Kernel characterization of an interval function

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Abstract. This paper proposes a set-membership approach to characterize the kernel of an interval-valued function. In the context of a bounded-error estimation, this formulation makes it possible to embed all uncertainties of the problem inside the interval function and thus to avoid bisections with respect to all these uncertainties. To illustrate the principle of the approach, two testcases taken from robotics will be presented. The first testcase deals with the characterization of all loops of a mobile robot from proprioceptive measurements only. The second testcase is the localization of a robot from range-only measurements.

Keywords. interval analysis, kernel, robotics, loop closure, localization.

1. Introduction

Consider a mapping $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. The kernel of $\mathbf{f}$ is defined [5] by

$$\ker \mathbf{f} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) = \mathbf{0}\} = \mathbf{f}^{-1}(\mathbf{0}). \tag{1.1}$$

In mathematics, the kernel of a mapping $\mathbf{f}$ is a subset of the domain of $\mathbf{f}$ that measures how far $\mathbf{f}$ is from being injective. When $\mathbf{f}$ depends on some unknown but bounded parameter vectors, the kernel of $\mathbf{f}$ is known as *united solution set* [14] and its characterization has been studied by many authors (see e.g. [7], [6], [2], [3]), all of them using interval methods. More formally, we define the *united solution set* of a function $\mathbf{f} : [\mathbf{x}] \times [\mathbf{v}] \to \mathbb{R}^n$ as

$$\begin{aligned} \mathbb{X} &= \{\mathbf{x} \in [\mathbf{x}] \subset \mathbb{R}^n \mid \exists \mathbf{v} \in [\mathbf{v}],\ \mathbf{f}(\mathbf{x}, \mathbf{v}) = \mathbf{0}\} \\ &= \{\mathbf{x} \in [\mathbf{x}] \subset \mathbb{R}^n \mid \mathbf{0} \in \mathbf{f}(\mathbf{x}, [\mathbf{v}])\}, \end{aligned} \tag{1.2}$$

where $[\mathbf{x}]$ is a box of $\mathbb{R}^n$ and $[\mathbf{v}] \subset \mathbb{V}$ is a box of $\mathbb{R}^m$. The function $\mathbf{f}$: $[\mathbf{x}] \times [\mathbf{v}] \to \mathbb{R}^n$ is assumed to be continuous with respect to both $\mathbf{v}$ and $\mathbf{x}$ and differentiable with respect to $\mathbf{x}$. Now, if we want to obtain an accurate approximation of $\mathbb{X}$, we generally have to bisect the $\mathbf{v}$-space, which limits the resolution to small dimensional $\mathbf{v}$. Here, we shall assume that the function $\mathbf{f}$ belongs to an interval function [11] denoted by $[\mathbf{f}] = \left[\mathbf{f}^-, \mathbf{f}^+\right]$. Which means that the uncertainties vector $\mathbf{v}$ of the function $\mathbf{f}(\mathbf{x}, [\mathbf{v}])$ is embedded in the interval function $[\mathbf{f}](\mathbf{x})$, i.e. $\mathbf{f}(\mathbf{x}, [\mathbf{v}]) \subset [\mathbf{f}](\mathbf{x})$.

Definition 1.1. The kernel of an interval function $[\mathbf{f}]$ is defined by

$$\ker [\mathbf{f}] := \bigcup_{\mathbf{f} \in [\mathbf{f}]} \ker \mathbf{f} = \{\mathbf{x} \in [\mathbf{x}] \subset \mathbb{R}^n \mid \mathbf{0} \in [\mathbf{f}](\mathbf{x})\}. \tag{1.3}$$

The quantity $\mathbf{f}$ corresponds to an unknown function, the true value of which will be denoted by $\mathbf{f}^*$. We only know that $\mathbf{f}^*$ is inside $[\mathbf{f}]$ which corresponds to a bounded-error assumption [15]. Interval methods have been shown to be very efficient in this context (see e.g. [4]).

Remark 1.1. In the particular case where the function $\mathbf{f}(\mathbf{x}, \mathbf{v})$ is box conservative with respect to $\mathbf{v}$, i.e. for all boxes $[\mathbf{v}]$, and for all $\mathbf{x}$, the set $\mathbf{f}(\mathbf{x}, [\mathbf{v}]) = \{\mathbf{f}(\mathbf{x}, \mathbf{v}), \mathbf{v} \in [\mathbf{v}]\}$ is a box $[\mathbf{f}](\mathbf{x})$ of $\mathbb{R}^n$, the problem of characterizing the united solution set is equivalent to characterizing the kernel of an interval function.

Problem. In this paper, we shall consider the problem of finding an inner and an outer approximation of $\mathbb{X} = \ker[\mathbf{f}]$, i.e., two subpavings $\mathbb{X}^-, \mathbb{X}^+$ such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+. \tag{1.4}$$

As a consequence, we have

$$\ker \mathbf{f}^* \subset \mathbb{X}^+. \tag{1.5}$$

To solve the problem, we shall consider interval analysis which has already been shown very efficient in the context of control theory [13] and bounded error estimation [12], [10].

This paper is stuctured as follows. Section 2 presents the characterization of the kernel of an interval function. Section 3 gives an overview of the algorithm that will be used in Sections 4 and 5 which develop two testcases to show the efficiency of the approach. Finally, Section 6 summerizes this work.

2. Characterization of the kernel of an interval function

In this section, we propose a method to characterize $\ker[\mathbf{f}]$ as defined by Equation (1.3). Denote by $\mathbf{f}^-(\mathbf{x})$ and $\mathbf{f}^+(\mathbf{x})$ upper and lower bounds of $[\mathbf{f}](\mathbf{x})$. These two functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfy

$$\forall \mathbf{x},\ [\mathbf{f}](\mathbf{x}) = \left[\mathbf{f}^-(\mathbf{x}), \mathbf{f}^+(\mathbf{x})\right]. \tag{2.1}$$

This is illustrated by Figure 1(a). Moreover, we assume that we have two convergent inclusion functions for $\mathbf{f}^-(\mathbf{x})$ and $\mathbf{f}^+(\mathbf{x})$ (see Figure 1(b) and Figure 2). Define the two interval functions:

$$\left[\mathbf{f}^{\subset}\right]([\mathbf{x}]) = \left[\mathrm{ub}\left(\mathbf{f}^-([\mathbf{x}])\right), \mathrm{lb}\left(\mathbf{f}^+([\mathbf{x}])\right)\right] \tag{2.2}$$

$$\left[\mathbf{f}^{\supset}\right]([\mathbf{x}]) = \left[\mathrm{lb}\left(\mathbf{f}^-([\mathbf{x}])\right), \mathrm{ub}\left(\mathbf{f}^+([\mathbf{x}])\right)\right] \tag{2.3}$$

as illustrated by Figure 3. The quantity $\left[\left[\mathbf{f}^{\subset}\right]([\mathbf{x}]), \left[\mathbf{f}^{\supset}\right]([\mathbf{x}])\right]$ is an interval in the lattice of boxes of $\mathbb{R}^n$ equipped with the inclusion.

Proposition 2.1. *We have*

$$\mathbf{x} \in [\mathbf{x}] \Rightarrow \left[\mathbf{f}^{\subset}\right]([\mathbf{x}]) \subset [\mathbf{f}](\mathbf{x}) \subset \left[\mathbf{f}^{\supset}\right]([\mathbf{x}]). \tag{2.4}$$

Proof. Since

$$\mathbf{x} \in [\mathbf{x}] \Rightarrow \begin{cases} \mathbf{f}^-(\mathbf{x}) \leq \mathrm{ub}\left(\mathbf{f}^-([\mathbf{x}])\right) \\ \mathbf{f}^+(\mathbf{x}) \geq \mathrm{lb}\left(\mathbf{f}^+([\mathbf{x}])\right) \end{cases}, \tag{2.5}$$

we have

$$\mathbf{x} \in [\mathbf{x}] \Rightarrow \left[\mathbf{f}^{\subset}\right]([\mathbf{x}]) \subset [\mathbf{f}](\mathbf{x}).$$

And since

$$\mathbf{x} \in [\mathbf{x}] \Rightarrow \begin{cases} \mathbf{f}^-(\mathbf{x}) \geq \mathrm{lb}\left(\mathbf{f}^-([\mathbf{x}])\right) \\ \mathbf{f}^+(\mathbf{x}) \leq \mathrm{ub}\left(\mathbf{f}^+([\mathbf{x}])\right) \end{cases}, \tag{2.6}$$

we have

$$\mathbf{x} \in [\mathbf{x}] \Rightarrow [\mathbf{f}](\mathbf{x}) \subset \left[\mathbf{f}^{\supset}\right]([\mathbf{x}]).$$

$\square$

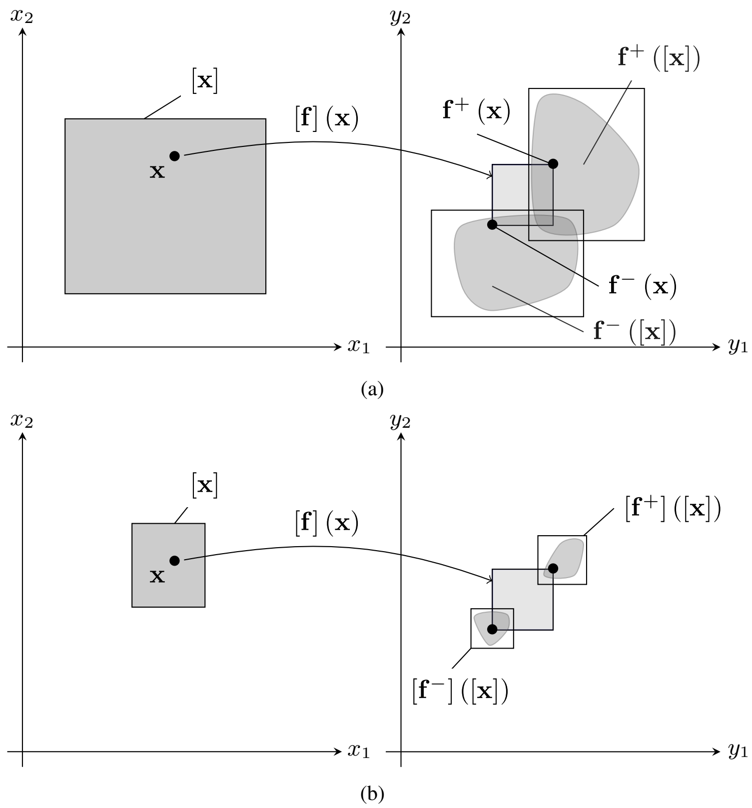


FIGURE 1. Inclusion functions associated with the two bound functions $\mathbf{f}^-(\mathbf{x})$ and $\mathbf{f}^+(\mathbf{x})$.

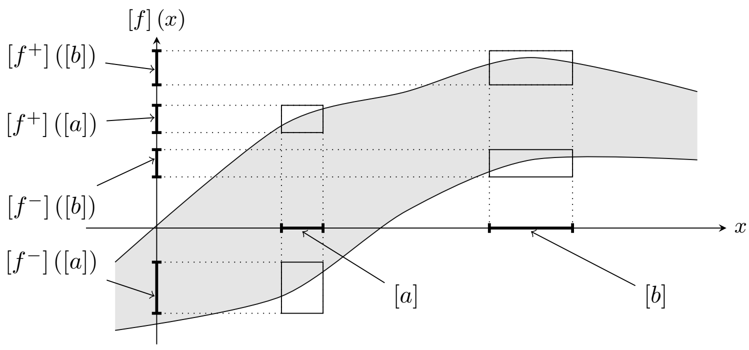


FIGURE 2. Illustration of $f^-(x)$ and $f^+(x)$ with $[f](x) \in \mathbb{IR}$.

Proposition 2.2.

$$\begin{pmatrix} \text{(i)} & \mathbf{0} \in \left[\mathbf{f}^{\subset}\right]([\mathbf{x}]) & \Rightarrow [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & \mathbf{0} \notin \left[\mathbf{f}^{\supset}\right]([\mathbf{x}]) & \Rightarrow [\mathbf{x}] \cap \mathbb{X} = \emptyset \end{pmatrix}. \tag{2.7}$$

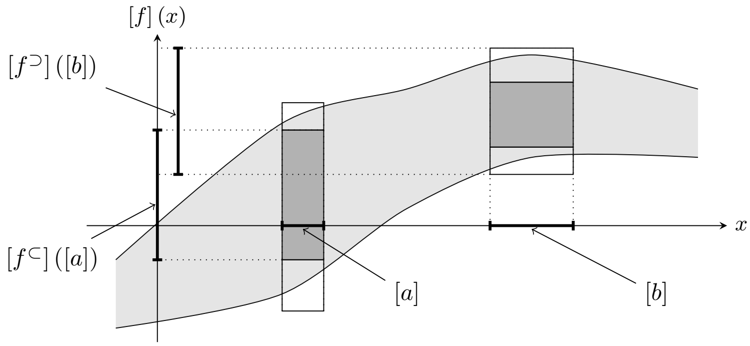


FIGURE 3. Illustration of $\left[f^{\subset}\right]([x])$ and $\left[f^{\supset}\right]([x])$ with $[f](x) \in \mathbb{IR}$.

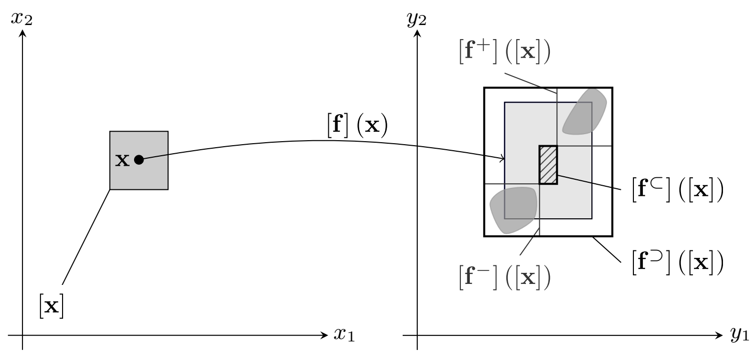


FIGURE 4. Illustration of the inclusion $\left[\mathbf{f}^{\subset}\right]([\mathbf{x}]) \subset [\mathbf{f}](\mathbf{x}) \subset \left[\mathbf{f}^{\supset}\right]([\mathbf{x}])$.

Proof. Let us first prove *(i)*. We have

$$\begin{aligned} \mathbf{x} \in \mathbb{X} \quad &\overset{(1.3)}{\Rightarrow} \quad \mathbf{x} \in \ker[\mathbf{f}] \Leftrightarrow \exists \mathbf{f} \in [\mathbf{f}],\ \mathbf{f}(\mathbf{x}) = \mathbf{0} \\ &\Leftrightarrow \quad \mathbf{0} \in [\mathbf{f}](\mathbf{x}) \Leftrightarrow \mathbf{0} \in \left[\mathbf{f}^{-}(\mathbf{x}), \mathbf{f}^{+}(\mathbf{x})\right]. \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned} \mathbf{0} \in \left[\mathbf{f}^{\subset}\right]([\mathbf{x}]) \quad &\overset{(2.4)}{\Rightarrow} \quad \forall \mathbf{x} \in [\mathbf{x}], \mathbf{0} \in [\mathbf{f}](\mathbf{x}) \\ &\Rightarrow \quad \forall \mathbf{x} \in [\mathbf{x}], \mathbf{0} \in \left[\mathbf{f}^{-}(\mathbf{x}), \mathbf{f}^{+}(\mathbf{x})\right] \\ &\Rightarrow \quad \forall \mathbf{x} \in [\mathbf{x}], \mathbf{x} \in \mathbb{X} \Rightarrow [\mathbf{x}] \subset \mathbb{X}. \end{aligned} \tag{2.9}$$

We now have to prove (ii).

$$\begin{aligned} \mathbf{0} \notin \left[\mathbf{f}^{\supset}\right]([\mathbf{x}]) \quad &\overset{(2.4)}{\Rightarrow} \quad \forall \mathbf{x} \in [\mathbf{x}], \mathbf{0} \notin [\mathbf{f}](\mathbf{x}) \\ &\Rightarrow \quad \forall \mathbf{x} \in [\mathbf{x}], \mathbf{x} \notin \mathbb{X} \Rightarrow [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{aligned} \tag{2.10}$$

$\square$

Remark 2.1. The two tests of the Proposition 2.2 make it possible to have an inner and an outer subpaving for $\mathbb{X}$ using any paver.

3. Algorithm

The following algorithm proposes to characterize the kernel of an interval function $[\mathbf{f}]$. This algorithm is a branch and bound algorithm similar to SIVIA (Set Inverter Via Interval Analysis) [9]. The main idea of the algorithm is to bisect an initial domain of research into non-overlapping boxes using the two tests of Proposition 2.2. The inputs of the algorithm are: the interval function $[\mathbf{f}]$, the initial domain of research $[\mathbf{x_0}]$ and a precision ε. This algorithm returns three subpavings: $\mathbb{X}^{\text{in}}$ which encloses boxes $[\mathbf{x}]$ which have been proved to be inside $\mathbb{X}$; $\mathbb{X}^{\text{out}}$ which encloses boxes $[\mathbf{x}]$ which are outside $\mathbb{X}$ and $\mathbb{X}^{?}$ which contains small boxes $[\mathbf{x}]$ for which nothing is known. A box is considered as too small if the maximum width over all component is lower than the parameter ε . The algorithm is given by Table 1.

Algorithm KER(in: $\varepsilon, [\mathbf{x_0}], [\mathbf{f}]$;out: $\mathbb{X}^{\text{in}}, \mathbb{X}^{?}, \mathbb{X}^{\text{out}}$)	
1	$\mathcal{Q} := \{[\mathbf{x_0}]\};\ \mathbb{X}^{\text{in}} = \emptyset; \mathbb{X}^{?} = \emptyset; \mathbb{X}^{\text{out}} = \emptyset;$
2	If $\mathcal{Q} \neq \emptyset$, take an element $[\mathbf{x}]$ in $\mathcal{Q}$; else terminate.
3	If $\mathbf{0} \notin \left[\mathbf{f}^{\supset}\right]([\mathbf{x}])$, then $\mathbb{X}^{\text{out}} := \mathbb{X}^{\text{out}} \cup [\mathbf{x}]$; go to 2;
4	If $\mathbf{0} \in \left[\mathbf{f}^{\subset}\right]([\mathbf{x}])$, then $\mathbb{X}^{\text{in}} := \mathbb{X}^{\text{in}} \cup [\mathbf{x}]$; go to 2;
5	If $\max(\text{width}([\mathbf{x}])) < \varepsilon$, then $\mathbb{X}^{?} := \mathbb{X}^{?} \cup [\mathbf{x}]$; go to 2;
6	Bisect $[\mathbf{x}]$ and store the resulting boxes in $\mathcal{Q}$; go to 2.

TABLE 1. Algorithm KER

At Step 1, the list $\mathcal{Q}$ is initialized with a single box which corresponds to the initial domain of research. At Step 2, a box $[\mathbf{x}]$ is taken from $\mathcal{Q}$, if $\mathcal{Q}$ is non-empty. If the list is empty, the algorithm terminates. Steps 3 and 4 use the tests of Proposition 2.2 to show that the current box $[\mathbf{x}]$ is outside or inside $\mathbb{X}$. If one of the tests succeeds, the next box is taken from the list $\mathcal{Q}$ at Step 2. If the current box $[\mathbf{x}]$ is too small, the algorithm stores it into $\mathbb{X}^{?}$ at Step 5 and $[\mathbf{x}]$ will not be bisected anymore. If all tests failed and if $[\mathbf{x}]$ is still large enough, it is bisected at Step 6 into two boxes that are stored in the current list $\mathcal{Q}$. When the algorithm terminates, we have

$$\mathbb{X}^{-} \subset \mathbb{X} \subset \mathbb{X}^{+}.$$

where

$$\mathbb{X}^{-} = \mathbb{X}^{\text{in}}$$
$$\mathbb{X}^{+} = \mathbb{X}^{\text{in}} \cup \mathbb{X}^{?}$$

4. Loop detection

4.1. Loop detection problem

The loop detection problem [1] aims to detect loops in a mobile robot trajectory moving on a horizontal plane using proprioceptive sensors only. Denote by $\mathbf{v}(t) \in \mathbb{R}^2$ its velocity vector which is assumed to be measured. The time t belongs to $[0, t_{max}]$ where t_{max} is the duration of the mission. Figure 5 presents a robot which performs a single loop.Define the function:

$$\mathbf{p}(t) = \int_0^t \mathbf{v}(\tau)\, d\tau, \tag{4.1}$$

which corresponds to the centered position of the robot and which origin corresponds to the position of the robot at time $t = 0$. To detect loops, we have to characterize the set:

$$\mathbb{T}^{*} = \left\{(t_1, t_2) \in [0, t_{\max}]^2, \mathbf{p}(t_1) = \mathbf{p}(t_2)\right\}, \tag{4.2}$$

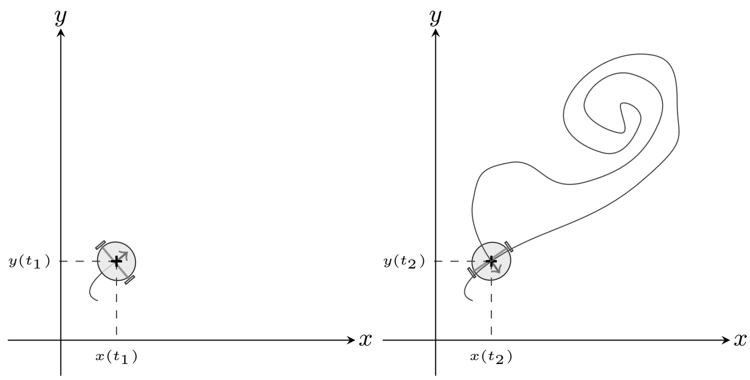


FIGURE 5. A robot trajectory with one single loop.

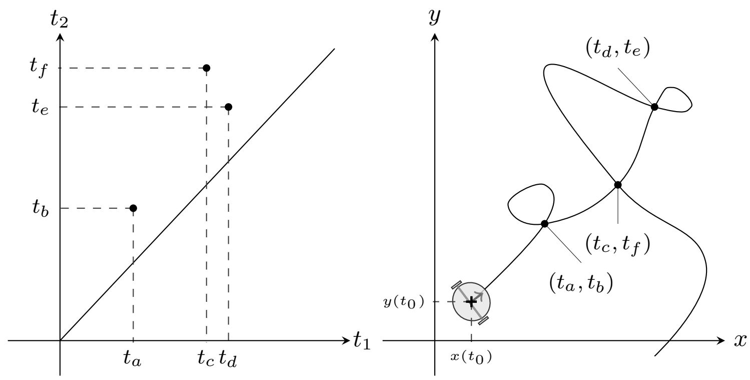


FIGURE 6. Left: t-plane, with three loops; Right: trajectory of the robot with two small loops (t_a, t_b), (t_d, t_e) and one complex loop (t_c, t_f). Note that $[t_d, t_e] \subset [t_c, t_f]$.

where a vector $\mathbf{t} = (t_1, t_2)$ is called a *t-pair*. The set of all t-pairs is called the *t-plane* (see Figure 6).Assume that the robot is only able to measure its velocity vector $\mathbf{v}$ with a known bounded error, i.e., a box $[\mathbf{v}](t)$ which contains $\mathbf{v}(t)$ is known for each $t \in [0, t_{max}]$. This velocity can be represented by a tube $[\mathbf{v}](t)$, i.e., an interval function from $\mathbb{R}$ to $\mathbb{IR}^2$. Thus, the set $\mathbb{T}^*$ is enclosed by the set $\mathbb{T}$ defined by:

$$\mathbb{T} = \left\{ (t_1, t_2) \in [0, t_{\max}]^2 \mid \exists \mathbf{v} \in [\mathbf{v}], \int_{t_1}^{t_2} \mathbf{v}(\tau) d\tau = \mathbf{0} \right\}. \tag{4.3}$$

We can deduce from the monotonicity of the integral operator that:

$$\int_{t_1}^{t_2} [\mathbf{v}](\tau)\, d\tau = \left[\int_{t_1}^{t_2} \mathbf{v}^-(\tau), \int_{t_1}^{t_2} \mathbf{v}^+(\tau) \right]. \tag{4.4}$$

Now, since

$$\exists \mathbf{v} \in [\mathbf{v}], \int_{t_1}^{t_2} \mathbf{v}(\tau) d\tau = \mathbf{0} \tag{4.5}$$

is equivalent to

$$\mathbf{0} \in \left[\int_{t_1}^{t_2} \mathbf{v}^-(\tau) d\tau, \int_{t_1}^{t_2} \mathbf{v}^+(\tau) d\tau \right], \tag{4.6}$$

we have

$$\mathbb{T} = \left\{ \mathbf{t} \in [0, t_{\max}]^2, \mathbf{0} \in [\mathbf{f}](\mathbf{t}) \right\} = \ker[\mathbf{f}], \tag{4.7}$$

where

$$[\mathbf{f}](\mathbf{t}) = \left[\int_{t_1}^{t_2} \mathbf{v}^-(\tau) d\tau, \int_{t_1}^{t_2} \mathbf{v}^+(\tau) d\tau \right]. \tag{4.8}$$

As a consequence, we can use the algorithm KER described in Section 3 to characterize the *loop-set* $\mathbb{T}$.

4.2. Testcase

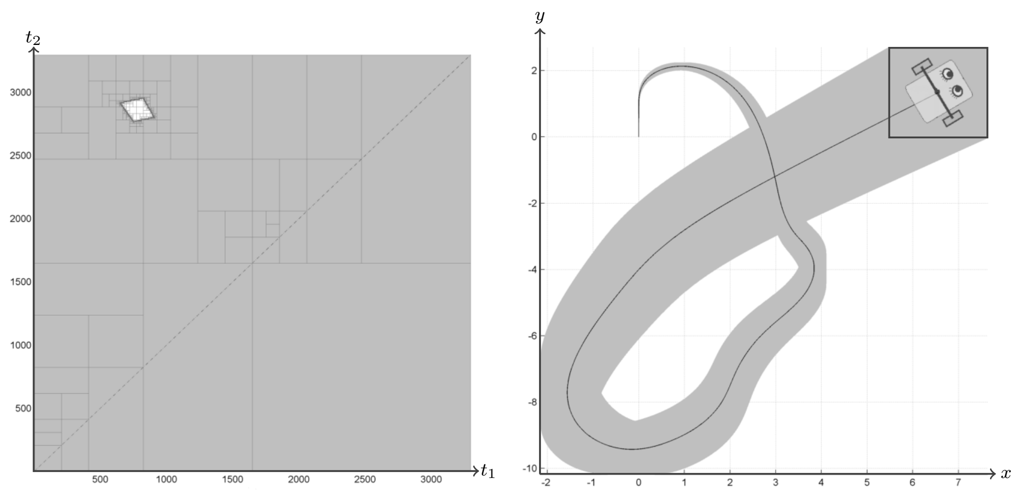
Consider a robot moving on a plane. The dynamic of this robot is described by the following state equations:

$$\mathbf{v} = \begin{pmatrix} u \cos \psi \\ u \sin \psi \end{pmatrix}, \tag{4.9}$$

where u is the linear velocity, ψ the heading angle. Assume that this robot is able to measure its velocity vector with a known bounded error. Figure 7(b) provides a representation of the robot that performs a single loop. The estimated trajectory given in gray is defined by:

$$[\mathbf{p}](t) = \int_0^t [\mathbf{v}](\tau)\, d\tau, \tag{4.10}$$

where $\mathbf{p}$ represents the position of the robot. The center of this trajectory is plotted in black.



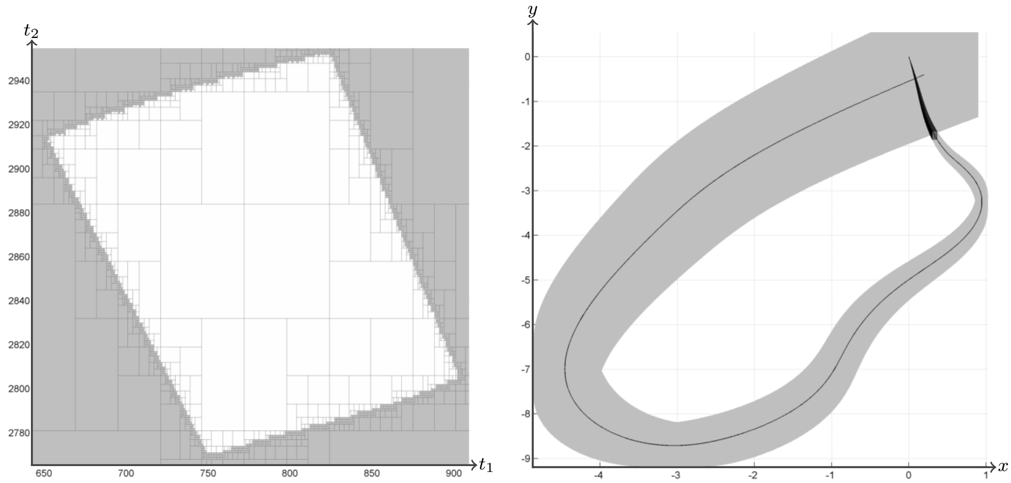
(a) (b)

FIGURE 7. Trajectory of the robot and associated t-plane.

The part of the trajectory from t_1 to t_2 chosen in $\ker[f]$ is depicted on Figure 8(b) and is defined by:

$$t \in [t_1, t_2], [\mathbf{p}](t) = \int_{t_1}^t [\mathbf{v}](\tau)\, d\tau. \tag{4.11}$$

We use the algorithm KER with inputs $[\mathbf{p}](t)$, $[\mathbf{x_0}] = [0, t_{max}] \times [0, t_{max}]$ and a precision ε equal to the acquisition time of the velocity data. Figures 7(a) and 8(a) give us the characterization of the set $\mathbb{T}$: in white, $\mathbb{T}^{\text{in}}$ which encloses boxes $[\mathbf{t}]$ which have been proved to be inside $\mathbb{T}$; in gray, $\mathbb{T}^{\text{out}}$ which encloses boxes $[\mathbf{t}]$ which are outside $\mathbb{T}$ and, a thin area, $\mathbb{T}^?$ at the frontier between $\mathbb{T}^{\text{in}}$



(a) (b)

FIGURE 8. Zoom on the t-plane containing $\ker[\mathbf{f}]$trajectory of the robot between t_1 and t_2.

and $\mathbb{T}^{\text{out}}$ which contains small boxes $[\mathbf{t}]$ for which nothing is known. When the algorithm terminates, we get

$$\mathbb{T}^{\text{in}} \subset \mathbb{T} \subset \mathbb{T}^{\text{out}} \tag{4.12}$$

where

$$\mathbb{T}^{\text{out}} = \mathbb{T}^{\text{in}} \cup \mathbb{T}^{?}$$

5. Localization

5.1. Localization problem

Consider k marks $\mathbf{m}(i)$ located at positions $(m_1(i), m_2(i))$ and a robot located at a distance $d(i)$ of the i^{th} mark. We assume that these quantities are only known to belong to intervals, i.e. $m_1(i) \in [m_1](i)$, $m_2(i) \in [m_2](i)$, $d(i) \in [d](i)$. The position $\mathbf{x} = (x_1, x_2)$ for the robot is consistent with the i^{th} distance if

$$\begin{array}{c} \exists m_1 \in [m_1](i), \ \exists m_2 \in [m_2](i), \ \exists d \in [d](i) \\ \text{such that } (x_1 - m_1)^2 + (x_2 - m_2)^2 = d^2, \end{array} \tag{5.1}$$

or equivalently

$$0 \in [f_i](\mathbf{x}), \tag{5.2}$$

with

$$[f_i](\mathbf{x}) = (x_1 - [m_1](i))^2 + (x_2 - [m_2](i))^2 - [d(i)]^2. \tag{5.3}$$

Using the symbolic interval arithmetic [8], we get

$$[f_i](\mathbf{x}) = \left[f_i^-(\mathbf{x}), f_i^+(\mathbf{x})\right], \tag{5.4}$$

where

$$\begin{aligned} f_i^-(\mathbf{x}) = &\max\Big(0, \mathrm{sign}\Big(\left(x_1 - m_1^+(i)\right)\left(x_1 - m_1^-(i)\right)\Big)\Big) \\ &\cdot \min\left(\left(x_1 - m_1^+(i)\right)^2, \left(x_1 - m_1^-(i)\right)^2\right) \\ &+ \max\Big(0, \mathrm{sign}\Big(\left(x_2 - m_2^+(i)\right)\left(x_2 - m_2^-(i)\right)\Big)\Big) \\ &\cdot \min\left(\left(x_2 - m_2^+(i)\right)^2, \left(x_2 - m_2^-(i)\right)^2\right) - \left(d^+(i)\right)^2, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} f_i^+(\mathbf{x}) = &\max\left(\left(x_1 - m_1^+(i)\right)^2, \left(x_1 - m_1^-(i)\right)^2\right) \\ &+ \max\left(\left(x_2 - m_2^+(i)\right)^2, \left(x_2 - m_2^-(i)\right)^2\right) \\ &- \left(d^-(i)\right)^2. \end{aligned} \tag{5.6}$$

The set of all feasible positions of the robot is thus

$$\mathbb{X} = \{\mathbf{x} \mid \mathbf{0} \in [\mathbf{f}](\mathbf{x})\} = \ker[\mathbf{f}]. \tag{5.7}$$

As the distance function, defined by Equation 5.3, can be described with upper and lower bound functions, and as the localization problem can be resolved by finding the kernel of this distance function, we can use algorithm KER of Section 3 to characterize the set $\mathbb{X}$ by

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+. \tag{5.8}$$

with

$$\mathbb{X}^+ = \mathbb{X}^- \cup \mathbb{X}^?$$

5.2. Testcase

Consider a robot moving on a plane. This robot have to compute its position $\mathbf{x} = (x_1, x_2)$ relative to three marks whose badly known positions will be represented by boxes. The robot is equipped with a rangefinder that measures the time flight of a signal emitted from a mark (e.g. an acoustic signal) in order to compute the distance to this mark. Assume also that the measured signal is affected by some perturbations in the environment that can be bounded. We can thus represent the distance from the robot to a mark by an interval.

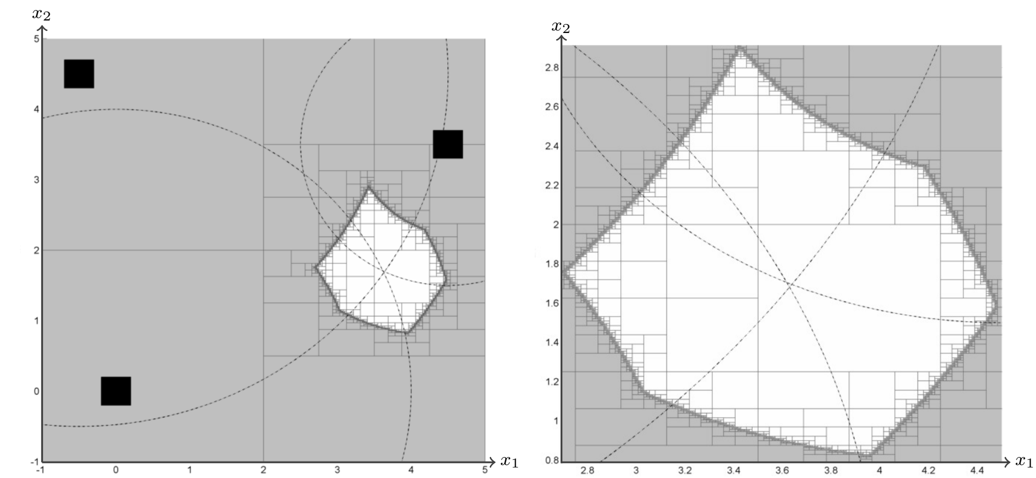
Figure 9(a) illustrates the localization problem. The black boxes represent the uncertain positions $m_1(i)$ and $m_2(i)$ of the marks. The dashed black circles represent the centers of the measured distances $[d(i)]$ plotted from the center of the box corresponding to the i^{th} mark position (as an indication of the distance). To characterize the set $\mathbb{X}$, we use the algorithm KER presented in Section 3 with inputs $[f_i](\mathbf{x})$ for $i \in \{1,2,3\}$, $[\mathbf{x_0}] = [-1,5] \times [-1,5]$ and a precision $\varepsilon = 0.01$. The gray boxes represent the boxes that satisfy condition $\mathbf{0} \notin \left[\mathbf{f}^{\supset}\right]([\mathbf{x}])$. The white boxes match the test $\mathbf{0} \in \left[\mathbf{f}^{\subset}\right]([\mathbf{x}])$ and represent the inner subpaving $\mathbb{X}^-$ for $\mathbb{X}$. The boxes between this two zones (see Figure 9(b)) represent uncertain solutions that do not match neither (i) nor (ii) from Proposition 2.2.

At the end of the algorithm, we get

$$\ker[\mathbf{f}] \subset \mathbb{X}^+ \tag{5.9}$$

or

$$\ker[\mathbf{f}] \subset \left(\mathbb{X}^? \cup \mathbb{X}^{in}\right). \tag{5.10}$$



(a) (b)

FIGURE 9. Position of a robot from 3 range only measurements and zoom on this position.

6. Conclusion

This paper has presented an approach to compute the kernel of interval valued functions. We only assume that the unknown function belongs to an interval function, which corresponds to a bounded error assumption. The algorithm presented builds three subpavings to approximate the kernel of the function.

The first testcase corresponds to the loop detection problem in the trajectory of a mobile robot using proprioceptive measurements only. In this situation, we have to find the *united solution set* with infinite dimension parameters. The use of our representation allows us to avoid bisecting this infinite parameter space and to enclose the kernel of the function. The second testcase deals with the localization of a robot from range only measurements. Symbolic interval arithmetic in the localization problem remove quantifiers from the distance function and format it into a tube. Doing so, we can apply an inner test as given by the algorithm and find a guaranteed positioning of the robot despite the uncertainties of mark positions and distance measurements.

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