Using interval arithmetic to prove that a set is path-connected

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Abstract

In this paper, we give a numerical algorithm able to prove whether a set $\mathbb{S}$ described by nonlinear inequalities is path-connected or not. To our knowledge, no other algorithm (numerical or symbolic) is able to deal with this type of problem. The proposed approach uses interval arithmetic to build a graph which has exactly the same number of connected components than $\mathbb{S}$. Examples illustrate the principle of the approach.

Key words: interval arithmetic, graph theory, connected set, topology, set computation, automatic proof.

Introduction

Topology is the mathematical study of properties of objects which are preserved through deformations, twistings, and stretchings (mathematically, through functions called homeomorphism). Because spaces by themselves are very complicated, they are unmanageable without looking at particular aspects. One of the topological aspects of a set is its number of path-connected components.

Proving that a set is connected is an important problem already considered for robotics (e.g. for path planning) and identifiability applications ([8],[12]). In [2], Stander guarantees the topology of an implicit surface defined by only one inequality by combining Morse theory [9] and interval analysis to find critical points. Nevertheless, this approach is limited since it cannot be applied

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to set defined by more than one inequality, or in higher dimension.
In Section 1, some notions of topology are recalled. Section 2 deals with lattices and intervals. Most of the examples presented in this section are useful to understand the proposed reliable method. The third section shows how a specific problem of topology (proving that a set is star-shaped) can be solved by resolving a constraint satisfaction problem [6]. The sufficient condition given in this section will be the key of the discretization presented in section 4. The idea is to build a finite set preserving some topological properties of a given set. In the last part, the method is given and illustrated by examples.

1 Topology recall

Definition A topological set $\mathbb{S}$ is *path-connected* [5] if for every two points $x, y \in \mathbb{S}$, there is a continuous function f from $[0,1]$ to $\mathbb{S}$ such that $f(0) = x$ and $f(1) = y$. Path-connected sets are also called 0-connected[1]. The set represented on the left of Figure 1 is path connected whereas the right one is not (it has 4 connected components).

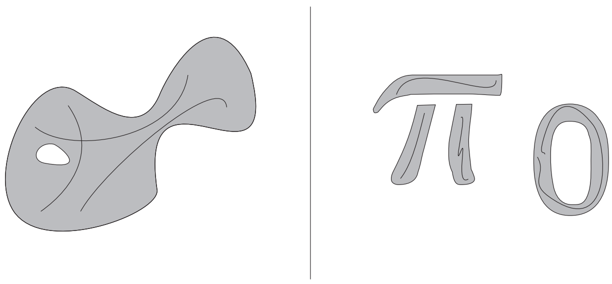

Fig. 1. Example of a set which is path-connected and a set which is not.

Definition The point v^* is a *star* for a subset X of an Euclidean set if X contains all the line segments connecting any of its points and v^*.

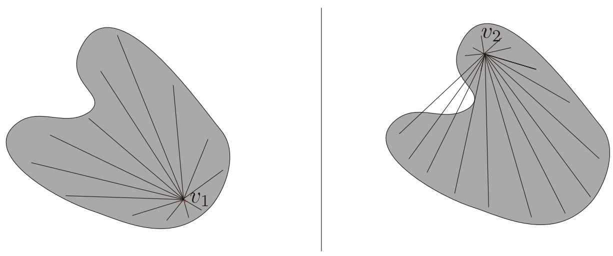

Fig. 2. v_1 is a star for this subset of $\mathbb{R}^2$ and v_2 is not.

[1] In algebraic topology, $\pi_0(\mathbb{S})$ is the classical notation for the number of connected components of $\mathbb{S}$.

Definition A subset X of an Euclidean set is *star-shaped* or *v^*-star-shaped* if there exists $v^* \in X$ such that v^* is a star for X.

Proposition 1.1 *A star-shaped set is a path-connected set.*

Proposition 1.2 *Let X and Y two v^*-star-shaped set, then $X \cap Y$ and $X \cup Y$ are also v^*-star-shaped.*

2 Intervals

This section recalls some definitions and properties related to lattices. It introduces the notion of graph interval which be used in the last section.

Definition A *lattice* $(X, \leq)$ is a partially ordered set satisfying:
$\forall x, y \in X, x \vee y \in X$ and $x \wedge y \in X$, where $x \wedge y$ is the greatest lower bound and is called the *meet*, $x \vee y$ is the least upper bound and is called the *join*. See [1] and [3] for more details.

Example Let E be a set. A *simple* [2] graph on E is a symmetric relation on E, i.e. a subset of $E \times E$. Let G be the set of all simple graphs on E, G is a lattice with respect to the partial order: $g_1, g_2 \in G$. (See [4])
$g_1 \leq g_2 \Leftrightarrow g_1 \subset g_2$.

Definition An *interval* I of a lattice ξ is a subset of ξ which satisfies
$I = \{x \in \xi \text{ s.t. } \wedge I \leq x \leq \vee I\}$. The interval I is generally represented by its bounds, using the following notation [3]: $I = [\wedge I, \vee I]$.
Both $\emptyset$ and ξ are intervals of ξ. The set of all intervals of ξ will be denoted $\mathcal{I}(\xi)$. Note that $\mathcal{I}(\xi)$ is a subset of $\mathcal{P}(\xi)$.

Example Let consider the Figure 3, $[g_1, g_2]$ is an interval of $(\mathcal{G}, \leq)$, this interval contains 4 elements:

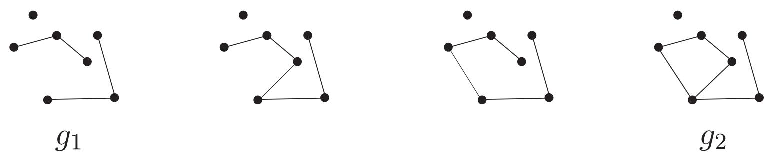

Fig. 3. Example of an interval in $(\mathcal{G}, \leq)$

[2] Non simple graphs can have different edges connecting the same pair of vertices.
[3] If $\wedge I = \vee I$, the interval I is said punctual.

3 Proving that v^* is a star

This section shows that, when $\mathbb{S}$ is defined by an inequality ($\mathbb{S} \subset \mathbb{R}^n$), proving that $\mathbb{S}$ is v^*-star-shaped often amounts to prove the inconsistency of inequalities. It is really attractive because the inconsistency of inequalities can be proven thanks to an interval method (see [10], [6]). In this section, Df denotes the gradient of a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$.

Proposition 3.1 *Let us define* $\mathbb{S} = \{x \in D \subset \mathbb{R}^n | f(x) \leq 0\}$ *where* f *is a* C^1 *function from* D *to* $\mathbb{R}$, *and* D *a convex set. Let* v^* *be in* $\mathbb{S}$. *If*

$$f(x) = 0,\ Df(x) \cdot (x - v^*) \leq 0,\ x \in D \tag{1}$$

is inconsistent then v^* *is star for* $\mathbb{S}$.

Proof The proof is by reduction to a contradiction. Suppose that v^* is not a star for $\mathbb{S}$, then there exists $x_0 \in \mathbb{S}$ such that the segment $[v^*, x_0] \not\subset \mathbb{S}$. Thus, since D is convex, there exists $x_1 \in [v^*, x_0]$ such that $f(x_1) > 0$. Let g denote the function: $g : [0, 1] \to \mathbb{R},\ \ t \mapsto g(t) = f((1 - t)v^* + tx_0)$. Since the numeric function f is a $\mathcal{C}^1$ function, g is differentiable. Moreover, it satisfies the following inequalities: $g(0) \leq 0,\ g(1) \leq 0,\ g(t_1) > 0$ where t_1 is such that $x_1 = (1 - t_1)v^* + t_1 x_0$.
Since g is continuous, the intermediate value theorem guarantees that there exists $t_2 \in [t_1, 1]$ such that $g(t_2) = 0$. In the case where there is more than one real in $[t_1, 1]$ which satisfies $g(t) = 0$, let t_2 be the infinium of them. Thus we have: $g(t_2) = 0$ and $\forall t \in]t_1, t_2[,\ \ g(t) > 0$. Since g is differentiable on the open interval]0,1[,

$$g'(t_2) = \lim_{h\to 0} \frac{g(t_2 + h) - g(t_2)}{h} = \lim_{h\to 0^-} \frac{g(t_2 + h)}{h}.$$

There exists $\epsilon > 0$ such that $\forall h < 0, |h| < \epsilon \Rightarrow t_2 + h \in [t_1, t_2]$ (take $\epsilon = (t_1 - t_2)/2$). So

$$\forall h < 0, |h| < \epsilon, \frac{g(t_2 + h)}{h} < 0.$$

We deduce that $g'(t_2) \leq 0$. In conclusion, taking $x_2 = (1 - t_2)v^* + t_2 x_0$, $x_2 \in D$ is such that: $f(x_2) = 0$ and $Df(x_2) \cdot (x_2 - v^*) \leq 0$. $\square$

An geometric interpretation, of this last proposition is that a set is star-shaped if all light rays coming from v^* cross the boundary at most once (from inside to outside).

Example Consider the problem of proving that $v_1 = (0, 0.7)$ is a star for the subset $\mathbb{S}$ of $\mathbb{R}^2$ defined by $f(x_1, x_2) \leq 0$ where f is the C^1 function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by: $f(x_1, x_2) = -e^{-(2x_1)^2} - e^{-(2x_1-2.8)^2} + 0.1 + x_2^2$.

Using the proposition 3.1, v_1 is a star for $\mathbb{S}$ if

$$\begin{cases} \partial_1 f(x_1, x_2).(x_1 - 0) + \partial_2 f(x_1, x_2).(x_2 - 0.7) \leq 0 \\ f(x_1, x_2) = 0 \end{cases} \tag{2}$$

is inconsistent. The gradient $Df(x)$ and light rays $x - v_1$ are represented on the boundary of $\mathbb{S}$ (where $f(x) = 0$) on Figure 4.

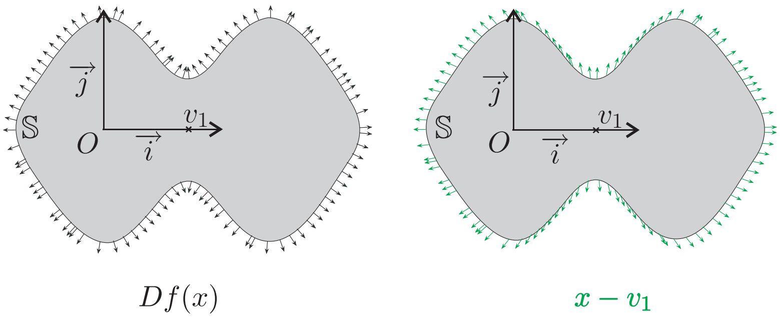

Fig. 4. Fields unit vector which represents $Df(x)$ and $x - v_1$ on the boundary.

Figure 5 illustrates that for all x satisfying $f(x) = 0$, we have $Df(x)\cdot(x - v_1) > 0$, i.e. the angle between two vectors is an acute angle, i.e. all the light rays cross the boundary from inside to outside.

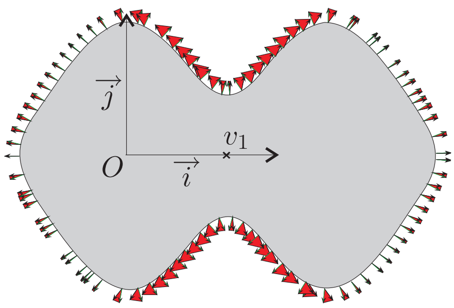

Fig. 5. All the light rays cross the boundary at most once (from inside to outside).

In the case showing in Figure 6, v_2 is not a star for $\mathbb{S}$ and there exists $x \in \mathbb{R}^2$ such that $f(x) = 0$ and $Df(x) \cdot (x - v_2) \leq 0$.

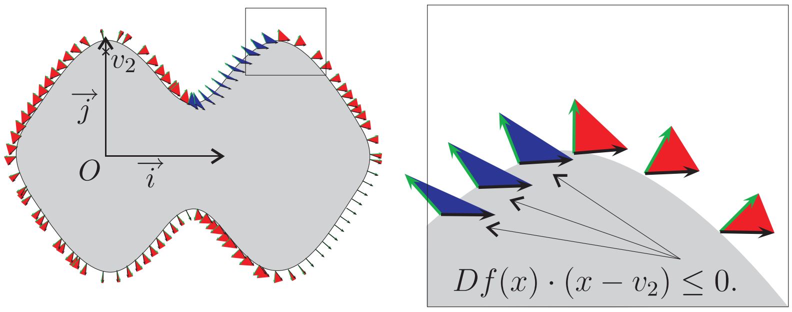

Fig. 6. v_2 is not a star.

4 Discretization

Since star-shaped sets are path-connected, the proposition 3.1 is also a sufficient condition to prove that a set is path-connected. But, most of the path-connected sets are not star-shaped as illustrated by Figure 7, i.e. it is not possible to find a point v^* which lights the set.

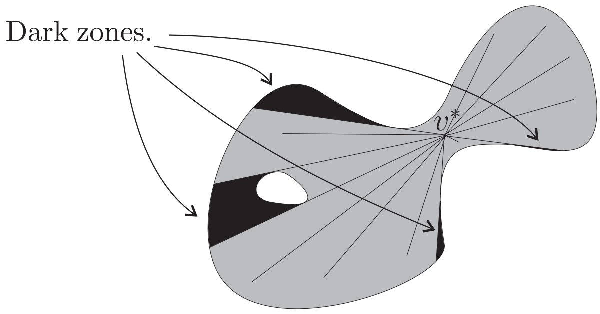

Fig. 7. Example of a path-connected set which is not star-shaped.

The idea of our approach, for proving that $\mathbb{S}$ is path-connected, is to divide it with a paving [6] $\mathcal{P}$ such that, on each part p, $\mathbb{S} \cap p$ is star-shaped (see Figure 8).

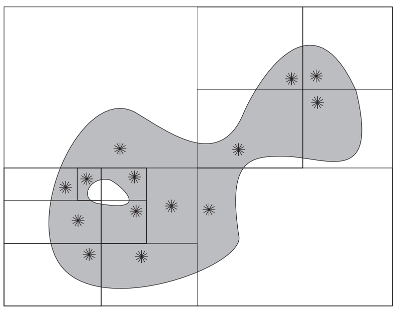

Fig. 8. Example of paving $\mathcal{P}$ satisfying $\forall p \in \mathcal{P}, \mathbb{S} \cap p$ is star-shaped.

In order to glue the pieces together, let us define the notion of *star-spangled* graph.

Definition A *star-spangled* graph of a set $\mathbb{S}$, noted by $\mathcal{G}_{\mathbb{S}}$, is a relation $\mathcal{R}$ on a paving $\mathcal{P}$ where :

- $\mathcal{P}$ is a paving, i.e. a finite collection of non overlapping n-boxes (Cartesian product of n intervals), $\mathcal{P} = (p_i)_{i \in I}$. Moreover, for all p of $\mathcal{P}$, $\mathbb{S} \cap p$ is star-shaped.
- $\mathcal{R}$ is the reflexive and symmetric relation on $\mathcal{P}$ defined by $p \,\mathcal{R}\, q \Leftrightarrow \mathbb{S} \cap p \cap q \neq \emptyset$.[4]
- $\mathbb{S} \subset \bigcup_{i \in I} p_i$

For instance, a star-spangled graph of $\mathbb{S}$ is given on Figure 9.

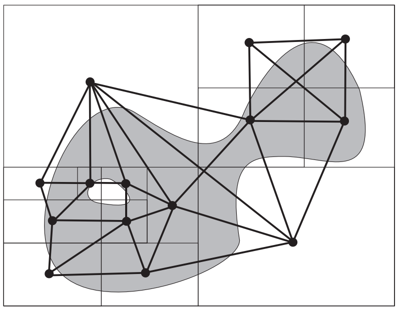

Fig. 9. A star-spangled graph $\mathcal{G}_{\mathbb{S}}$.

Definition The *support* of a star-spangled graph $\mathcal{G}_{\mathbb{S}}$ is the subset P of $\mathbb{R}^n$ defined by $P = \cup_{i \in I} p_i$.

Proposition 4.1 *Let $\mathcal{G}_{\mathbb{S}}$ be a star-spangled graph of a set $\mathbb{S}$.*
$\mathbb{S}$ is path-connected $\Leftrightarrow$ $\mathcal{G}_{\mathbb{S}}$ is connected .

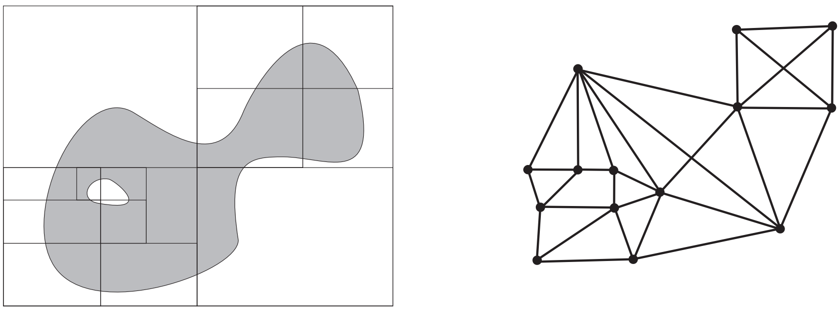

Fig. 10. If the graph $\mathcal{G}_{\mathbb{S}}$ is connected then $\mathbb{S}$ is path-connected.

[4] This is pairs of adjacent boxes in the paving whose common boundary intersects $\mathbb{S}$.

Proof If $\mathcal{G}_{\mathbb{S}}$ is connected, then there exists a path from any node to any other node in the graph. Let n be the number of nodes, and $\mathcal{N} = (\alpha_i)_{i\in\{1,\dots,n\}}$ be the nodes. Since $\mathcal{G}_{\mathbb{S}}$ is connected, for all i in $\{1,\dots,n-1\}$, there exists a path connecting α_i to α_{i+1}, i.e. there exists a finite sequence $\{\alpha_{i_1}, \alpha_{i_2}, .., \alpha_{i_k}\} \in \mathcal{N}^k$ such that $(\alpha_{i_1}, \alpha_{i_2}), (\alpha_{i_2}, \alpha_{i_3}), \dots, (\alpha_{i_{k-1}}, \alpha_{i_k})$ are edges of $\mathcal{G}_{\mathbb{S}}$ (with $\alpha_{i_1} = \alpha_i$, and $\alpha_{i_k} = \alpha_{i+1}$). Let $p(\alpha_i, \alpha_{i+1})$ denote this path.
Let $path_1$ and $path_2$ be two paths of $\mathcal{G}_{\mathbb{S}}$.
If one of the endpoints of $path_1$ is one of the endpoints of $path_2$, then it is possible to create a new path from $path_1$ and $path_2$, denoted by $path_1 + path_2$, which is the concatenation of $path_1$ and $path_2$.
Let p_{all} be the path defined by this associative operation:

$$p_{all} = p(\alpha_1, \alpha_2) + p(\alpha_2, \alpha_3) + \dots + p(\alpha_{n-1}, \alpha_n).$$

So p_{all} is a path of $\mathcal{G}_{\mathbb{S}}$ which visits each node at least once. Let $(\beta_i)_{i\in\{1,\dots,m\}}$ denote the sequence of nodes visited by p_{all} with $\beta_1 = \alpha_1$ and $\beta_m = \alpha_n$.
Thus the sequence of boxes $(p_i)_{i\in\{1,\dots,m\}}$, where p_i is the box associated to the node β_i, satisfies:

$$\begin{cases} \forall i \in \{1,\dots,m\}, & p_i \cap \mathbb{S} \text{ is path-connected } (p_i \cap \mathbb{S} \text{ is star-shaped}) \\ \forall i \in \{2,\dots,m\}, & \mathbb{S} \cap p_{i-1} \cap p_i \neq \emptyset. \end{cases}$$

Fact that for every denumerable family $(A_i)_{i\in I}$ of path-connected sets such that [5]: $\forall i \in I\backslash\{0\}, A_{i-1} \cap A_i \neq \emptyset$ the set $\bigcup\limits_{i\in I} A_i$ is path-connected, we can say that $\bigcup\limits_{i\in I}(\mathbb{S} \cap p_i) = \mathbb{S} \cap \bigcup\limits_{i\in I} p_i = \mathbb{S}$ is path-connected. $\square$

Corollary 4.2 *Let $\mathcal{G}_{\mathbb{S}}$ be a star-spangled graph of a set $\mathbb{S}$.
$\mathcal{G}_{\mathbb{S}}$ has the same number of connected components than $\mathbb{S}$. i.e. $\pi_0(\mathbb{S}) = \pi_0(\mathcal{G}_{\mathbb{S}})$.*

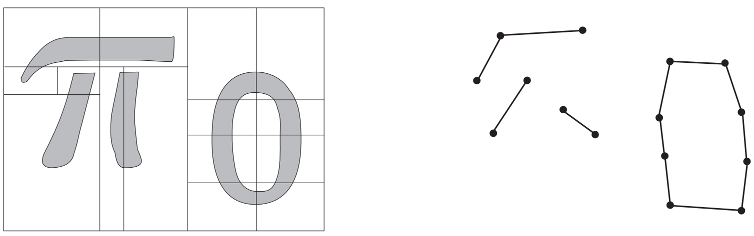

Fig. 11. The number of connected components of $\mathcal{G}_{\mathbb{S}}$ is the same that $\mathbb{S}$. In this example, this number is 4.

Proof The main idea is to break apart the star-spangled $\mathcal{G}_{\mathbb{S}}$ od $\mathbb{S}$ into star-spangled $(\mathcal{G}_i)_{1\leq i\leq n}$ following the graph connected components. (n is the number of the graph connected components.)

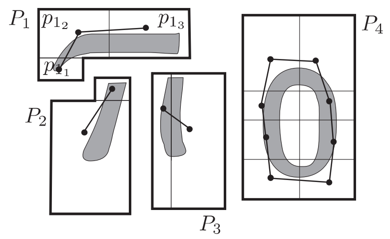

Fig. 12. Break apart the star-spangled $\mathcal{G}_{\mathbb{S}}$ following the graph connected components.

Let P_i be the support of $\mathcal{G}_i$, and $\mathcal{P}_i = \{p_{i_j}\}_{1 \leq j \leq n_j}$. For each star-spangled graph $\mathcal{G}_i$, we can apply Proposition 4.1, and affirm that $\mathbb{S} \cap P_i$ is connected. So the set $\mathbb{S}$ has n connected components at most.
The end of the proof is by reduction to a contradiction. Suppose that $\mathbb{S}$ has less than n connected components. i.e. there exists α, β in $1, .., n$ such that: $\alpha \neq \beta$ and $P_\alpha \cap P_\beta \cap \mathbb{S} \neq \emptyset$ i.e. there exists α_0 in $1, .., n_\alpha$ and β_0 in $1, .., n_\beta$ such that: $p_{\alpha_0} \cap p_{\beta_0} \cap \mathbb{S} \neq \emptyset$, i.e. $p_{\alpha_0} \mathcal{R} p_{\beta_0}$.
$p_{\alpha_0} \in \mathcal{P}_\alpha$, $p_{\alpha_0} \in \mathcal{P}_\beta$, $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are two connected components of $\mathcal{G}_{\mathbb{S}}$, so $p_{\alpha_0} \not\mathcal{R} p_{\beta_0}$. $\square$

Tarjan [7] analyzes a simple algorithm that finds the connected components of a simple undirected graph with n vertices in $O(n)$ expected time. In the next section, we present an algorithm which tries to create a star-spangled graph.

5 Algorithm for proving that a set is path-connected, or guaranteing its number of path-connected components and examples

This section presents a new algorithm called: `CIA` (path-**C**onnected using **I**nterval **A**nalysis). This algorithm tries to generate a star-spangled graph $\mathcal{G}_{\mathbb{S}}$ (Proposition 4.2). The main idea is to test a suggested paving $\mathcal{P}$ and, in the case where it does not satisfy the condition : $\forall p \in \mathcal{P},\ p \cap \mathbb{S}$ is star-shaped, to improve this one by bisecting any boxes responsible for this failure.
For a paving $\mathcal{P}$, we have to check for a box p of $\mathcal{P}$ whether $\mathbb{S} \cap p$ is star-shaped or not, and to build its associated graph with the relation $\mathcal{R}$ mentioned before. This two tasks will be done by Alg. 2 and Alg. 3 respectively.

In `CIA` Alg. 1, $\mathcal{P}_*$, $\mathcal{P}_{out}$, $\mathcal{P}_\Delta$ are three pavings such that $\mathcal{P}_* \cup \mathcal{P}_{out} \cup \mathcal{P}_\Delta = \mathcal{P}$, with $\mathcal{P}$ is a paving whose support is a (possibly very large) initial box X_0 (containing $\mathbb{S}$):

- the *star-spangled* paving $\mathcal{P}_*$ contains boxes p such that $\mathbb{S} \cap p$ is star-shaped.
- the *outer* paving $\mathcal{P}_{out}$ contains boxes p such that $\mathbb{S} \cap p$ is empty.
- the *uncertain* paving $\mathcal{P}_\Delta$, nothing is known about its boxes.

Alg. 1 `CIA` - path-Connected using Interval Analysis

Require: $\mathbb{S}$ a subset of $\mathbb{R}^n$, X_0 a box of $\mathbb{R}^n$
1: Initialization : $\mathcal{P}_* := \emptyset$, $\mathcal{P}_\Delta := \{X_0\}$, $\mathcal{P}_{out} := \emptyset$
2: **while** $\mathcal{P}_\Delta \neq \emptyset$ **do**
3: Pull the last element of $\mathcal{P}_\Delta$ into the box p
4: **if** "$\mathbb{S} \cap p$ is proven empty" **then**
5: Push $\{p\}$ into $\mathcal{P}_{out}$, Goto Step 2.
6: **end if**
7: **if** "$\mathbb{S} \cap p$ is proven star-shaped" and `Build_Graph_Interval`$(\mathbb{S}, \mathcal{P}_* \cup \{p\})$ is punctual **then**
8: Push $\{p\}$ into $\mathcal{P}_*$, Goto Step 2.
9: **end if**
10: $Bisect(p)$ and Push the two resulting boxes into $\mathcal{P}_\Delta$
11: **end while**
12: $n \leftarrow$ Number of connected components of $\underline{g}$
13: return "$\mathbb{S}$ has n path-connected components"

To bisect p into two boxes at step 10, we cut it at its centre, perpendicularly to one of its edges of maximum length. To prove "$\mathbb{S} \cap p$ is star-shaped", it suffices to check if one of the vertices v_p of p is a star for $\mathbb{S} \cap p$. The following algorithm called `Star-shaped` shows how this verification can be implemented.

Alg. 2 `Star-shaped`(p, f)

Require: f a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}$
Require: p a box of $\mathbb{R}^n$
if $f(p)$ can be proven to be inside $\mathbb{R}^{+*}$ **then**
  Return "$\mathbb{S} \cap p$ is empty thus it is not star-shaped"
else
  for all vertex v_p of p **do**
    if $\{x \in p, f(x) = 0, Df(x) \cdot (x - v_p) \leq 0\}$ is can be proven inconsistent **then**
      Return "$\mathbb{S} \cap p$ is star-shaped "
    end if
  end for
  Return "Failure"
end if

Remark If $\mathbb{S} = \bigcap_{i \in I} \mathbb{S}_i$, where $\mathbb{S}_i = f_i^{-1}(\mathbb{R}^-)$ and $(f_i)_{i \in I}$ is a finite collection of C_1 functions, a proof that $\mathbb{S} \cap p$ is v^*-star-shaped can be given by proving that for each $i \in I$, $\mathbb{S}_i \cap p$ is v^*-star-shaped. (See Proposition : 1.2). The same remark holds if $\mathbb{S} = \bigcup_{i \in I} \mathbb{S}_i$.
To build the associated graph of a paving $\mathcal{P}$, we have to check whether for each pair (p_i, p_j), of the paving $\mathcal{P}$, $\mathbb{S} \cap p_i \cap p_j$ is empty or not. When we do not know

whether $\mathbb{S} \cap p_i \cap p_j$ is empty or not, we create a graph interval which contains the true graph. The following algorithm called `Build_Graph_Interval` shows how the graph construction can be implemented:

Alg. 3 `Build_Graph_Interval`$(\mathbb{S}, \mathcal{P})$

Require: $\mathbb{S}$ a subset of $\mathbb{R}^n$, $\mathcal{P}$ a paving
Ensure: A graph interval $[\underline{g}, \overline{g}]$ associated to the paving $\mathcal{P}$.
1: Initialization : $\overline{g} := \emptyset$, $\underline{g} := \emptyset$
2: **for all** (p_i,p_j) in $\mathcal{P} \times \mathcal{P}$ **do**
3:    **if** $\mathbb{S} \cap p_i \cap p_j = \emptyset$ **then next**
4:    **if** for one of the vertices v of $p_i \cap p_j$, $v \in \mathbb{S}$ **then**
5:       add (p_i, p_j) to $\underline{g}$ and to $\overline{g}$
6:    **else**
7:       add (p_i, p_j) to $\overline{g}$ // *i.e.*(p_i,p_j) *is an undetermined edge of* $[\underline{g}, \overline{g}]$
8:    **end if**
9: **end for**

When $\mathbb{S}$ is defined by inequalities, condition at step 4 is checked using interval arithmetic. With this tool, we can also prove that $\mathbb{S} \cap p_i \cap p_j = \emptyset$ (step 3).

Example Figure 13 shows the paving generated for

$$\mathbb{S} = \left\{ (x,y) \in \mathbb{R}^2, \left\{ \begin{array}{ll} f_1(x,y) = x^2 + 4y^2 - 16 & \leq 0 \\ f_2(x,y) = 2\sin(x) - \cos(y) + y^2 - 1.5 & \leq 0 \\ f_3(x,y) = -(x+2.5)^2 - 4(y-0.4)^2 + 0.3 & \leq 0 \end{array} \right. \right\} \quad (3)$$

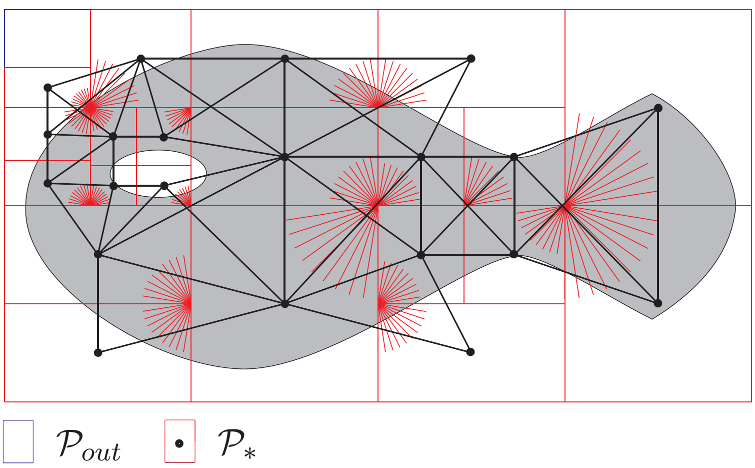

Fig. 13. Example of star-spangled graph generated by `CIA`.

Example Figure 14 shows the paving generated for $\mathbb{S} = \bigcup_{i=1}^{i=4} \mathbb{S}_i$ where

$$\begin{aligned}
&D = [-5, 5] \times [-4.6, 4.6] \\
&\mathbb{S}_1 = \{(x, y) \in D, \quad f_1(x, y) = -x^2 - y^2 + 9 \leq 0\} \\
&\mathbb{S}_2 = \{(x, y) \in D, \quad f_2(x, y) = (x - 1)^2 + (y - 1.5)^2 - 0.5 \leq 0\} \\
&\mathbb{S}_3 = \{(x, y) \in D, \quad f_3(x, y) = (x + 1)^2 + (y - 1.5)^2 - 0.5 \leq 0\} \\
&\mathbb{S}_4 = \{(x, y) \in D, \quad f_4(x, y) = \cos^2(x + 1.5) + 4(y + 2)^2 - 0.5 \leq 0\}
\end{aligned} \tag{4}$$

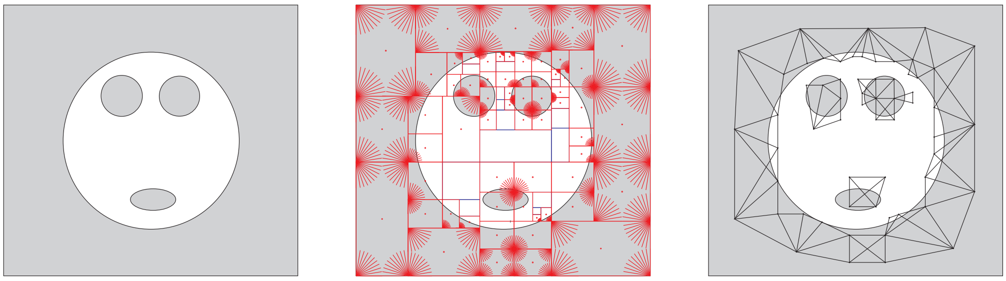

Fig. 14. Star-spangled graph generated by `CIA`. $\mathbb{S}$ and $\mathcal{G}_{\mathbb{S}}$ has 4 connected components.

When the solver proves that a vertex of a box p is a star for $\mathbb{S} \cap p$, it uses the same representation as the one presented at Figure 2 to display it. (This solver can be downloaded from http://www.istia.univ-angers.fr/~delanoue/)

6 Conclusion

In this paper, an approach has been proposed to prove that a set $\mathbb{S}$ defined by inequalities is or not path-connected. Combining tools from interval arithmetic and graph theory, an algorithm has been presented to create a graph which has some topological properties in common with $\mathbb{S}$. For instance, the number of path-connected components of $\mathbb{S}$ is the same as its associated graph. One of the main limitation of the proposed approach is that the computing time increases exponentially with respect to the dimension of $\mathbb{S}$.
At the moment, we do not have a sufficient condition about f to ensure that our algorithm `CIA` will terminate.
The condition : $f^{-1}(\{0\}) \cap (x \mapsto Df(x))^{-1}(\{0\}) \neq \emptyset$ seems to be a good one but a more thorough study must be made. An extension of this work is the problem of the computation of a triangulation homeomorphic to $\mathbb{S}$. Roughly speaking, a triangulation is a nonoverlapping union of simplexes. This would make possible to get more topological properties of the set, for example its homology groups. We hope that this problem could be solved by combining the

tools presented in this paper with algorithms arising out from Computational Topology [11].

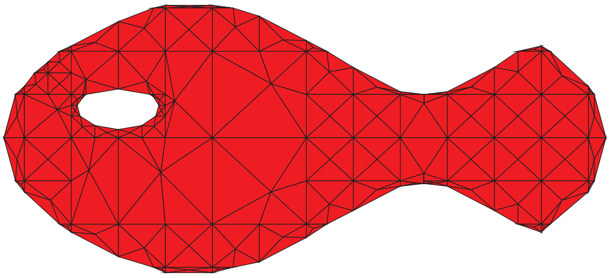

Fig. 15. Example of triangulation which is homeomorphic to $\mathbb{S}$ defined by (3).

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