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Brief paper

Interval analysis and dioid: application to robust controller design for timed event graphs☆

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Abstract

This paper deals with feedback controller synthesis for timed event graphs, where the number of initial tokens and time delays are only known to belong to intervals. We discuss here the existence and the computation of a robust controller set for uncertain systems that can be described by parametric models, the unknown parameters of which are assumed to vary between known bounds. Each controller is computed in order to guarantee that the closed-loop system behavior is greater than the lower bound of a reference model set and is lower than the upper bound of this set. The synthesis presented here is mainly based on dioid, interval analysis and residuation theory.

Keywords: Discrete-event systems; Timed event graphs; Dioid; Interval analysis; Residuation theory; Feedback synthesis

1. Introduction

Discrete event systems (DES) appear in many applications in manufacturing systems (Ayhan & Wortman, 1999), computer and communication systems (LeBoudec & Thiran, 2002) and are often described by the Petri net formalism. Timed-event graphs (TEG) are Timed Petri nets in which all places have single upstream and single downstream transitions and appropriately model DES characterized by delay and synchronization phenomena. TEG can be described by linear equations in the dioid algebra (Baccelli, Cohen, Olsder, & Quadrat, 1992; Cohen, Moller, Quadrat, & Viot, 1989) and this fact has permitted many important achievements on the control of DES modelled by TEG (Cohen et al., 1989; Cottenceau, Hardouin, Boimond, & Ferrier, 2001; Menguy, Boimond, Hardouin, & Ferrier, 2000; Lüders & Santos-Mendes, 2002). TEG control problems are usually stated in a just-in-time context. The design goal is to achieve some performance while minimizing internal stocks. In Baccelli et al. (1992) and Menguy et al. (2000) an optimal open-loop control law is given. In Cottenceau et al. (2001) linear closed-loop controllers synthesis are given in a model matching objective, i.e. the controller synthesis is done in order that the controlled system will behave as close as possible to a reference model and will delay as much as possible the token input in the system. The reference model is a priori known and depicts the desired behavior of the corrected system.

This paper aims at designing robust feedback controller when the system includes some parametric uncertainties which can be described by intervals. First, by using interval analysis, we give a model to depict TEG with number of tokens and time delays which are assumed to vary between known bounds.[1] Next, we consider a controller synthesis for these uncertain systems. The controller synthesis is done in order to maintain the controlled system in a set of admissible behaviors. We assume that the upper and lower bound of

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[1] In a manufacturing context, these systems can represent production systems in which the number of resources varies with time (e.g. due to some maintenance operations, or to some machines breakdowns, etc.) or in which the processing times are not well known but vary in known intervals.

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this specification set are a priori known, the synthesis yields a controller set[2] which guarantees that the closed-loop system behavior is both greater than the lower bound of the specification set and lower than the upper bound of this same set. Controller synthesis is obtained by considering residuation theory which allows the inversion of mapping defined over ordered sets, and interval analysis which is known to be efficient to characterize set of robust controllers in a guaranteed way (Jaulin, Kieffer, Dirit, & Walter, 2001).

2. Dioids and residuation

Definition 1. A dioid $\mathscr{D}$ is a set endowed with two internal operations denoted by $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted by ε and e, respectively, such that $\oplus$ is also commutative and idempotent (i.e. $a \oplus a = a$). The $\otimes$ operation is distributive with respect to $\oplus$, and ε is absorbing for the product (i.e. $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a$). When $\otimes$ is commutative, the dioid is said to be commutative. The symbol $\otimes$ is often omitted.

Dioids can be endowed with a natural *order*: $a \succcurlyeq b$ iff $a = a \oplus b$. Then they become sup-semilattices and $a \oplus b$ is the least upper bound of a and b. A dioid is *complete* if sums of infinite number of terms are always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all elements of the dioid is defined and denoted by $\top$ (for 'top'). A complete dioid (sup-semilattice) becomes a lattice by constructing the greatest lower bound of a and b, denoted by $a \wedge b$, as the least upper bound of the (nonempty) subset of all elements which are less than a and b (see Baccelli et al., 1992, Section 4).

Definition 2 (Subdioid). A subset $\mathscr{C}$ of a dioid is called a subdioid of $\mathscr{D}$ if

- $\varepsilon \in \mathscr{C}$ and $e \in \mathscr{C}$;
- $\mathscr{C}$ is closed for $\oplus$ and $\otimes$, i.e. $\forall a, b \in \mathscr{C}$, $a \oplus b \in \mathscr{C}$ and $a \otimes b \in \mathscr{C}$.

Theorem 3 (Cottenceau et al., 2001). *Over a complete dioid $\mathscr{D}$, the implicit equation $x = ax \oplus b$ admits $x = a^*b$ as least solution, where $a^* = \oplus_{i \in \mathbb{N}} a^i$ (Kleene star operator) with $a^0 = e$. In the following this operator will sometimes be represented by the mapping $\mathscr{K} : \mathscr{D} \to \mathscr{D}$, $x \mapsto x^*$. Furthermore, letting $x, y \in \mathscr{D}$, we have*

$$x(yx)^* = (xy)^*x, \tag{1}$$

$$(x^*)^* = x^*. \tag{2}$$

[2] It is a set of robust controllers which ensures that, for all the possible behaviors of the uncertain system, the controlled system is slower than a reference model (a specification which is described as a TEG) and is faster than another one.

The residuation theory provides, under some assumptions, *optimal* solutions to inequalities such as $f(x) \preccurlyeq b$, where f is an isotone mapping (f s.t. $a \preccurlyeq b \Rightarrow f(a) \preccurlyeq f(b)$) defined over ordered sets.

Definition 4 (Residual and residuated mapping). An isotone mapping $f : \mathscr{D} \to \mathscr{E}$, where $\mathscr{D}$ and $\mathscr{E}$ are ordered sets, is a residuated mapping if for all $y \in \mathscr{E}$, the least upper bound of the subset $\{x | f(x) \preccurlyeq y\}$ exists and belongs to this subset. It is then denoted by $f^\sharp(y)$. Mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique isotone mapping such that

$$f \circ f^\sharp \preccurlyeq \mathsf{Id}_{\mathscr{E}} \quad \text{and} \quad f^\sharp \circ f \succcurlyeq \mathsf{Id}_{\mathscr{D}}, \tag{3}$$

where Id is the identity mapping, respectively, on $\mathscr{D}$ and $\mathscr{E}$.

Property 5. *Let $f : \mathscr{D} \to \mathscr{E}$ be a residuated mapping, then $y \in f(\mathscr{D}) \Leftrightarrow f(f^\sharp(y)) = y$.*

Property 6 (Baccelli et al., 1992, Theorem 4.56). *If $h : \mathscr{D} \to \mathscr{C}$ and $f : \mathscr{C} \to \mathscr{B}$ are residuated mappings, then $f \circ h$ is also residuated and*

$$(f \circ h)^\sharp = h^\sharp \circ f^\sharp. \tag{4}$$

Theorem 7 (Baccelli et al., 1992, Section 4.4.2). *Consider the mapping $f : \mathscr{E} \to \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are complete dioids. Their bottom elements are, respectively, denoted by $\varepsilon_{\mathscr{E}}$ and $\varepsilon_{\mathscr{F}}$. Then, f is residuated iff $f(\varepsilon_{\mathscr{E}}) = \varepsilon_{\mathscr{F}}$ and $f(\oplus_{x \in \mathscr{G}} x) = \oplus_{x \in \mathscr{G}} f(x)$ for each $\mathscr{G} \subseteq \mathscr{E}$ (i.e., f is lower-semicontinuous).*

Corollary 8. *The mappings $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ defined over a complete dioid $\mathscr{D}$ are both residuated.*[3] *Their residuals are usually denoted, respectively, by $L_a^\sharp(x) = a \circ\!\!\!\!\backslash\, x$ and $R_a^\sharp(x) = x \circ\!\!\!\!/\, a$ in (max, +) literature.*[4]

Theorem 9 (Baccelli et al., 1992, Section 4.4.4; MaxPlus, 1991). *The mappings $x \mapsto a \circ\!\!\!\!\backslash\, x$ and $x \mapsto x \circ\!\!\!\!/\, a$ satisfy the following properties*:

$$a^*x = a^* \circ\!\!\!\!\backslash\,(a^*x), \quad xa^* = (xa^*) \circ\!\!\!\!/\, a^*, \tag{5}$$

$$(ab) \circ\!\!\!\!\backslash\, x = b \circ\!\!\!\!\backslash\,(a \circ\!\!\!\!\backslash\, x), \quad x \circ\!\!\!\!/\,(ba) = (x \circ\!\!\!\!/\, a) \circ\!\!\!\!/\, b, \tag{6}$$

$$a \circ\!\!\!\!\backslash\,(x \wedge y) = a \circ\!\!\!\!\backslash\, x \wedge a \circ\!\!\!\!\backslash\, y, \quad (x \wedge y) \circ\!\!\!\!/\, a = x \circ\!\!\!\!/\, a \wedge y \circ\!\!\!\!/\, a, \tag{7}$$

$$a \circ\!\!\!\!\backslash\, a = (a \circ\!\!\!\!\backslash\, a)^*, \quad a \circ\!\!\!\!/\, a = (a \circ\!\!\!\!/\, a)^*. \tag{8}$$

The problem of mapping restriction and its connection with the residuation theory is now addressed.

[3] This property also concerns the matrix dioid, for instance $X \mapsto AX$ where $A, X \in \mathscr{D}^{n \times n}$. See Baccelli et al. (1992) for the computation of $A \circ\!\!\!\!\backslash\, B$ and $B \circ\!\!\!\!/\, A$.

[4] $a \circ\!\!\!\!\backslash\, b$ is the greatest solution of $ax \preccurlyeq b$.

Proposition 10 (Blyth & Janowitz, 1972). *Let* $\mathsf{Id}_{|\mathscr{D}_{\text{sub}}} : \mathscr{D}_{\text{sub}} \to \mathscr{D}$, $x \mapsto x$ *be the canonical injection from a complete subdioid into a complete dioid. The injection* $\mathsf{Id}_{|\mathscr{D}_{\text{sub}}}$ *is residuated and its residual is a projector which will be denoted by* Pr_{sub}, *therefore*,

$$\mathsf{Pr}_{\text{sub}} = (\mathsf{Id}_{|\mathscr{D}_{\text{sub}}})^{\sharp} = \mathsf{Pr}_{\text{sub}} \circ \mathsf{Pr}_{\text{sub}}.$$

Definition 11 (Restricted mapping). Let $f : \mathscr{E} \to \mathscr{F}$ be a mapping and $\mathscr{A} \subseteq \mathscr{E}$. We will denote[5] $f_{|\mathscr{A}} : \mathscr{A} \to \mathscr{F}$ the mapping defined by $f_{|\mathscr{A}} = f \circ \mathsf{Id}_{|\mathscr{A}}$ where $\mathsf{Id}_{|\mathscr{A}} : \mathscr{A} \to \mathscr{E}$. Identically, let $\mathscr{B} \subseteq \mathscr{F}$ with $\mathsf{Im} f \subseteq \mathscr{B}$. Mapping ${}_{\mathscr{B}|}f : \mathscr{E} \to \mathscr{B}$ is defined by $f = \mathsf{Id}_{|\mathscr{B}} \circ {}_{\mathscr{B}|} f$, where $\mathsf{Id}_{|\mathscr{B}} : \mathscr{B} \to \mathscr{F}$.

Proposition 12. *Let* $f : \mathscr{D} \to \mathscr{E}$ *be a residuated mapping and* $\mathscr{D}_{\text{sub}}$ *(resp.* $\mathscr{E}_{\text{sub}}$*) be a complete subdiod of* $\mathscr{D}$ *(resp.* $\mathscr{E}$*).*

1. *Mapping* $f_{|\mathscr{D}_{\text{sub}}}$ *is residuated and its residual is given by*
   $$(f_{|\mathscr{D}_{\text{sub}}})^{\sharp} = (f \circ \mathsf{Id}_{|\mathscr{D}_{\text{sub}}})^{\sharp} = \mathsf{Pr}_{\text{sub}} \circ f^{\sharp}.$$
2. *If* $\mathsf{Im} f \subset \mathscr{E}_{\text{sub}}$ *then mapping* ${}_{\mathscr{E}_{\text{sub}}|}f$ *is residuated and its residual is given by*
   $$({}_{\mathscr{E}_{\text{sub}}|}f)^{\sharp} = f^{\sharp} \circ \mathsf{Id}_{|\mathscr{E}_{\text{sub}}} = (f^{\sharp})_{|\mathscr{E}_{\text{sub}}}.$$

Proof. Statement 1 follows directly from Property 6 and Proposition 10. Statement 2 is obvious since f is residuated and $\mathsf{Im} f \subset \mathscr{E}_{\text{sub}} \subset \mathscr{E}$. □

Definition 13 (Closure mapping). An isotone mapping $f : \mathscr{E} \to \mathscr{E}$ defined on an ordered set $\mathscr{E}$ is a closure mapping if $f \succcurlyeq \mathsf{Id}_{\mathscr{E}}$ and $f \circ f = f$.

Proposition 14 (Cottenceau et al., 2001). *Let* $f : \mathscr{E} \to \mathscr{E}$ *be a closure mapping. A closure mapping restricted to its image* ${}_{\mathsf{Im} f|}f$ *is a residuated mapping whose residual is the canonical injection* $\mathsf{Id}_{|\mathsf{Im} f} : \mathsf{Im} f \to \mathscr{E}$, $x \mapsto x$.

Corollary 15. *The mapping* ${}_{\mathsf{Im}\mathscr{K}|}\mathscr{K}$ *is a residuated mapping whose residual is* $({}_{\mathsf{Im}\mathscr{K}|}\mathscr{K})^{\sharp} = \mathsf{Id}_{|\mathsf{Im}\mathscr{K}}$. *This means that* $x = a^*$ *is the greatest solution to inequality* $x^* \preccurlyeq a^*$. *Actually, the greatest solution achieves equality.*

Proposition 16. *Let* $M_A : x \mapsto (ax)^* a$, *be a mapping defined over a complete dioid. Consider* $g \in \mathscr{D}$ *and* $d \in \mathscr{D}$. *Let us consider the following sets*:

$$\begin{aligned} &\mathscr{G}_1 = \{g \mid \exists d \text{ s.t. } g = d^* a\} \\ \text{and} \quad &\mathscr{G}_2 = \{g \mid \exists d \text{ s.t. } g = a d^*\}. \end{aligned} \tag{9}$$

The mappings ${}_{\mathscr{G}_1|}M_a$ *and* ${}_{\mathscr{G}_2|}M_a$ *are both residuated. Their residuals are such that* $({}_{\mathscr{G}_1|}M_a)^{\sharp}(x) = ({}_{\mathscr{G}_2|}M_a)^{\sharp}(x) = a \backslash\!\!\!\!\circ\, x \,/\!\!\!\!\circ\, a$. *Furthermore,* $\mathsf{Im} M_a \subseteq (\mathscr{G}_1 \cap \mathscr{G}_2)$.

Proof. According to Definition 4, since the mapping L_a is residuated (cf. Corollary 8) and according to (1), we have

$$(ax)^* a = a(xa)^* \preccurlyeq d^* a \Leftrightarrow (xa)^* \preccurlyeq a \backslash\!\!\!\!\circ\, (d^* a).$$

According to (5) and (6), we can rewrite $a \backslash\!\!\!\!\circ\, (d^* a) = a \backslash\!\!\!\!\circ\, (d^* \backslash\!\!\!\!\circ\, (d^* a)) = (d^* a) \backslash\!\!\!\!\circ\, (d^* a)$. According to (8), this last expression shows that $a \backslash\!\!\!\!\circ\, (d^* a)$ belongs to the image of $\mathscr{K}$. Since ${}_{\mathsf{Im}\mathscr{K}|}\mathscr{K}$ is residuated (cf. Corollary 15), there is also the following equivalence:

$$(xa)^* \preccurlyeq a \backslash\!\!\!\!\circ\, (d^* a) \Leftrightarrow xa \preccurlyeq a \backslash\!\!\!\!\circ\, (d^* a).$$

Finally, since R_a is also residuated (cf. Corollary 8), we verify that $x = a \backslash\!\!\!\!\circ\, (d^* a) \,/\!\!\!\!\circ\, a$ is the greatest solution of $(ax)^* a \preccurlyeq d^* a$, $\forall d \in \mathscr{D}$. That amounts to saying that ${}_{\mathscr{G}_1|}M_a$ is residuated. Similarly, one can show that ${}_{\mathscr{G}_2|}M_a$ is residuated, then if $g \in (\mathscr{G}_1 \cup \mathscr{G}_2)$, $M_a(x) \preccurlyeq g$ admits a greatest solution.

Moreover, Eq. (1) leads to $(ax)^* a = a(xa)^*$, by choosing $d = ax$ or $d = xa$, it comes $\mathsf{Im} M_a \subseteq (\mathscr{G}_1 \cap \mathscr{G}_2)$. □

Corollary 17. *If* $g \in \mathsf{Im} M_a$, *then* $x = a \backslash\!\!\!\!\circ\, g \,/\!\!\!\!\circ\, a$ *is the greatest solution to the equation* $(ax)^* a = g$.

Proof. First $\mathsf{Im} M_a \subseteq (\mathscr{G}_1 \cap \mathscr{G}_2)$, thus ${}_{\mathsf{Im} M_a|}M_a$ is residuated. Furthermore, $\forall y \in \mathsf{Im} M_a$, $M_a(x) = y$ admits a solution, i.e. ${}_{\mathsf{Im} M_a|}M_a$ is surjective, then $({}_{\mathsf{Im} M_a|}M_a)^{\sharp}$ provides the greatest solution (see Property 5). □

3. Dioid of pairs

The set of pairs (x', x'') with $x' \in \mathscr{D}$ and $x'' \in \mathscr{D}$ endowed with two coordinate-wise algebraic operations

$$(x', x'') \oplus (y', y'') = (x' \oplus y', x'' \oplus y'')$$

$$\text{and} \quad (x', x'') \otimes (y', y'') = (x' \otimes y', x'' \otimes y''),$$

is a dioid denoted by $\mathscr{C}(\mathscr{D})$ with $(\varepsilon, \varepsilon)$ as the zero element and (e, e) as the identity element (see Definition 1).

Remark 18. The operation $\oplus$ generates the corresponding canonical partial order $\preccurlyeq_{\mathscr{C}}$ in $\mathscr{C}(\mathscr{D})$: $(x', x'') \oplus (y', y'') = (y', y'') \Leftrightarrow (x', x'') \preccurlyeq_{\mathscr{C}} (y', y'') \Leftrightarrow x' \preccurlyeq_{\mathscr{D}} y'$ and $x'' \preccurlyeq_{\mathscr{D}} y''$ where $\preccurlyeq_{\mathscr{D}}$ is the order relation in $\mathscr{D}$.

Proposition 19 (Litvinov & Sobolevski, 2001). *If the dioid* $\mathscr{D}$ *is complete, then the dioid* $\mathscr{C}(\mathscr{D})$ *is complete and its top element is given by* $(\top, \top)$.

[5] These notations are borrowed from classical linear system theory see Wonham (1985).

Notation 20. Let us consider the following mappings over $\mathscr{C}(\mathscr{D})$:

$$L_{(a',a'')} : (x',x'') \mapsto (a',a'') \otimes (x',x''),$$

$$R_{(a',a'')} : (x',x'') \mapsto (x',x'') \otimes (a',a'').$$

Proposition 21. *The mappings $L_{(a',a'')}$ and $R_{(a',a'')}$ defined over $\mathscr{C}(\mathscr{D})$ are both residuated. Their residuals are equal to $L^{\sharp}_{(a',a'')}(b',b'') = (a',a'') \mathbin{\backslash\!\!\!\circ} (b',b'') = (a' \mathbin{\backslash\!\!\!\circ} b', a'' \mathbin{\backslash\!\!\!\circ} b'')$ and $R^{\sharp}_{(a',a'')}(b',b'') = (b',b'') \mathbin{/\!\!\!\circ} (a',a'') = (b' \mathbin{/\!\!\!\circ} a', b'' \mathbin{/\!\!\!\circ} a'')$.*

Proof. Observe that $L_{(a',a'')}(\bigoplus_{(x',x'')\in X}(x',x''))=\bigoplus_{(x',x'')\in X} L_{(a',a'')}(x',x'')$, (for every subset X of $\mathscr{C}(\mathscr{D})$), moreover $L_{(a',a'')}(\varepsilon,\varepsilon) = (a'\varepsilon, a''\varepsilon) = (\varepsilon,\varepsilon)$. Then $L_{(a',a'')}$ is residuated (follows from Theorem 7). Therefore, we have to find, for given (b',b'') and (a',a''), the greatest solution (x',x'') for inequality $(a',a'') \otimes (x',x'') \preccurlyeq_{\mathscr{C}} (b',b'') \Leftrightarrow (a' \otimes x', a'' \otimes x'') \preccurlyeq_{\mathscr{C}} (b',b'')$, moreover according to Remark 18 on the order relation induced by $\oplus$ on $\mathscr{C}(\mathscr{D})$ we have,

$$a' \otimes x' \preccurlyeq_{\mathscr{D}} b' \quad \text{and} \quad a'' \otimes x'' \preccurlyeq_{\mathscr{D}} b''.$$

Since the mappings $x' \mapsto a' \otimes x'$ and $x'' \mapsto a'' \otimes x''$ are residuated over $\mathscr{D}$ (cf. Corollary 8), we have $x' \preccurlyeq_{\mathscr{D}} a' \mathbin{\backslash\!\!\!\circ} b'$ and $x'' \preccurlyeq_{\mathscr{D}} a'' \mathbin{\backslash\!\!\!\circ} b''$. Then, we obtain $L^{\sharp}_{(a',a'')}(b',b'') = (a' \mathbin{\backslash\!\!\!\circ} b', a'' \mathbin{\backslash\!\!\!\circ} b'')$. □

Notation 22. The set of pairs $(\widetilde{x}',\widetilde{x}'')$ s.t. $\widetilde{x}' \preccurlyeq \widetilde{x}''$ is denoted by $\mathscr{C}_O(\mathscr{D})$.

Proposition 23. *Let $\mathscr{D}$ be a complete dioid. The set $\mathscr{C}_O(\mathscr{D})$ is a complete subdioid of $\mathscr{C}(\mathscr{D})$.*

Proof. Clearly $\mathscr{C}_O(\mathscr{D}) \subset \mathscr{C}(\mathscr{D})$ and it is closed for $\oplus$ and $\otimes$ since: $\widetilde{x}' \oplus \widetilde{y}' \preccurlyeq \widetilde{x}'' \oplus \widetilde{y}''$ and $\widetilde{x}' \otimes \widetilde{y}' \preccurlyeq \widetilde{x}'' \otimes \widetilde{y}''$ whenever $\widetilde{x}' \preccurlyeq \widetilde{x}''$ and $\widetilde{y}' \preccurlyeq \widetilde{y}''$. Moreover, zero element $(\varepsilon,\varepsilon)$, unit element (e,e) and top element $(\top,\top)$ of $\mathscr{C}(\mathscr{D})$ are in $\mathscr{C}_O(\mathscr{D})$. □

Proposition 24. *The canonical injection $\mathsf{Id}_{|\mathscr{C}_O(\mathscr{D})} : \mathscr{C}_O(\mathscr{D}) \to \mathscr{C}(\mathscr{D})$ is residuated. Its residual $(\mathsf{Id}_{|\mathscr{C}_O(\mathscr{D})})^{\sharp}$ is a projector denoted by $\mathsf{Pr}_{\mathscr{C}_O(\mathscr{D})}$. Its practical computation is given by*

$$\mathsf{Pr}_{\mathscr{C}_O(\mathscr{D})}((x',x'')) = (x' \wedge x'', x'') = (\widetilde{x}',\widetilde{x}''). \tag{10}$$

Proof. It is a direct application of Proposition 10, since $\mathscr{C}_O(\mathscr{D})$ is a subdioid of $\mathscr{C}(\mathscr{D})$. Practically, let $(x',x'') \in \mathscr{C}(\mathscr{D})$, we have $\mathsf{Pr}_{\mathscr{C}_O(\mathscr{D})}((x',x'')) = (\widetilde{x}',\widetilde{x}'') = (x' \wedge x'', x'')$, which is the greatest pair such that:

$$\widetilde{x}' \preccurlyeq x', \quad \widetilde{x}'' \preccurlyeq x'' \quad \text{and} \quad \widetilde{x}' \preccurlyeq \widetilde{x}''. \quad \square$$

Definition 25. An isotone mapping f defined over $\mathscr{D}$ admits a natural extension over $\mathscr{C}_O(\mathscr{D})$, which is defined as $f(\widetilde{x}',\widetilde{x}'') = (f(\widetilde{x}'), f(\widetilde{x}''))$. For example, the Kleene star mapping in $\mathscr{C}_O(\mathscr{D})$ is defined by $\mathscr{K}(\widetilde{x}',\widetilde{x}'') = (\mathscr{K}(\widetilde{x}'), \mathscr{K}(\widetilde{x}'')) = (\widetilde{x}'^{*}, \widetilde{x}''^{*})$.

Proposition 26. *Let $(\widetilde{a}',\widetilde{a}'') \in \mathscr{C}_O(\mathscr{D})$, mapping ${}_{\mathscr{C}_O(\mathscr{D})|}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_O(\mathscr{D})}} : \mathscr{C}_O(\mathscr{D}) \to \mathscr{C}_O(\mathscr{D})$ is residuated. Its residual is given by*

$$({}_{\mathscr{C}_O(\mathscr{D})|}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_O(\mathscr{D})}})^{\sharp} = \mathsf{Pr}_{\mathscr{C}_O(\mathscr{D})} \circ (L_{(\widetilde{a}',\widetilde{a}'')})^{\sharp} \circ I_{|\mathscr{C}_O(\mathscr{D})}.$$

Proof. Since $(\widetilde{a}',\widetilde{a}'') \in \mathscr{C}_O(\mathscr{D}) \subset \mathscr{C}(\mathscr{D})$, it follows directly from Proposition 21 that mapping $L_{(\widetilde{a}',\widetilde{a}'')}$ defined over $\mathscr{C}(\mathscr{D})$ is residuated. Furthermore, $\mathscr{C}_O(\mathscr{D})$ being closed for $\otimes$ we have $\mathsf{Im} L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_O(\mathscr{D})}} \subset \mathscr{C}_O(\mathscr{D})$, it follows from Definition 11 and Proposition 12 that

$$\begin{aligned}({}_{\mathscr{C}_O(\mathscr{D})|}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_O(\mathscr{D})}})^{\sharp} &= (L_{(\widetilde{a}',\widetilde{a}'')} \circ I_{|\mathscr{C}_O(\mathscr{D})})^{\sharp} \circ I_{|\mathscr{C}_O(\mathscr{D})} \\ &= \mathsf{Pr}_{\mathscr{C}_O(\mathscr{D})} \circ (L_{(\widetilde{a}',\widetilde{a}'')})^{\sharp} \circ I_{|\mathscr{C}_O(\mathscr{D})}.\end{aligned}$$

Then, by considering $(\widetilde{b}',\widetilde{b}'') \in \mathscr{C}_O(\mathscr{D}) \subset \mathscr{C}(\mathscr{D})$, the greatest solution in $\mathscr{C}_O(\mathscr{D})$ of $L_{(\widetilde{a}',\widetilde{a}'')}((\widetilde{x}',\widetilde{x}''))=(\widetilde{a}',\widetilde{a}'')\otimes(\widetilde{x}',\widetilde{x}'') \preccurlyeq (\widetilde{b}',\widetilde{b}'')$ is $L^{\sharp}_{(\widetilde{a}',\widetilde{a}'')}((\widetilde{b}',\widetilde{b}'')) = (\widetilde{x}',\widetilde{x}'') = (\widetilde{a}',\widetilde{a}'') \mathbin{\backslash\!\!\!\circ} (\widetilde{b}',\widetilde{b}'') = \mathsf{Pr}_{\mathscr{C}_O(\mathscr{D})}((\widetilde{a}' \mathbin{\backslash\!\!\!\circ} \widetilde{b}', \widetilde{a}'' \mathbin{\backslash\!\!\!\circ} \widetilde{b}'')) = (\widetilde{a}' \mathbin{\backslash\!\!\!\circ} \widetilde{b}' \wedge \widetilde{a}'' \mathbin{\backslash\!\!\!\circ} \widetilde{b}'', \widetilde{a}'' \mathbin{\backslash\!\!\!\circ} \widetilde{b}'')$. □

4. Dioid and interval mathematics

Interval mathematics was pioneered by R.E. Moore as a tool for bounding and rounding errors in computer programs. Since then, interval mathematics had been developed into a general methodology for investigating numerical uncertainty in numerous problems and algorithms, and is a powerful numerical tool for calculating guaranteed bounds on functions using computers.

In Litvinov and Sobolevski (2001) the problem of interval mathematics in dioids is addressed. The authors give a weak interval extensions of dioids and show that idempotent interval mathematics appears to be remarkably simpler than its traditional analog. For example, in the traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, while idempotent interval arithmetic keeps this distributivity. Below, we state that residuation theory has a natural extension in dioid of intervals.

Definition 27. A (closed) interval in dioid $\mathscr{D}$ is a set of the form $\mathbf{x} = [\underline{x},\bar{x}] = \{t \in \mathscr{D} \mid \underline{x} \preccurlyeq t \preccurlyeq \bar{x}\}$, where $(\underline{x},\bar{x}) \in \mathscr{C}_O(\mathscr{D})$, $\underline{x}$ (respectively, $\bar{x}$) is said to be lower (respectively, upper) bound of the interval $\mathbf{x}$.

Proposition 28. *The set of intervals, denoted by $\mathrm{I}(\mathscr{D})$, endowed with two coordinate-wise algebraic operations*

$$\begin{aligned} &\mathbf{x} \bar{\oplus} \mathbf{y} = [\underline{x} \oplus \underline{y}, \bar{x} \oplus \bar{y}] \\ \text{and} \quad &\mathbf{x} \bar{\otimes} \mathbf{y} = [\underline{x} \otimes \underline{y}, \bar{x} \otimes \bar{y}] \end{aligned} \tag{11}$$

is a dioid, where the interval $\varepsilon=[\varepsilon,\varepsilon]$ (respectively, $\mathbf{e}=[e,e]$) is zero (respectively, unit) element of $\mathrm{I}(\mathscr{D})$. Moreover, the dioid $\mathrm{I}(\mathscr{D})$ is isomorphic to $\mathscr{C}_O(\mathscr{D})$.

Proof. First, $\underline{x} \oplus \underline{y} \preccurlyeq \bar{x} \oplus \bar{y}$ and $\underline{x} \otimes \underline{y} \preccurlyeq \bar{x} \otimes \bar{y}$ whenever $\underline{x} \preccurlyeq \bar{x}$ and $\underline{y} \preccurlyeq \bar{y}$, then $\mathrm{I}(\mathscr{D})$ is closed with respect to the operations $\bar{\oplus}, \bar{\otimes}$. From Definition 1, it follows directly that it is a dioid. Obviously, it is isomorphic to $\mathscr{C}_O(\mathscr{D})$ (see Proposition 23). □

Remark 29. Let $\mathscr{D}$ be a complete dioid and $\{\mathbf{x}_\alpha\}$ be an infinite subset of $\mathrm{I}(\mathscr{D})$, the infinite sum of elements of this subset is

$$\bar{\bigoplus_\alpha} \mathbf{x}_\alpha = \left[\bigoplus_\alpha \underline{x}_\alpha, \bigoplus_\alpha \bar{x}_\alpha\right].$$

Remark 30. If $\mathscr{D}$ is a complete dioid then $\mathrm{I}(\mathscr{D})$ is a complete dioid by considering Definition 29. Its top element is given by $\top = [\top, \top]$.

Note that if $\mathbf{x}$ and $\mathbf{y}$ are intervals in $\mathrm{I}(\mathscr{D})$, then $\mathbf{x} \subset \mathbf{y}$ iff $\underline{y} \preccurlyeq \underline{x} \preccurlyeq \bar{x} \preccurlyeq \bar{y}$. In particular, $\mathbf{x} = \mathbf{y}$ iff $\underline{x} = \underline{y}$ and $\bar{x} = \bar{y}$. An interval for which $\underline{x} = \bar{x}$ is called degenerate. Degenerate intervals allow to represent numbers without uncertainty. In this case we identify $\mathbf{x}$ with its element by writing $\mathbf{x} \equiv x$.

Proposition 31. *Mapping* $L_{\mathbf{a}}: \mathrm{I}(\mathscr{D}) \to \mathrm{I}(\mathscr{D}), \mathbf{x} \mapsto \mathbf{a} \bar{\otimes} \mathbf{x}$ *is residuated. Its residual is equal to* $L_{\mathbf{a}}^\sharp(\mathbf{b}) = \mathbf{a} \lcirclearrowright \mathbf{b} = [\underline{a} \lcirclearrowright \underline{b} \wedge \bar{a} \lcirclearrowright \bar{b}, \bar{a} \lcirclearrowright \bar{b}]$.

Proof. Let $\Psi: \mathscr{C}_O(\mathscr{D}) \to \mathrm{I}(\mathscr{D}), (\tilde{x}', \tilde{x}'') \mapsto [\underline{x}, \bar{x}] = [\tilde{x}', \tilde{x}'']$ be the mapping which maps an interval to an ordered pair. This mapping defines an isomorphism, since it is sufficient to handle the bounds to handle an interval. Then the result follows directly from Proposition 26. □

Remark 32. We would show in the same manner that mapping $R_{\mathbf{a}}: \mathrm{I}(\mathscr{D}) \to \mathrm{I}(\mathscr{D}), \mathbf{x} \mapsto \mathbf{x} \bar{\otimes} \mathbf{a}$ is residuated.

Remark 33. It is possible to extend the Kleene star operator over $\mathrm{I}(\mathscr{D})$ (see Example 25). Then ${}_{\mathrm{Im}\mathbf{K}}|\mathbf{K}$ is also a residuated mapping (see Corollary 15) whose residual is $({}_{\mathrm{Im}\mathbf{K}|}\mathbf{K})^\sharp = \mathrm{Id}_{|\mathrm{Im}\mathbf{K}}$. This means that $\mathbf{x} = \mathbf{a}^*$ is the greatest solution to inequality $\mathbf{x}^* = [\underline{x}^*, \bar{x}^*] \preccurlyeq \mathbf{a}^* = [\underline{a}^*, \bar{a}^*]$.

5. Interval arithmetic and TEG

It is well known that the behavior of a TEG can be expressed by linear state equations over some dioids, e.g. over dioid of formal power series with coefficients in $\overline{\mathbb{Z}}_{\max}$ and exponents in $\mathbb{Z}$ namely $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

$$X = AX \oplus BU, \tag{12}$$

$$Y = CX, \tag{13}$$

where $X \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^n$ represents the internal transitions behavior, $U \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^p$ represents the input transitions behavior, $Y \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^q$ represents the output transitions behavior, and $A \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times p}$ and $C \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{q\times n}$ represent the link between transitions.

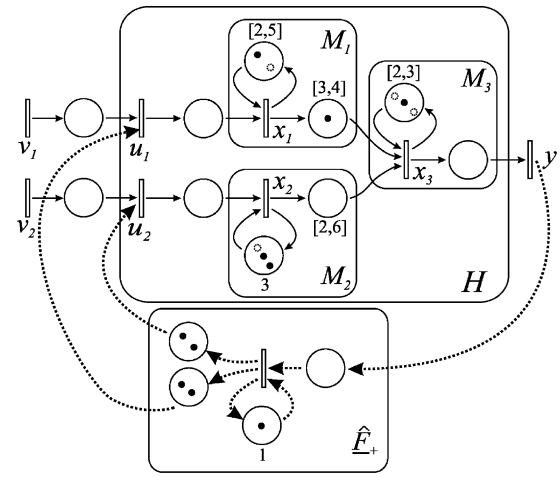

Fig. 1. An uncertain TEG with a controller (bold dotted lines).

Remark 34. A, B, C entries are periodic and causal series (i.e., rational and realizable series see Baccelli et al., 1992), and then are in subdioid $\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]$, which is the subset of causal element in $\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]$ (We refer the reader to Cohen et al. (1989) and Cottenceau et al. (2001) for a complete presentation). According to Proposition 10, the canonical injection $\mathrm{Id}_{|\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]}: \overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!] \to \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is residuated. Its residual is denoted by Pr_+ and its computation for all $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is given by

$$\mathsf{Pr}_+\left(\bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k\right) = \bigoplus_{k\in\mathbb{Z}} s_+(k)\gamma^k$$

where

$$s_+(k) = \begin{cases} s(k) & \text{if } (k, s(k)) \geqslant (0,0), \\ \varepsilon & \text{otherwise.} \end{cases}$$ □

The uncertain systems, which will be considered, are TEG where the number of tokens and time delays are only known to belong to intervals. Therefore, uncertainties can be described by intervals with known lower and upper bounds and the matrices of Eqs. (12) and (13) are such that $A \in \mathbf{A} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{n\times n}$, $B \in \mathbf{B} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{n\times p}$ and $C \in \mathbf{C} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times n}$, each entry of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are intervals with bounds in dioid $\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]$ with only non-negative exponents and integer coefficients. By Theorem 3, Eq. (12) has the minimum solution $X = A^*BU$. Therefore, $Y = CA^*BU$ and the transfer function of the system is $H = CA^*B \in \mathbf{H} = \mathbf{C}\mathbf{A}^*\mathbf{B} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times p}$, where $\mathbf{H}$ represents the interval in which the transfer function will lie for all the variations of the parameters.

Fig. 1 shows a TEG with 2 inputs and 1 output, which may represent a manufacturing system with 3 machines. Machines M_1 and M_2 produce parts assembled on machine M_3. The token in dotted line means that the resource may or may not be available to manufacture parts (e.g. a machine may be disabled for maintenance operations ...). Durations in brackets give the interval in which the temporization of the place may evolve. This may represent an operation with a processing time which is not well known (e.g. a task executed by a human, etc.). For instance, machine M_1 can manufacture 1 or 2 parts and each processing time will last between 2 and 5 time units, this leads to a parameter which evolves in interval $\mathbf{A}_{1,1}=[2\gamma^2, 5\gamma]$. The exponent in γ denotes resource number, and the coefficient depicts the processing time. Therefore, we obtain the following interval matrices:

$$\mathbf{A} = \begin{pmatrix} [2\gamma^2, 5\gamma] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \\ [\varepsilon,\varepsilon] & [3\gamma^3, 3\gamma^2] & [\varepsilon,\varepsilon] \\ [3\gamma, 4\gamma] & [2,6] & [2\gamma^3, 3\gamma] \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} [e,e] & [\varepsilon,\varepsilon] \\ [\varepsilon,\varepsilon] & [e,e] \\ [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \end{pmatrix}$$

$$\mathbf{C} = ([\varepsilon,\varepsilon]\ [\varepsilon,\varepsilon]\ [e,e]). \tag{14}$$

It follows from Theorem 3 that the transfer function H belongs to the interval matrix $\mathbf{H}$ given below. It characterizes the whole transfer functions arising from (14)

$$\mathbf{H} = \mathbf{C}\mathbf{A}^*\mathbf{B} = ([3\gamma(2\gamma^2)^*, 4\gamma(5\gamma)^*]\ [2(3\gamma^3)^*, 6(3\gamma)^*]).$$

Remark 35. We can easily check that $\mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])$ is a subdioid of $\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])$ and that the residual of the canonical injection $\mathsf{Id}_{|\mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])}$ is given by $\mathsf{IPr}_+ : \mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]) \to \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]), \mathbf{x} \mapsto \mathsf{IPr}_+(\mathbf{x}) = [\mathsf{Pr}_+(\underline{x}), \mathsf{Pr}_+(\bar{x})]$.

6. Robust feedback controller synthesis

We consider the behavior of a p-input q-output TEG by a state representation such as (12) and (13). We focus here on output feedback controller synthesis denoted by F, added between the output Y and the input U of the system (see Fig. 2). Therefore, the process input satisfies $U = V \oplus FY$, and the output is described by $Y = H(V \oplus FY)$. According to Theorem 3, the closed-loop transfer relation (depending on F) is then equal to $Y = (HF)^*HV$, where $H \in \mathbf{H}$ is the uncertain system transfer.

The objective of the robust feedback synthesis is to compute a controller F which is realizable (i.e., F is periodic and causal) and which imposes a desired behavior (a specification) to the uncertain system. The first problem addressed here, consists in computing the greatest interval (in the sense of the order relation $\preccurlyeq_{\mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{p\times q}}$), denoted by $\hat{\mathbf{F}}$, which guarantees that the behavior of the closed-loop system is lower than $\mathbf{G}_{\mathrm{ref}} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times p}$ (a specification which is defined as an interval of causal and periodic elements) for all $H \in \mathbf{H}$. Formally the problem consists in computing the upper bound of the following set:

$$\{\mathbf{F} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{p\times q} | (\mathbf{H}\mathbf{F})^*\mathbf{H} \preccurlyeq \mathbf{G}_{\mathrm{ref}}\}. \tag{15}$$

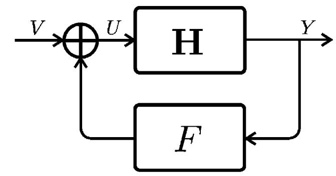

Fig. 2. An uncertain system with a feedback controller.

Proposition 36 shows that this problem admits a solution for some reference models.

Proposition 36. *Let* $M_{\mathbf{H}} : \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{p\times q} \to \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times p}$, $\mathbf{F} \mapsto (\mathbf{H}\mathbf{F})^*\mathbf{H}$ *be a mapping. Let us consider the following sets*:

$$\mathbf{G}_1 = \{\mathbf{G} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times p} \quad | \quad \exists \mathbf{D} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times q} \text{ such that } \mathbf{G} = \mathbf{D}^*\mathbf{H}\},$$

$$\mathbf{G}_2 = \{\mathbf{G} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{q\times p} \quad | \quad \exists \mathbf{D} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{p\times p} \text{ such that } \mathbf{G} = \mathbf{H}\mathbf{D}^*\},$$

If $\mathbf{G}_{\mathrm{ref}} \in \mathbf{G}_1 \cup \mathbf{G}_2$, *there exists a greatest* $\mathbf{F} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{p\times q}$ *such that* $M_{\mathbf{H}}(\mathbf{F}) \preccurlyeq \mathbf{G}_{\mathrm{ref}}$, *and it is given by*

$$\hat{\mathbf{F}} = \oplus\{\mathbf{F} \in \mathrm{I}(\overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!])^{p\times q} \,|\, (\mathbf{H}\mathbf{F})^*\mathbf{H} \preccurlyeq \mathbf{G}_{\mathrm{ref}}\}$$
$$= \mathbf{H}\,\overline{\circ\!\!\!\backslash}\,\mathbf{G}_{\mathrm{ref}}\,\overline{\circ\!\!\!/}\,\mathbf{H}. \tag{16}$$

Proof. Follows directly from Proposition 16. □

Below, we consider the robust controllers set, denoted by $\mathscr{F}$, such that the transfer of the closed-loop system be in $\mathbf{G}_{\mathrm{ref}}$ for all $H \in \mathbf{H}$

$$\mathscr{F} = \{F \in \overline{\mathbb{Z}}^+_{\max}[\![\gamma]\!]^{p\times q} | (\mathbf{H}F)^*\mathbf{H} \subset \mathbf{G}_{\mathrm{ref}}\}.$$

Proposition 37. *If* $\mathbf{G}_{\mathrm{ref}} \in \mathsf{Im}M_{\mathbf{H}}$, *then* $\hat{\mathbf{F}} \subset \mathscr{F}$.

Proof. If $\mathbf{G}_{\mathrm{ref}} \in \mathsf{Im}M_{\mathbf{H}}$, then $M_{\mathbf{H}}(\hat{\mathbf{F}}) = \mathbf{G}_{\mathrm{ref}}$ due to Corollary 17, thus $(\mathbf{H}\hat{\mathbf{F}})^*\mathbf{H} \subset \mathbf{G}_{\mathrm{ref}}$. Obviously, this is equivalent to $\forall F \in \hat{\mathbf{F}}, (\mathbf{H}F)^*\mathbf{H} \subset \mathbf{G}_{\mathrm{ref}}$, which leads to the result. □

Proposition 37 shows that if $\mathbf{G}_{\mathrm{ref}} \in \mathsf{Im}M_{\mathbf{H}}$ each feedback controller $F \in \hat{\mathbf{F}}$ is also in $\mathscr{F}$. From a practical point of view this means that for all number of tokens and holding time belonging to the given intervals the closed-loop system will be in the specification interval.

Remark 38. From a computational point of view we have

$$\hat{\mathbf{F}} = \mathbf{H}\,\overline{\circ\!\!\!\backslash}\,\mathbf{G}_{\mathrm{ref}}\,\overline{\circ\!\!\!/}\,\mathbf{H} = [\underline{H}, \bar{H}]\,\overline{\circ\!\!\!\backslash}\,[\underline{G}_{\mathrm{ref}}, \bar{G}_{\mathrm{ref}}]\,\overline{\circ\!\!\!/}\,[\underline{H}, \bar{H}]$$
$$= [\mathsf{Pr}_+(\underline{H} \circ\!\!\!\backslash\, \underline{G}_{\mathrm{ref}}) \wedge \mathsf{Pr}_+(\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\mathrm{ref}}), \mathsf{Pr}_+(\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\mathrm{ref}})]\,\overline{\circ\!\!\!/}\,[\underline{H}, \bar{H}]$$
$$= [\mathsf{Pr}_+(\underline{H} \circ\!\!\!\backslash\, \underline{G}_{\mathrm{ref}} \wedge \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\mathrm{ref}}), \mathsf{Pr}_+(\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\mathrm{ref}})]\,\overline{\circ\!\!\!/}\,[\underline{H}, \bar{H}]$$

$$= [\mathsf{Pr}_+((\mathsf{Pr}_+(\underline{H} \circ\!\!\!\backslash\, \underline{G}_{\text{ref}} \wedge \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}})) \circ\!\!\!/\, \underline{H}) \wedge \mathsf{Pr}_+(\bar{H} \circ\!\!\!\backslash\, \mathsf{Pr}_+ (\bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H})), \mathsf{Pr}_+(\bar{H} \circ\!\!\!\backslash\, \mathsf{Pr}_+(\bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H}))]$$

$$= [\mathsf{Pr}_+(\underline{H} \circ\!\!\!\backslash\, \underline{G}_{\text{ref}} \circ\!\!\!/\, \underline{H} \wedge \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \underline{H} \wedge \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H}), \mathsf{Pr}_+(\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H})] \quad \text{follows from (7).}$$

The last equation may be simplified since $(\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}}) \circ\!\!\!/\, \underline{H} \succcurlyeq (\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}}) \circ\!\!\!/\, \bar{H}$ due to the antitony of mapping $a \circ\!\!\!/\, x$ (i.e., $x_1 \succcurlyeq x_2 \Rightarrow a \circ\!\!\!/\, x_1 \preccurlyeq a \circ\!\!\!/\, x_2$), then $\bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \underline{H} \wedge \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H} = \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H}$. Therefore,

$$\hat{\mathbf{F}} = [\underline{\hat{F}}, \overline{\hat{F}}] = \mathbf{H} \overline{\circ\!\!\!\backslash}\, \mathbf{G}_{\text{ref}} \overline{\circ\!\!\!/}\, \mathbf{H}$$

$$= \mathsf{IPr}_+([\underline{H} \circ\!\!\!\backslash\, \underline{G}_{\text{ref}} \circ\!\!\!/\, \underline{H} \wedge \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H}, \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H}]).$$

Corollary 39. *If $\mathbf{G}_{\text{ref}} \in \mathsf{Im} M_{\mathbf{H}}$, then the upper bound of the interval $\hat{\mathbf{F}}$, denoted by $\overline{\hat{F}}$, is the upper bound of the set $\mathscr{F}$.*

Proof. Proposition 37 yields $(\mathbf{H}\hat{\mathbf{F}})^*\mathbf{H} = \mathbf{G}_{\text{ref}}$, i.e., $[(\underline{H}\underline{\hat{F}})^*\underline{H}, (\bar{H}\overline{\hat{F}})^*\bar{H}] = [\underline{G}_{\text{ref}}, \bar{G}_{\text{ref}}]$. Furthermore, $\mathbf{G}_{\text{ref}} \in \mathsf{Im} M_{\mathbf{H}}$ implies that there exists F such that $\bar{G}_{\text{ref}} = (\bar{H}F)^*\bar{H}$, i.e., $\bar{G}_{\text{ref}} \in \mathsf{Im} M_H$ then, due to Corollary 17, $\overline{\hat{F}} = \bar{H} \circ\!\!\!\backslash\, \bar{G}_{\text{ref}} \circ\!\!\!/\, \bar{H}$ is the greatest feedback such that $\bar{G}_{\text{ref}} = (\bar{H}\overline{\hat{F}})^*\bar{H}$, thus the greatest in $\mathscr{F}$. □

7. Example: output feedback synthesis

We describe a complete synthesis of a controller for the uncertain TEG depicted with solid black lines in Fig. 1. The reference model chosen is

$$G_{\text{ref}} = \left(\mathbf{H}\begin{pmatrix} \gamma^2 \\ \gamma^2 \end{pmatrix}\right)^* \mathbf{H}$$

$$= \begin{pmatrix} [3\gamma \oplus 5\gamma^3(1\gamma)^*, 4\gamma(5\gamma)^*] \\ [2 \oplus (4\gamma^2)(1\gamma)^*, 6 \oplus 9\gamma \oplus 12\gamma^2 \oplus 15\gamma^3 \oplus 18\gamma^4 \oplus 21\gamma^5 \oplus 25\gamma^6(5\gamma)^*] \end{pmatrix}^t.$$

This specification means that no more than two tokens can input in the TEG at the same moment. We refer the reader to Cottenceau et al. (2001), Cottenceau, Lhommeau, Hardouin, and Boimond (2003) and Maia, Hardouin, Santos-Mendes, and Cottenceau (2003) for a discussion about reference model choice. We aim at computing the greatest interval of robust controllers which keep the same objective.

According to Proposition 36 and solution (16), the controller is obtained by computing $\mathbf{H} \overline{\circ\!\!\!\backslash}\, \mathbf{G}_{\text{ref}} \overline{\circ\!\!\!/}\, \mathbf{H}$. Therefore, we obtain (see (38))

$$\hat{\mathbf{F}} = \begin{pmatrix} [\gamma^2(1\gamma)^*, \gamma^2(5\gamma)^*] \\ [\gamma^2(1\gamma)^*, \gamma^2 \oplus 3\gamma^3 \oplus 6\gamma^4 \oplus 9\gamma^5 \oplus 13\gamma^6(5\gamma)^*] \end{pmatrix}.$$

All controllers in this interval allow to achieve the objective given in Proposition 37. For the realization, it is necessary to choose one feedback in the interval $\hat{\mathbf{F}}$. Its upper bound, $\overline{\hat{F}}$, leads to a closed-loop behavior which is in $[(\underline{H}\overline{\hat{F}})^*\underline{H}, (\bar{H}\overline{\hat{F}})^*\bar{H}]$, i.e. an interval which have the same upper bound than $\mathbf{G}_{\text{ref}}$. The lower bound, $\underline{\hat{F}}$, leads to a closed-loop behavior which is in $[(\underline{H}\underline{\hat{F}})^*\underline{H}, (\bar{H}\underline{\hat{F}})^*\bar{H}]$, i.e. an interval which have the same lower bound than $\mathbf{G}_{\text{ref}}$. Consequently the choice of the controller may be done by considering the desired location in $\mathbf{G}_{\text{ref}}$ of the interval depicting the closed-loop system. Here we choose $\underline{\hat{F}}$ which allows to delaying as much as possible the input of tokens while preserving the possibility to match the lower bound of $\mathbf{G}_{\text{ref}}$, i.e. the fastest behavior allowed by the specification. Fig. 1 shows one realization of this controller (bold dotted lines), which is equal to

$$\underline{\hat{F}} = (\gamma^2(1\gamma)^* \ \gamma^2(1\gamma)^*)^t.$$

Remark 40. The reader can find Scilab and $C++$ toolbox in order to handle periodic series (see SW2001, 2001). The script allowing to compute the controller of the illustration are also available.

8. Conclusion

In this paper we assumed that the TEG includes some parametric uncertainties in a bounded context. We have given a robust feedback controller synthesis which ensures that the closed-loop system transfer is in a given interval for all feasible values for the parameters. The next step is to extend this work to other control structures such as the one given in Maia et al. (2003). The traditional interval theory is very effective for parameter estimation, it would be interesting to apply the results of this paper to the TEG parameter estimation such as the one studied in Jaulin, Boimond, and Hardouin (1999).

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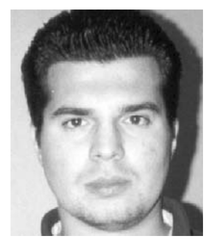

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