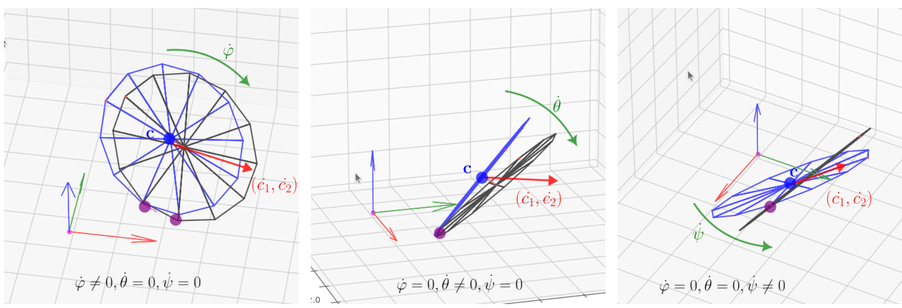

Fig. 2: Deriving the non holonomic constraints

2.5 D'Alembert's principle

We need to find an expression for $\boldsymbol{\tau}$ which occurs in the right hand side of the Euler-Lagrange equation (10). The components for $\boldsymbol{\tau}$ correspond to the generalized forces applied to our system. Now, in our specific case, we have no friction no external forces to thrust or slow down the disk. The only forces that apply are the reaction of the ground on the disk.

In order to use this information, let us to recall the principle of d'Alembert: *for arbitrary virtual displacements, the constraint forces don't do any work.*

The virtual displacements are infinitesimal changes $\delta \mathbf{q} = \mathbf{q}(t + dt) - \mathbf{q}(t)$ for $\mathbf{q}(t)$ that should be consistent with some feasible trajectories. For our rolling disk, the virtual displacements satisfy

$$\begin{array}{rcll} \delta c_1 - r \sin \psi \cdot \delta \varphi - r \cos \psi \cos \theta \cdot \delta \theta + r \sin \psi \sin \theta \cdot \delta \psi & = & 0 & (i) \\ \delta c_2 + r \cos \psi \cdot \delta \varphi - r \sin \psi \cos \theta \cdot \delta \theta - r \cos \psi \sin \theta \cdot \delta \psi & = & 0 & (ii) \end{array} \tag{21}$$

for the same reasons than those used to derive (17). The fact that there is no work translates into

$$\delta W = \boldsymbol{\tau}^{\mathrm{T}} \delta \mathbf{q} = \tau_{c_1} \cdot \delta c_1 + \tau_{c_2} \cdot \delta c_2 + \tau_{\varphi} \cdot \delta \varphi + \tau_{\theta} \cdot \delta \theta + \tau_{\psi} \cdot \delta \psi \quad = \quad 0. \tag{22}$$

Equivalently, Equation (22) is a linear combination of the two equations (21), *i.e.*, (22) = $\lambda_1 \cdot$(21,i)+$\lambda_2 \cdot$(21,ii) and the λ_i are the Lagrange parameters. Therefore:

$$\begin{pmatrix} \tau_{c_1} \\ \tau_{c_2} \\ \tau_{\varphi} \\ \tau_{\theta} \\ \tau_{\psi} \end{pmatrix} = \underbrace{\lambda_1 \cdot \begin{pmatrix} 1 \\ 0 \\ -r \sin \psi \\ -r \cos \psi \cos \theta \\ r \sin \psi \sin \theta \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 0 \\ 1 \\ r \cos \psi \\ -r \sin \psi \cos \theta \\ -r \cos \psi \sin \theta \end{pmatrix}}_{\boldsymbol{\tau}(\mathbf{q}, \boldsymbol{\lambda})} \tag{23}$$

Using (14) and (23), we get that the Euler Lagrange equation rewrites into

$$\mathcal{Q}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) - \boldsymbol{\tau}(\mathbf{q}, \boldsymbol{\lambda}) = \mathbf{0} \tag{24}$$

i.e.,

$$\begin{array}{rcl}
m\ddot{c}_1 - \lambda_1 & = & 0 \\
m\ddot{c}_2 - \lambda_2 & = & 0 \\
\frac{1}{2}mr^2(\ddot{\varphi} - \ddot{\psi}\sin\theta - \dot{\theta}\dot{\psi}\cos\theta) + \lambda_1 r\sin\psi - \lambda_2 r\cos\psi & = & 0 \\
\frac{1}{8}mr\left(-8g\sin\theta + 8r\ddot{\theta}\sin^2\theta + 2r\ddot{\theta} + 4r\dot{\theta}^2\sin(2\theta) + 4r\dot{\varphi}\dot{\psi}\cos\theta - r\dot{\psi}^2\sin(2\theta)\right) & & \\
+\lambda_1 r\cos\psi\cos\theta + \lambda_2 r\sin\psi\cos\theta & = & 0 \\
\frac{1}{4}mr^2\left(-2\ddot{\varphi}\sin\theta + \ddot{\psi}\sin^2\theta + \ddot{\psi} - 2\dot{\theta}\dot{\varphi}\cos\theta + \dot{\theta}\dot{\psi}\sin(2\theta)\right) - \lambda_1 r\sin\psi\sin\theta + \lambda_2 r\cos\psi\sin\theta & = & 0
\end{array} \tag{25}$$

This system is made of 5 equations which are linear in 7 variables : $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\ddot{\mathbf{q}} = (\ddot{c}_1, \ddot{c}_2, \ddot{\varphi}, \ddot{\theta}, \ddot{\psi})$.
In order to square the system, we need to add two equations (to get 7 equations). They can be derived from non-holonomic constraints (17) given by

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 0 & -r\sin\psi & -r\cos\psi\cos\theta & r\sin\psi\sin\theta \\ 0 & 1 & r\cos\psi & -r\sin\psi\cos\theta & -r\cos\psi\sin\theta \end{pmatrix}}_{\mathbf{A}(\mathbf{q})} \cdot \dot{\mathbf{q}}}_{\mathbf{a}(\mathbf{q},\dot{\mathbf{q}})} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let us differentiate this equation. We get:

$$\underbrace{\frac{\partial \mathbf{a}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial \mathbf{a}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \cdot \ddot{\mathbf{q}}}_{=\frac{d}{dt}\mathbf{a}(\mathbf{q},\dot{\mathbf{q}})} = \mathbf{0} \tag{26}$$

We add these two equations to (25) to get

$$\underbrace{\begin{pmatrix} \frac{d}{dt}\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathcal{Q}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) - \boldsymbol{\tau}(\mathbf{q}, \boldsymbol{\lambda}) \end{pmatrix}}_{=\mathcal{S}(\mathbf{q},\dot{\mathbf{q}},\boldsymbol{\lambda},\ddot{\mathbf{q}})} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{27}$$

Now, $\frac{d}{dt}\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}})$ is linear in $\ddot{\mathbf{q}}$ (see 26), $\mathcal{Q}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ is linear in $\ddot{\mathbf{q}}$, $\boldsymbol{\tau}(\mathbf{q}, \boldsymbol{\lambda})$ is linear in $\boldsymbol{\lambda}$. Therefore $\mathcal{S}(\boldsymbol{\lambda}, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ is linear in $(\boldsymbol{\lambda}, \ddot{\mathbf{q}})$, *i.e.*,

$$\mathcal{S}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\lambda}, \ddot{\mathbf{q}}) = \mathbf{M}(\mathbf{q}, \dot{\mathbf{q}}) \begin{pmatrix} \boldsymbol{\lambda} \\ \ddot{\mathbf{q}} \end{pmatrix} - \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}). \tag{28}$$

where $\mathbf{M}(\mathbf{q}, \dot{\mathbf{q}})$ is called the *mass matrix*. Since it does not depend on only depends on $\dot{\mathbf{q}}$, we will write $\mathbf{M}(\mathbf{q})$ instead of $\mathbf{M}(\mathbf{q}, \dot{\mathbf{q}})$. An expression for $\mathbf{M}(\mathbf{q})$ and $\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})$ can be obtained from an expression for $\mathcal{S}$ by (see 28):

$$\begin{array}{rcl}
\mathbf{M}(\mathbf{q}) & = & \frac{\partial \mathcal{S}}{\partial(\boldsymbol{\lambda}, \ddot{\mathbf{q}})} \\
\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) & = & \mathbf{M}(\mathbf{q}) \begin{pmatrix} \boldsymbol{\lambda} \\ \ddot{\mathbf{q}} \end{pmatrix} - \mathcal{S}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\lambda}, \ddot{\mathbf{q}})
\end{array}$$

Thus, (28) rewrites into

$$\mathbf{M}(\mathbf{q}) \begin{pmatrix} \boldsymbol{\lambda} \\ \ddot{\mathbf{q}} \end{pmatrix} = \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}), \tag{29}$$

An expression for $\mathbf{M}(\mathbf{q})$ and $\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})$ are obtained by

```
A=Matrix([[1,0,-r*sin(ψ),-r*cos(ψ)*cos(θ),r*sin(ψ)*sin(θ)],
          [0,1, r*cos(ψ),-r*sin(ψ)*cos(θ),-r*cos(ψ)*sin(θ)]])
τ=λ1*A[0,:]+λ2*A[1,:]
a=A*dq
da=diff(a,t)
S=Matrix([da,*list(Q-τ)])
M=S.jacobian([λ1,λ2,ddq])
b=M*Matrix([λ1,λ2,ddq])-S
```

We get

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 & -r\sin\psi & -r\cos\theta\cos\psi & r\sin\theta\sin\psi \\ 0 & 0 & 0 & 1 & r\cos\psi & -r\sin\psi\cos\theta & -r\sin\theta\cos\psi \\ -1 & 0 & m & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & m & 0 & 0 & 0 \\ r\sin\psi & -r\cos\psi & 0 & 0 & \frac{mr^2}{2} & 0 & -\frac{mr^2\sin\theta}{2} \\ r\cos\theta\cos\psi & r\sin\psi\cos\theta & 0 & 0 & 0 & mr^2(\sin^2\theta+\frac{1}{4}) & 0 \\ -r\sin\theta\sin\psi & r\sin\theta\cos\psi & 0 & 0 & -\frac{mr^2}{2}\sin\theta & 0 & \frac{mr^2(\sin^2\theta+1)}{4} \end{pmatrix} \tag{30}$$

and

$$\mathbf{b} = \begin{pmatrix} r(-\dot{\theta}^2\sin\theta\cos\psi - 2\dot{\theta}\dot{\psi}\sin\psi\cos\theta + \dot{\varphi}\dot{\psi}\cos\psi - \dot{\psi}^2\sin\theta\cos\psi) \\ r(-\dot{\theta}^2\sin\theta\sin\psi + 2\dot{\theta}\dot{\psi}\cos\theta\cos\psi + \dot{\varphi}\dot{\psi}\sin\psi - \dot{\psi}^2\sin\theta\sin\psi) \\ 0 \\ 0 \\ mr^2\dot{\theta}\dot{\psi}\cos\theta/2 \\ \frac{mr}{8}(8g\sin\theta - 4r\dot{\theta}^2\sin(2\theta) - 4r\dot{\varphi}\dot{\psi}\cos\theta + r\dot{\psi}^2\sin(2\theta)) \\ \frac{mr^2}{2}(\dot{\varphi} - \dot{\psi}\sin\theta)\dot{\theta}\cos\theta \end{pmatrix} \tag{31}$$

We isolate $\boldsymbol{\lambda}$ and $\ddot{\mathbf{q}}$ by

```
λ1,λ2,ddc1,ddc2,ddφ,ddθ,ddψ=list((M.inv()*b))
```

and we get

$$\begin{array}{rcc} \lambda_1 & = & m\frac{6g\sin(2\theta)\cos\psi - 15r\dot{\theta}^2\sin\theta\cos\psi - 5r\dot{\theta}\dot{\psi}\sin\psi\cos\theta + 18r\dot{\varphi}\dot{\psi}\sin^2\theta\cos\psi - 3r\dot{\varphi}\dot{\psi}\cos\psi - 15r\dot{\psi}^2\sin^3\theta\cos\psi}{15} \\ \lambda_2 & = & m\frac{6g\sin(2\theta)\sin\psi - 15r\dot{\theta}^2\sin\theta\sin\psi + 5r\dot{\theta}\dot{\psi}\cos\theta\cos\psi + 18r\dot{\varphi}\dot{\psi}\sin^2\theta\sin\psi - 3r\dot{\varphi}\dot{\psi}\sin\psi - 15r\dot{\psi}^2\sin^3\theta\sin\psi}{15} \\ \ddot{c}_1 & = & \frac{2g\sin(2\theta)\cos\psi}{5} - r\dot{\theta}^2\sin\theta\cos\psi - \frac{r}{3}\dot{\theta}\dot{\psi}\sin\psi\cos\theta + \frac{6}{5}r\dot{\varphi}\dot{\psi}\sin^2\theta\cos\psi - \frac{r}{5}\dot{\varphi}\dot{\psi}\cos\psi - r\dot{\psi}^2\sin^3\theta\cos\psi \\ \ddot{c}_2 & = & \frac{2g\sin(2\theta)\sin\psi}{5} - r\dot{\theta}^2\sin\theta\sin\psi + \frac{r}{3}\dot{\theta}\dot{\psi}\cos\theta\cos\psi + \frac{6}{5}r\dot{\varphi}\dot{\psi}\sin^2\theta\sin\psi - \frac{r}{5}\dot{\varphi}\dot{\psi}\sin\psi - r\dot{\psi}^2\sin^3\theta\sin\psi \\ \ddot{\varphi} & = & 2\dot{\varphi}\dot{\theta}\tan\theta + \frac{5}{3}\dot{\theta}\dot{\psi}\cos\theta \\ \ddot{\theta} & = & \frac{4}{5r}g\sin\theta - \frac{6}{5}\dot{\varphi}\dot{\psi}\cos\theta + \frac{1}{2}\dot{\psi}^2\sin(2\theta) \\ \ddot{\psi} & = & \frac{2\dot{\varphi}\dot{\theta}}{\cos\theta} \end{array} \tag{32}$$

Finally, the `Python` function associated to the evolution equation of the rolling disk is

```
lambdify((c1,c2,φ,θ,ψ,dφ,dθ,dψ,m,g,r),
       (dc1-a[0],dc2-a[1],dφ,dθ,dψ,ddφ,ddθ,ddψ))
```

or equivalently, the state equation are

$$
\begin{aligned}
\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} &= r \begin{pmatrix} \sin\psi & \cos\psi\cos\theta & -\sin\psi\sin\theta \\ -\cos\psi & \sin\psi\cos\theta & \cos\psi\sin\theta \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} && \text{(see (17))} \\
\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} &= \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} && (33) \\
\begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} &= \begin{pmatrix} 2\dot{\varphi}\dot{\theta}\tan\theta + \frac{5}{3}\dot{\theta}\dot{\psi}\cos\theta \\ \frac{4}{5r}g\sin\theta - \frac{6}{5}\dot{\varphi}\dot{\psi}\cos\theta + \frac{1}{2}\dot{\psi}^2\sin(2\theta) \\ \frac{2\dot{\varphi}\dot{\theta}}{\cos\theta} \end{pmatrix} && \text{(see (32))}
\end{aligned}
$$

3 Illustrations

Let us simulate the rolling disk with the following initial state

$$(c_1, c_2, \varphi, \theta, \psi, \dot{\varphi}, \dot{\theta}, \dot{\psi}) = (2, 0, 0, 0.1, 0, 2.5, 0, 0) \tag{34}$$

for $t \in [0, 10]$. The simulation generates Figure 3 where the blue disk corresponds to the initial state. The behavior is consistent with the intuition we could have for a rolling disk. The precession effect is visible by the fact that the trajectory is not perfectly circular.

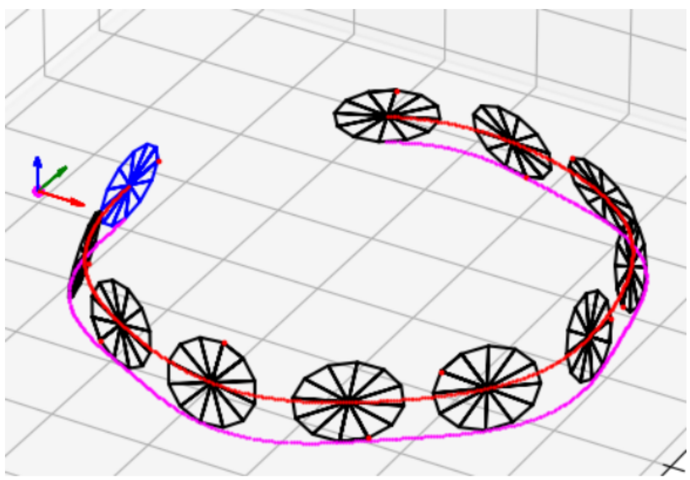

Fig. 3: Disk rolling on a plane with precession

To have a better understanding of the state model we have derived for the rolling disk, let us consider different cases.

Case 1: The disk has a vertical motion.

It means that $\theta = \dot{\theta} = \ddot{\theta} = 0$. From (33), we have

$$\begin{pmatrix} \ddot{\varphi} \\ 0 \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{\varphi}\dot{\psi} \\ 0 \end{pmatrix}. \tag{35}$$

We have either

- $\dot{\varphi} = 0$: the contact point is static and the disk spins around its vertical axis
- $\dot{\psi} = 0$: the disk moves straight forward

which is what we could have expected.

Case 2: The disk is horizontal.

In this case, $\theta = \pm\frac{\pi}{2}$. The state equations are not valid since we have a singularity. Indeed for this specific situation, the heading and the spin is even not clearly defined.

Case 3. The disk has a circular trajectory.

It means that $\ddot{\psi} = 0$ with $\theta \neq 0$ and $\dot{\varphi} \neq 0$. From (33), we get

$$\begin{pmatrix} \ddot{\varphi} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\dot{\varphi}\dot{\theta}\tan\theta + \frac{5}{3}\dot{\theta}\dot{\psi}\cos\theta \\ \frac{4}{5r}g\sin\theta - \frac{6}{5}\dot{\varphi}\dot{\psi}\cos\theta + \frac{1}{2}\dot{\psi}^2\sin(2\theta) \\ \dot{\theta} \end{pmatrix} \tag{36}$$

i.e.,

$$\begin{pmatrix} \ddot{\varphi} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{5r}g\sin\theta - \frac{6}{5}\dot{\varphi}\dot{\psi}\cos\theta + \frac{1}{2}\dot{\psi}^2\sin(2\theta) \\ \dot{\theta} \end{pmatrix}. \tag{37}$$

To illustrate this situation, let us simulate the rolling disk with the following initial state

$$\begin{cases} (c_1, c_2, \varphi, \theta, \psi, \dot{\theta}, \dot{\psi}) = (2, 0, 0, 0.5, 0, 0, 1) \\ \dot{\varphi} = \frac{2}{3r\dot{\psi}}g\tan\theta + \frac{5}{6}\dot{\psi}\sin\theta \end{cases} \tag{38}$$

to satisfy the circular condition. For a time $t \in [0, 6]$, the simulation generates Figure 4 which corresponds indeed to a circular motion.

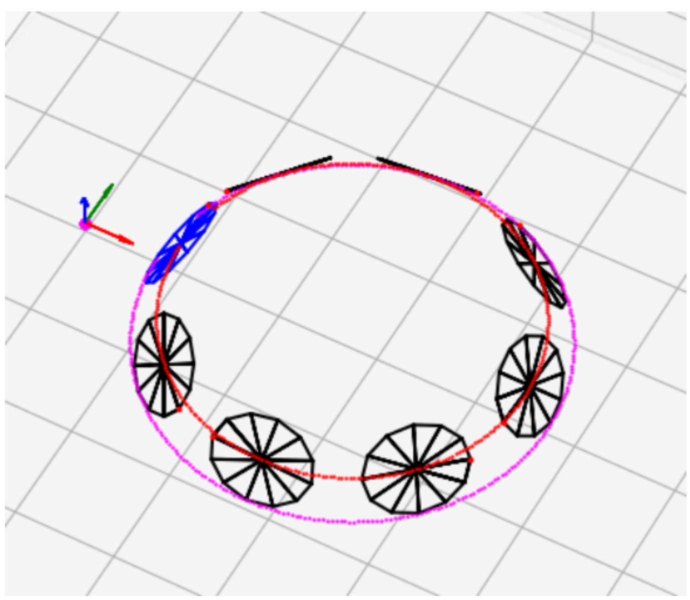

Fig. 4: Disk rolling on a plane and performing a circle

The `Python` code associated to all examples can be found here:

https://www.ensta-bretagne.fr/jaulin/rollingdisk.html

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