Assessment of the Accuracy of Interval Observers by a Zonotopic Approach

Andreas Rauh[1], Luc Jaulin[2] and Marit Lahme[1]

[1] Carl von Ossietzky Universität Oldenburg
Distributed Control in Interconnected Systems
D-26111 Oldenburg, Germany
{Andreas.Rauh, Marit.Lahme}@uni-oldenburg.de

[2] Lab-STICC, ROBEX Team, ENSTA
F-29806 Brest Cedex 09, France
luc.jaulin@ensta.fr

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Approaches for Bounded Error State Estimation

State estimation for dynamic systems with bounded uncertainty in initial conditions, parameters, and external disturbances can be classified generally into predictor–corrector approaches [7] and interval observers [6].

In the case of predictor–corrector approaches, the open-loop dynamics are evaluated with the help of set-based methods (employing, for example, intervals, ellipsoids, or zonotopes) until the point in time at which measured data become available. Then, the predicted state enclosures are contracted with the help of measurement information (being subject to bounded uncertainty). This contraction procedure can be replaced by a Luenberger-like observer approach as shown in [1].

In all approaches listed above, it is necessary to tune the estimators, especially by choosing an observer gain matrix so that stability of the error dynamics can be ensured and that the computed bounds tightly enclose the sets of reachable states that are compatible with the system dynamics, the uncertainty model, and the measured data.

Pessimism of Interval Observer Synthesis

In the frame of the design of interval observers, further restrictions need to be accounted for. These are essentially order relations between the lower and upper bounding trajectories. These relations can be ensured if the observer dynamics represent a cooperative system model, i.e., a model that shows monotonicity of each component of the state trajectory with respect to the initial conditions [9]. A sufficient condition for cooperativity is that, in the case of a linear observer (here stated for a continuous-time model)

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A} \cdot \hat{\mathbf{x}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{H} \cdot (\mathbf{y}_{\mathrm{m}}(t) - \mathbf{C} \cdot \hat{\mathbf{x}}(t)) \tag{1}$$

with the measurement $\mathbf{y}_{\mathrm{m}}(t)$, the matrix $\mathbf{A} - \mathbf{HC}$ is Metzler [2]. This means that all off-diagonal elements of this matrix need to be non-negative. Moreover, the diameters of the computed interval enclosures only remain bounded if the matrix is additionally Hurwitz.

Especially the prerequisite of finding a parameterization that leads to a Metzler matrix is extremely challenging if the matrices $\mathbf{A}$ and $\mathbf{C}$ are subject to bounded uncertainty or if they are derived from nonlinear systems that are reformulated into the quasi-linear form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\left(\mathbf{x}(t)\right) \cdot \mathbf{x}(t) + \mathbf{B}\left(\mathbf{x}(t)\right) \cdot \mathbf{u}(t) + \mathbf{w}(t) \tag{2}$$

$$\mathbf{y}(t) = \mathbf{C}\left(\mathbf{x}(t)\right) \cdot \mathbf{x}(t) + \mathbf{v}(t) \ . \tag{3}$$

Although coordinate transformations, rendering a non-cooperative realization cooperative [3] or an extension of the observer structure by further degrees of freedom [10] mitigate this issue to some extent, there still exist system models for which the tuning of the observer gain is demanding even though cooperativity can be ensured by a design based on linear matrix inequalities in combination with a polytopic uncertainty model [4]. This polytopic model overapproximates the possible ranges of each of the matrices included in (2) and (3).

Extracting lower and upper bounding trajectories for the observer (1) with a polytopic model for each of the matrices and an element-wise minimization (resp., maximization) of each of the equations introduces a further degree of pessimism. This procedure corresponds to the application of Müller's theorem [5]. However, the resulting bounding models may reflect combinations of values for the state variables that cannot be reached in reality.

In our contribution, we aim at quantifying this pessimism by a direct zonotopic evaluation of the observer dynamics without firstly extracting the bounding trajectories. In addition, we make a comparison between parameterizations for the observer gains $\mathbf{H}$ that are chosen so that cooperativity is enforced in addition to the optimization of a (typically H_∞) performance criterion and for a choice of the gain in which the cooperativity constraint is removed. This goes along with a thorough analysis of the influence of order reduction strategies for zonotopes which become necessary to limit the computational effort and memory requirements.

Use of the Zonotopic Observer as a Contractor

As the open-loop simulation, the interval observer, and the zonotopic observer all provide guaranteed enclosures for the state, the different solution approaches can mutually serve as contractors for their results. We demonstrate how the different state estimation approaches can mutually benefit from the properties of the individual implementations. This is realized in the form of a contractor scheme [8].

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