Inner and outer approximations of probabilistic sets

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ABSTRACT

This paper proposes a set-membership method to characterize a *probabilistic set*, i.e., a set enclosing the true value for the parameter vector of a parametric system with a given probability. The approach assumes that all errors are independent and an interval for the error is known. To each error interval, a probability to be an outlier is provided. It is shown that characterizing the probabilistic set is a set inversion problem. The main contribution of the paper is to provide a method to characterize the inner part of the probabilistic set. As an illustration, an application to the static localization of a mobile robot is considered.

INTRODUCTION

Parameter set estimation aims at characterizing a set which encloses an unknown parameter vector $\mathbf{p}$ of a parametric model $\mathcal{M}(\mathbf{p})$ from a data vector $\mathbf{y}$ collected on the system (26). In the context of bounded-error estimation, the measurement error is assumed to be bounded and characterizing the posterior feasible set amounts to solving a set inversion problem for which interval methods (23) have been shown to be particularly efficient (5; 25), even when the model is nonlinear. In a probabilistic context, the measurement error is not anymore described by membership intervals, but by *probability density functions* (PDF). When some prior PDF for $\mathbf{p}$ is available, the Bayes rule makes it possible to obtain the posterior PDF. The set to be estimated becomes the *minimal volume credible set* (4) which corresponds to the minimal volume set enclosing the associate random vector with a given probability. Unfortunately, this problem cannot be cast into a set inversion problem and the use of interval methods, which can still be useful to characterize credible sets (15), are limited to small dimensional problems with few data.

In this paper, we follow the *probabilistic-set approach* (16; 17), which does not assume that some prior PDF are available for the vector to be estimated. Instead, we fix a given probability α which corresponds to a tiny positive number. Then, we choose a collection of rare events for the error such that the prior probability of occurrence of one of these events is equal to α. We assume that the rare events will never occur and we solve the associated set inversion problem using an interval approach. Interval methods have already been combined with uncertainty theories (18; 10; 3; 24; 27; 28) in order to solve estimation problems (1). The main difference between our approach and the above mentioned papers is that here, we solve a traditional probabilistic estimation problem using interval tools and thus our approach is fully consistent with traditional probabilistic estimation. The problem to be considered is represented by an *error model equation*

$$\mathbf{e} = \mathbf{f}(\mathbf{y}, \mathbf{p}) = \mathbf{f}_{\mathbf{y}}(\mathbf{p}),$$

where $\mathbf{e} \in \mathbb{R}^m$ is the error vector, $\mathbf{y} \in \mathbb{R}^m$ is the collected data vector (with the same dimension as $\mathbf{e}$) and $\mathbf{p} \in \mathbb{R}^n$ is the parameter vector to be estimated. The parameter estimation problem amounts to finding $\mathbf{p}$ from $\mathbf{y}$ and some assumptions on the error $\mathbf{e}$. The principle of the probabilistic-set approach is to partition the error space into two subsets: a subset $\mathbb{E}$ on which we bet that the error vector $\mathbf{e}$ will belong and its complementary set $\overline{\mathbb{E}}$. The prior probability of the event $\mathbf{e} \in \mathbb{E}$ is $1 - \alpha$. The set $\mathbb{E}$ is chosen such that α is almost equal to 0 and we are almost certain that $\mathbf{e} \in \mathbb{E}$. The event $\mathbf{e} \in \overline{\mathbb{E}}$ is considered as rare (it has a probability α) and we bet that it will not occur. Once the data vector $\mathbf{y}$ is collected, we compute the *probabilistic set* $\mathbb{P} = \mathbf{f}_{\mathbf{y}}^{-1}(\mathbb{E})$. If $\mathbb{P}$ is not empty, then we still bet that the rare event did not occur and we conclude that $\mathbf{p} \in \mathbb{P}$ with a probability of $1 - \alpha$. If $\mathbb{P} = \emptyset$, then we conclude that the rare event occurred.

The main contribution of the paper is to propose a contractor-based approach to compute an inner approximation of the probabilistic set $\mathbb{P}$ (characterizing an outer approximation has already been done in this context (9; 2)). Moreover, contractor techniques make it possible to be more efficient than a pure branch and bound interval algorithm such as presented in (16).

SET INVERSION AND PROBABILITIES

In this section, we show how the probabilistic-set approach can be used for estimation problems where outliers are involved (see e.g. (19)). Consider again the error model $\mathbf{e} = \mathbf{f}_{\mathbf{y}}(\mathbf{p})$. The ith component $e_i, i \in \{1, \ldots, m\}$ of $\mathbf{e}$ is said to be an *inlier* if $e_i \in [e_i]$ and an outlier otherwise. Assume that the probability for e_i to be an inlier is π (it does not depend on i) and assume also that all e_i's are independent. The probability of having exactly k inliers among m is $\frac{m!}{k!(m-k)!}\pi^k.(1-\pi)^{m-k}$, which is a binomial distribution. As a consequence, the probability of having strictly more than q outliers is

$$\gamma(q, m, \pi) = \sum_{k=0}^{m-q-1} \frac{m!}{k!(m-k)!}\pi^k.(1-\pi)^{m-k}. \tag{1}$$

Define the set $\mathbb{E}_i = \{\mathbf{e} \in \mathbb{R}^m \mid e_i \in [e_i]\}$ of all error vectors consistent with the ith data. Denote by $\mathbb{E}^{\{q\}}$ the set of all $\mathbf{e} \in \mathbb{R}^m$ such that the number of outliers is smaller (or equal) than q. For instance if $m = 3$, we have

$$\begin{array}{lll} \mathbb{E}^{\{0\}} & = & \mathbb{E}_1 \cap \mathbb{E}_2 \cap \mathbb{E}_3 = [e_1] \times [e_2] \times [e_3] \\ \mathbb{E}^{\{1\}} & = & (\mathbb{E}_1 \cap \mathbb{E}_2) \cup (\mathbb{E}_2 \cap \mathbb{E}_3) \cup (\mathbb{E}_1 \cap \mathbb{E}_3) \\ \mathbb{E}^{\{2\}} & = & \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_3 \\ \mathbb{E}^{\{3\}} & = & \mathbb{R}^3. \end{array} \quad (2)$$

We have

$$\begin{array}{lll} \text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{q\}}\right) = 1 - \gamma(q, m, \pi) & \text{with} & \mathbb{P}^{\{q\}} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}^{\{q\}}\right) \\ \text{prob}\left(\mathbf{p} \in \overline{\mathbb{P}^{\{q\}}}\right) = \gamma(q, m, \pi) & \text{with} & \overline{\mathbb{P}^{\{q\}}} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\overline{\mathbb{E}^{\{q\}}}\right). \end{array} \quad (3)$$

As a consequence, the set $\mathbb{P}^{\{q\}}$ and $\overline{\mathbb{P}^{\{q\}}}$ are inverse of sets and contractors for these two complementary sets can be provided.

RELAXED INTERSECTION

This section recalls the notion of relaxed intersection and shows that the complementary set of the relaxed intersection of m sets $\mathbb{X}_i$ can also be expressed as the relaxed intersection of the m complementary sets $\overline{\mathbb{X}}_i$. This new result will be used to build an inner approximation for the probabilistic set. Consider m sets $\mathbb{X}_1, \dots, \mathbb{X}_m$ of $\mathbb{R}^n$. The q-relaxed intersection (14) $\bigcap^{\{q\}} \mathbb{X}_i$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ which belong to all $\mathbb{X}_i$'s, except q at most. We have

$$\mathbf{x} \in \bigcap^{\{q\}} \mathbb{X}_i \Leftrightarrow \# \{i | \mathbf{x} \in \mathbb{X}_i\} \geq m - q, \quad (4)$$

where $\#\mathbb{A}$ denotes the cardinal of the finite set $\mathbb{A}$.

Example 1. If the $\mathbb{X}_i$'s are the intervals of $\mathbb{R}$ given by $\mathbb{X}_1 = [1,4], \mathbb{X}_2 = [2,4], \mathbb{X}_3 = [2,7], \mathbb{X}_4 = [6,9], \mathbb{X}_5 = [3,4]$ and $\mathbb{X}_6 = [3,7]$, we have $\bigcap^{\{0\}} \mathbb{X}_i = \emptyset$, $\bigcap^{\{1\}} \mathbb{X}_i = [3,4]$, $\bigcap^{\{2\}} \mathbb{X}_i = [3,4]$, $\bigcap^{\{3\}} \mathbb{X}_i = [2,4] \cup [6,7]$, $\bigcap^{\{4\}} \mathbb{X}_i = [2,7]$, $\bigcap^{\{5\}} \mathbb{X}_i = [1,9]$, $\bigcap^{\{6\}} \mathbb{X}_i = \mathbb{R}$. ■

An efficient algorithm ($n \log n$) to compute the relaxed intersection of n intervals has been proposed by Marzullo in his thesis (21).

Example 2. Consider the example of Equation (2), we have $\mathbb{E}^{\{q\}} = \bigcap^{\{q\}} \mathbb{E}_i$. ■

We can define more formally, the relaxed intersection and the relaxed union as follows:

$$\begin{aligned}\bigcap^{\{q\}}\mathbb{X}_i &= \bigcup_{\{\sigma_1,\dots,\sigma_{m-q}\}\subset\{1,\dots,m\}} \mathbb{X}_{\sigma_1}\cap\dots\cap\mathbb{X}_{\sigma_{m-q}} \\ \bigcup^{\{q\}}\mathbb{X}_i &= \bigcap_{\{\sigma_1,\dots,\sigma_{m-q}\}\subset\{1,\dots,m\}} \mathbb{X}_{\sigma_1}\cup\dots\cup\mathbb{X}_{\sigma_{m-q}}.\end{aligned} \tag{5}$$

The following proposition is a generalization of the De Morgan's low, in the context of relaxed intersection.

Proposition. (De Morgan's law). We always have

$$\begin{array}{lrcl}\text{(i)} & \overline{\bigcap^{\{q\}}\mathbb{X}_i} &=& \bigcup^{\{q\}}\overline{\mathbb{X}_i} \\ \text{(ii)} & \overline{\bigcup^{\{q\}}\mathbb{X}_i} &=& \bigcap^{\{q\}}\overline{\mathbb{X}_i}.\end{array} \tag{6}$$

Proof. Let us first prove (i)

$$\begin{aligned}\overline{\bigcap^{\{q\}}\mathbb{X}_i} &\overset{(5)}{=} \overline{\bigcup_{\{\sigma_1,\dots,\sigma_{m-q}\}}\mathbb{X}_{\sigma_1}\cap\dots\cap\mathbb{X}_{\sigma_{m-q}}} = \bigcap_{\{\sigma_1,\dots,\sigma_{m-q}\}}\overline{\mathbb{X}_{\sigma_1}\cap\dots\cap\mathbb{X}_{\sigma_{m-q}}} \\ &= \bigcap_{\{\sigma_1,\dots,\sigma_{m-q}\}}\overline{\mathbb{X}_{\sigma_1}}\cup\dots\cup\overline{\mathbb{X}_{\sigma_{m-q}}} \overset{(5)}{=} \bigcup^{\{q\}}\overline{\mathbb{X}_i}.\end{aligned}$$

A similar proof can be written to prove (ii). ■

Proposition (Dual rule). We have

$$\bigcap^{\{q\}}\mathbb{X}_i = \bigcup^{\{m-q-1\}}\mathbb{X}_i. \tag{7}$$

Proof. We have

$$\begin{aligned}\bigcup^{\{m-q-1\}}\mathbb{X}_i &\overset{(6)}{=} \overline{\bigcap^{\{m-q-1\}}\overline{\mathbb{X}_i}} \overset{(4)}{=} \overline{\left\{\mathbf{x}\mid\#\left\{i|\mathbf{x}\in\overline{\mathbb{X}_i}\right\}\geq m-(m-q-1)\right\}} \\ &= \left\{\mathbf{x}\mid\#\left\{i|\mathbf{x}\in\overline{\mathbb{X}_i}\right\}<q+1\right\}.\end{aligned}$$

Now, since $\#\left\{i|\mathbf{x}\in\overline{\mathbb{X}_i}\right\}+\#\left\{i|\mathbf{x}\in\mathbb{X}_i\right\}=m$, we get

$$\begin{aligned}\bigcup^{\{m-q-1\}}\mathbb{X}_i &= \{\mathbf{x}\mid m-\#\{i|\mathbf{x}\in\mathbb{X}_i\}<q+1\} = \{\mathbf{x}\mid\#\{i|\mathbf{x}\in\mathbb{X}_i\}>m-q-1\} \\ &= \{\mathbf{x}\mid\#\{i|\mathbf{x}\in\mathbb{X}_i\}\geq m-q\} \overset{(4)}{=} \bigcap^{\{q\}}\mathbb{X}_i. \blacksquare\end{aligned}$$

Corollary. From the De Morgan's law and the dual rules, we get

$$\overline{\bigcap^{\{q\}} \mathbb{X}_i} \overset{(7)}{=} \overline{\bigcup^{\{m-q-1\}} \mathbb{X}_i} \overset{(6)}{=} \bigcap^{\{m-q-1\}} \overline{\mathbb{X}_i}. \tag{8}$$

As it will be shown in the next section, this corollary will allow us to obtain an inner approximation of the posterior feasible set.

ROBUST SET INVERSION

A contractor $\mathcal{C}_{\mathbb{X}}$ associated with a set $\mathbb{X}$ is an operator which contracts a box $[\mathbf{x}]$ of $\mathbb{R}^n$ without removing any point in $\mathbb{X}$, i.e.,

$$\begin{cases} \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance),} \\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness).} \end{cases}$$

Contractor algebra (6) makes it possible to extend to contractors most of the operations that can be done for sets (such as intersection, union, relaxed intersection (14), inverse by a function (7), ...). For instance, the contractors associated to $\mathbb{P}^{\{q\}}$ and $\overline{\mathbb{P}}^{\{q\}}$, as defined by Equations (3), are defined by

$$\mathcal{C}_{\mathbb{P}^{\{q\}}} = \mathbf{f}_y^{-1}(\bigcap^{\{q\}} \mathcal{C}_{\mathbb{E}_i}) \text{ and } \mathcal{C}_{\overline{\mathbb{P}^{\{q\}}}} = \mathbf{f}_y^{-1}(\bigcap^{\{m-q-1\}} \mathcal{C}_{\overline{\mathbb{E}_i}}),$$

where $\mathcal{C}_{\mathbb{E}_i}$ and $\mathcal{C}_{\overline{\mathbb{E}_i}}$ are the atomic contractors associated to the sets $\mathbb{E}_i$ and $\overline{\mathbb{E}_i}$. The right hand expression is a direct consequence of (8). The contractors $\mathcal{C}_{\mathbb{P}^{\{q\}}}$ and $\mathcal{C}_{\overline{\mathbb{P}^{\{q\}}}}$ are called inner and outer contractors, respectively. A paver using $\mathcal{C}_{\mathbb{P}^{\{q\}}}$ and $\mathcal{C}_{\overline{\mathbb{P}^{\{q\}}}}$ can then be used to obtain inner and outer approximations for the probabilistic sets $\mathbb{P}^{\{q\}}$.

APPLICATION TO MOBILE ROBOT LOCALIZATION

Localization aims at estimating the position of a robot from a set of measurements performed by the robot. This problem can be cast into a parameter estimation problem (13) where the parameters correspond to the position of the robot. Interval analysis combined with probabilistic techniques has already been considered to deal with localization problem (9; 11; 12) . Here, we use the approach developed in this paper to obtain an inner and an outer approximation of all consistent positions for a robot in a probabilistic context.

Test case 1. A robot measures its own distance to three beacons. The intervals corresponding to the distances and the coordinates of the beacons are given by the

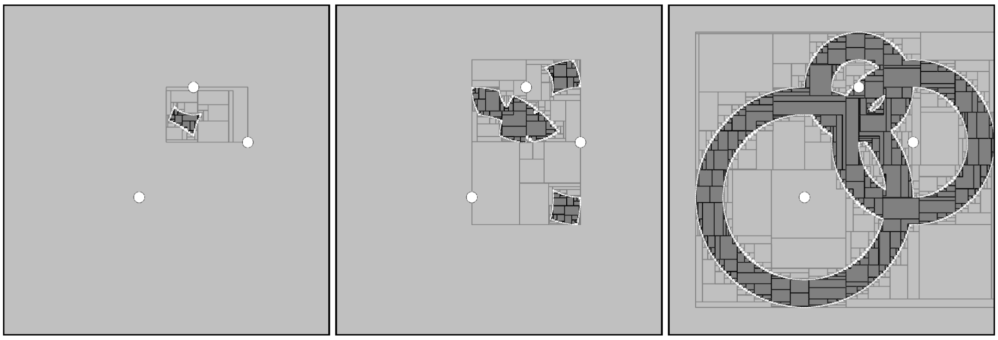

Figure 1. Probabilistic sets $\mathbb{P}^{\{q\}}$ obtained $q = 0, 1, 2$. The frame box is $[-6, 6]^3$. The three white disks represent the beacons.

following table.

beacons	x_i	y_i	$[d_i]$
1	1	3	$[1, 2]$
2	3	1	$[2, 3]$
3	-1	-1	$[3, 4]$

The collected intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$. The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5] \right\}$$

where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$. For $q = 0, 1, 2$, the algorithm SIVIA provides the probabilistic sets $\mathbb{P}^{\{q\}}$ represented on Figure 1. From Equation (1), we get $\text{prob}(\mathbf{p} \in \mathbb{P}^{\{0\}}) = 0.729$, $\text{prob}(\mathbf{p} \in \mathbb{P}^{\{1\}}) = 0.972$ and $\text{prob}(\mathbf{p} \in \mathbb{P}^{\{2\}}) = 0.999$. Note that Equation (8), which corresponds to the main contribution of this paper, made it possible to obtain the inner approximations represented by the darkgrey areas. The lightgrey zone which is outside $\mathbb{P}^{\{q\}}$ has been found using classical contractor techniques (see e.g. (9)).

Test case 2. We now consider the static wheeled robot represented on Figure 2. Note that indoor localization with a laser using the relaxed intersection and interval methods has already been considered by several authors (see e.g. (20)). The robot is equipped with a laser rangefinder (here an Hokuyo URG-04LX-UG01) and a compass. It moves inside a world made with four segments forming a 2m$\times$3m rectangle and one disk. On an experiment, that has been done in our robotics lab at ENSTA, the rangefinder was able to collect 143 distances as illustrated by Figure 3. The accuracy of the rangefinder is taken as 10cm, mainly in order to take into account the inaccuracy of the map. On our experiment, nine of 143 distances are outliers (but of course, this number is unknown). The probability of having one inlier for a given distance measurement is taken here as $\pi = 0.95$. Let us assume that we have no more than $q = 16$ outliers. From (1), the

Figure 2. Robot used to illustrate the localization method

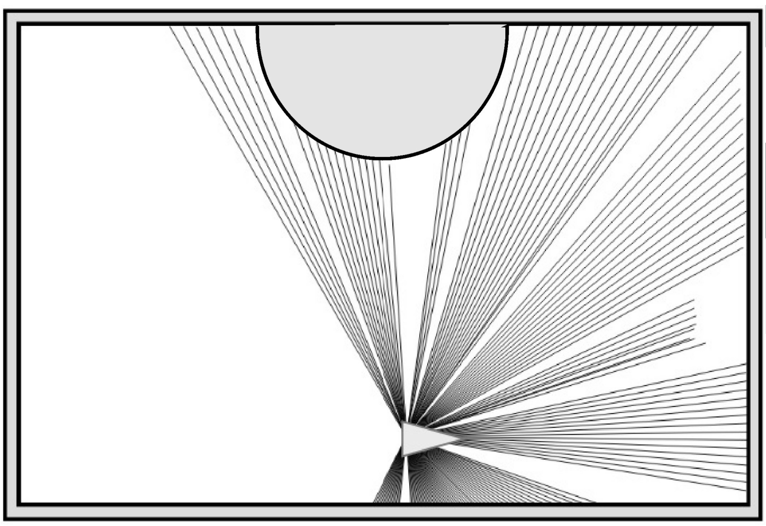

Figure 3. Distances collected by the rangefinder

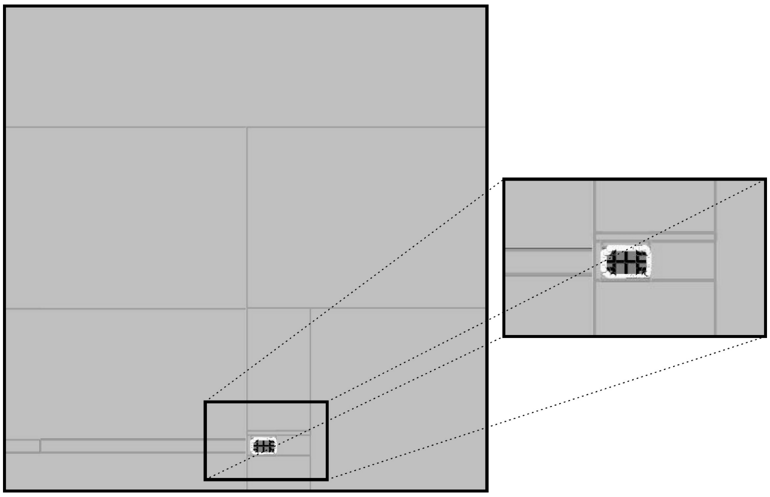

Figure 4. Probabilistic set obtained using a set inversion algorithm for $q = 16$. It contains the position for the robot with a probability 0.99915.

probability of having wrong assumption is

$$\begin{aligned}\alpha &= \gamma(q,m,\pi) = \sum_{k=0}^{m-q-1} \frac{m!}{k!\,(m-k)!}\pi^k \cdot (1-\pi)^{m-k} \\ &= \sum_{k=0}^{143-16-1} \frac{143!}{k!\,(143-k)!} 0.95^k \cdot 0.05^{143-k} \simeq 8.46 \times 10^{-4}.\end{aligned}$$

Note that this probability is reliable if the independence assumption is valid, which is not the case in our experiment. Assume here that the heading θ is measured with a good accuracy. The location of the robot $\mathbf{p} = (x,y)^{\mathrm{T}}$ is unknown and is estimated using the method presented in this paper. The resulting subpaving is depicted on Figure 4. The associated probabilistic set $\mathbb{P}^{\{16\}}$ encloses the true value for the position of the robot with a probability $1 - \alpha = 0.99915$.

CONCLUSION

In this paper, we have presented an approach for parameter estimation which combines interval propagation methods with a probabilistic modelization of uncertainty. The main idea is to transform a probabilistic problem into a set inversion problem. The resulting solution set, called *probabilistic set,* encloses the true value for the parameter vector with a probability which can be rigorously computed. The approach is able compute an inner and an outer approximation of probabilistic sets. The main contribution of the paper is to generalize the De Morgan's low in order to derive an inner contractor for the probabilistic set. The feasibility of the approach has been illustrated on a static localization problem (which corresponds to a specific nonlinear parameter estimation problem).

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