Fuzzy Matrix Contractor based Approach for Localization of Robots

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Abstract Localization consists of finding the pose of some robots with respect to its position and orientation. In case of localization of a group of robots over a planar surface, each robot is linked with other robots using constraints that may be considered in term of matrix equations. As such, this chapter deals with the localization of group of robots using angle and distance constraints associated with fuzzy matrix contractors. Matrix contractors based on Azimuth-Distance and Bearing-Distance constraints help efficient propagation of fuzzy uncertainties through a group of robots for localization purpose when no absolute frame is present. Finally, various group of robots have been considered for the verification of proposed contractors viz. azimuth, distance, azimuth-distance and bearing-distance contractors using Gaussian fuzzy uncertainty.

Keywords Localization, Pose estimation, Angle constraints, Angle-Distance localization, Matrix contractors, Azimuth angle, Bearing angle, Gaussian fuzzy number.

1 Introduction

Mobile robotics (Cook [1]; Jaulin [2]; Dudek and Jenkin [3]) help in navigation of dynamic robots within a frame of reference. Navigation helps a robot to navigate within its environment subject to external barriers and environmental conditions. Generally, navigation comprises of three fundamental problems (Nehmzow [4]) viz. self-localization, path planning and map-building. Further, Nehmzow [4] gave a detailed discussion regarding robot hardware, robot learning and navigation**.**

Generally, robots are determined using mechanical systems and the major problem in navigation consists of localization and mapping where, localization refers to the estimation of current position of the robot and mapping refers to the modeling of the environment. Simultaneous Localization And Mapping (SLAM) consists of building a map or updating the unknown environment of robot along with simultaneous determination of its location. Bailey and Durrant-Whyte [5] discussed Bayesian formulation of SLAM in terms of absolute or relative landmark locations. Further, the computational complexity has been studied through various approaches viz. linear-time state augmentation, sparsification, partitioned updating and sub-mapping. Basically, localization consists of finding the pose of robot with respect to its position (co-ordinates) and orientation within a given or unknown frame. Sometimes, localization also consists of estimation of robot's current location within the same frame.

Let us consider a robot $\boldsymbol{R}$ governed by the state equations,

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= u_1 \\ \dot{v} &= u_2 \end{aligned} \qquad (1)$$

where, (x, y) is the position of the robot, v is the speed of the robot and θ is the orientation of the robot. Then, the localization problem of the robot (1) will be the estimation of state or pose (x, y, θ). In case of a robot $\boldsymbol{R_1}$ as depicted in Fig. 1, the pose (x_1, y_1, θ_1) will be estimation of position (x_1, y_1) and orientation θ_1. As such, the problem of localization may also be referred as state or pose estimation problem.

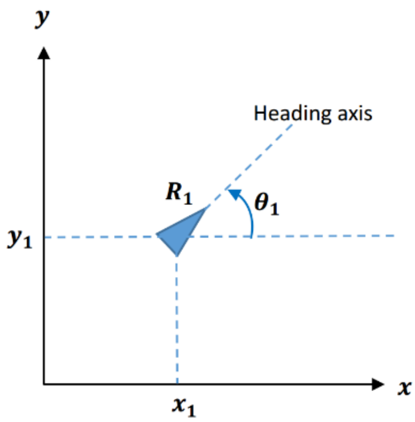


Fig. 1. Pose (x_1, y_1, θ_1) of a robot $\boldsymbol{R}_1$

Robots are often equipped with sensors such as compass, Global Positioning System (GPS), sonar, camera etc. Drumheller [6] proposed a method based on rangefinder (Polaroid Ultrasonic Rangefinder) using a sonar for estimation of 2-dimensional position and orientation of a mobile robot. Meizel [7] presented a method for solving initial localization problem using set-membership estimation. The advantage of the method presented by [7] is that it is robust to outliers and deals with nonlinear observation models equipped with sensors. There exist various other types of localization techniques (Jaulin [2]) viz. goniometric localization, multilateration, angle localization, distance localization etc.

Generally, for localization the measurements or observations using sensors, compass, cameras etc. are considered as crisp (exact) values (Halmos [8]). Measurements generally considered are uncertain with probability distribution errors or intervals. But in practice, the direction measured by the compass, angles measured via goniometric sensors (like cameras or microphones) may not be exact. Also, the interval uncertainty is not well known which could justify the use of fuzzy intervals. Due to such errors in measurements, the values are actually uncertain in nature which may be handled using fuzzy set theory based on the propagation of uncertainties in terms of fuzzy numbers. Fuzzy sets were introduced by Zadeh [9] as a generalization of classical sets having characteristic function varying over 0 to 1. The characteristic or membership function discussed by [9] depicts the extent to which an element belongs within the set. Hanss [10] presented standard and advanced fuzzy arithmetic with its applications in various engineering fields viz. mechanical, geotechnical, biomedical etc. Recently, Chakraverty et al. [11] presented systematic computational methods for solving fuzzy fractional differential equations governing uncertain models. Also, numerical techniques for solving fuzzy ordinary and partial differential equations, fuzzy nonlinear and fuzzy arbitrary order differential equations with its applications have been discussed by Chakraverty et al. [12]. Further, Anile et al. [13] demonstrated an application to environmental impact analysis based on high precision fuzzy arithmetic. Lee and Wu [14] proposed a fuzzy algorithm for navigation of a mobile robot by self-localization and environment recognition. A fuzzy triangulation approach for computing fuzzy position region has been used by Demirli and Türkşen [15] for identification of robot's pose based on sonar information.

Section 2 introduces the preliminaries related to fuzzy sets and fuzzy numbers. Also, the interval uncertainty is not well known which could justify the use of fuzzy intervals. Interval analysis is also a tool for studying propagation of uncertainties in terms of intervals using r-cut. Interval analysis yield rigorous enclosures of solutions of practical problems governed by mathematical equations. The interval number system, arithmetic, sequences, matrices, solution to integral and differential equations along with applications of interval analysis has been discussed in detail by Moore et al. [16]. Alefeld and Herzberger [17] presented a good discussion on interval arithmetic, interval matrices, fixed point iteration for nonlinear systems, order of convergence of iteration methods etc. Kieffer et al. [18] proposed method for determination of position and orientation of mobile robot based on distance measurements provided by sensors using interval analysis. The interval analysis approach discussed by [18] bypasses complex data-association step and also helps in handling nonlinearity of the problem. Jaulin [19] presented a set membership method based on interval analysis for solving SLAM problems of an underwater robot.

Contractors (Jaulin et al. [20]) associated with interval computations help in computing guaranteed enclosure of solution bounds. As such, Section 3 discusses the new notion of fuzzy contractors based on fuzzy constraints. In case of non-availability of state models for group of robots as given by Eq. (1), localization may be absolute or relative. Based on natural landmarks by laser range finder, Arsénio and Ribeiro [21] proposed an absolute localization procedure. Using bearing angles measured by the robot with respect to landmarks, the pose estimation has been performed by Betke and Leonid [22]. But, in case of group of underwater robots due to lack of absolute landmarks, the absolute localization is not possible. In such case, the relative localization helps in determination of the pose when no absolute frame or fixed robot is present. Yuqing [23] investigated a relative localization problem of multiple robots based on Bayesian theory satisfying Markov assumption. Then, the states and covariances obtained using odometry model is updated based on Kalman filter for state estimation. Zhou and Roumeliotis [24] determined a 2-dimensional pose based on distance measurements between the robots. The localization of a mobile robot using an onboard-angular measuring device with respect to indistinguishable (not distinct) landmarks has been proposed by Hanebeck and S. Günther [25]. Using the bearing (heading) information, localization of robot networks has been investigated by Eren [26]. Very recently, Mahato et al. [27] studied a relative localization procedure for group of robots based on geometric measurements (angles and distances) among the robots. As such, Section 4 gives the localization of group of robots using matrix contractors based on fuzzy constraints in terms of angles and distances among the robots. Finally, numerical examples of group of robots have been considered for verification of the proposed localization.

2 Preliminaries

In this section, fuzzy sets and fuzzy numbers have been introduced in Subsection 2.1. Further, in terms of partial ordering properties satisfied by fuzzy sets, fuzzy lattices have been introduced in Subsection 2.2.

2.1 Fuzzy sets

A crisp (classical) set is a collection of well-defined objects from a universal set $\mathbb{X}$. The characteristic function associated with a crisp set A is a mapping χ_A such that,

$$\begin{aligned} \chi_A &: \mathbb{X} \to \{0,1\} \\ x &\mapsto \chi_A(x) \,. \end{aligned} \tag{2}$$

Characteristic function helps in determination of the extent of belonging of an element within the set. For instance

$$\chi_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases} . \tag{3}$$

But, if there exist a possibility of dependency for the elements of the sets, then the characteristic function may vary over 0 to 1. In such case, fuzzy sets may be preferred for handling the propagation of the uncertainties and the extent of propagation may be handled using associated characteristic function.

Fuzzy set: (Zadeh [9]) A fuzzy set is a set of ordered pairs such that

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in \mathbb{X}, \mu_{\tilde{A}}(x) \in [0,1]\} \,, \tag{4}$$

where $\mu_{\tilde{A}}(x)$ is referred as the membership or characteristic function over universal set $\mathbb{X}$, defined as a mapping,

$$\mu_{\tilde{A}} : \mathbb{X} \to [0,1] \,.$$

In case of localization of group of robots as discussed in Section 4, the uncertainties in angle and distance measurements are expressed in terms of fuzzy numbers and the extent of uncertainties are expressed using associated membership functions.

Fuzzy hypograph: (Wang and Syau [28]) A hypograph $hyp\tilde{A}$ of the fuzzy set given in (4), associated with membership function $\mu_{\tilde{A}}$ is defined as

$$hyp\tilde{A} = \{(x, y) | x \in \mathbb{X}, y \in (0, \mu_{\tilde{A}}(x))\} \subset \mathbb{X} \times [0,1]. \tag{5}$$

Convex fuzzy set: A fuzzy set $\tilde{A}$ is convex if $\forall x, y \in \mathbb{X}$, the membership function $\mu_{\tilde{A}}$ satisfies

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\} \tag{6}$$

where, $\lambda \in [0,1]$.

Fuzzy number: A convex fuzzy set $\tilde{A}$ satisfying (6) is a fuzzy number $\tilde{a}$ if,

- $\tilde{A}$ is normalized, $\sup \mu_{\tilde{A}}(x) = 1$,
- $\mu_{\tilde{A}}(x)$ is piecewise continuous,
- $\exists$ at least one x such that $\mu_{\tilde{A}}(x) = 1$.

$\forall x \in \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers.

There exist various types of fuzzy numbers (Hanss [10]) but below only Triangular Fuzzy Number (TFN) and Gaussian Fuzzy Number (GFN) have been discussed for the sake of completeness.

Triangular Fuzzy Number (TFN): A TFN $\tilde{a} = tfn(a,b,c)$ as shown in Fig. 2 is a special case of fuzzy number having membership function given by $\mu_{\tilde{a}_{tfn}}(x)$ such that

$$\mu_{\tilde{a}_{tfn}}(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , x \in [a,b] \\ \frac{c-x}{c-b} & , x \in [b,c] \\ 0 & , x > c \end{cases} \quad \text{or} \tag{7}$$

$$\mu_{\tilde{a}_{tfn}}(x) = \max\left\{\min\left\{\frac{x-a}{b-a}, \frac{c-x}{c-b}\right\}, 0\right\} \text{ for } \forall x \in \mathbb{R} \text{ where } a < b < c. \tag{8}$$

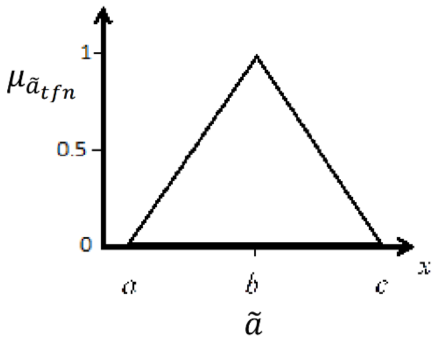


Fig. 2. Triangular fuzzy number

Gaussian Fuzzy Number (GFN): This is a GFN or bell shaped fuzzy number, $\tilde{a} = gfn(a, \sigma_1, \sigma_2)$ as shown in Fig. 3. The membership function is given by $\mu_{\tilde{A}}(x)$ such that $\forall x \in \mathbb{R}$,

$$\mu_{\tilde{A}}(x) = \begin{cases} e^{-\frac{(x-a)^2}{2\sigma_1^2}} & ,x < a \\ e^{-\frac{(x-a)^2}{2\sigma_2^2}} & ,x \geq a \end{cases} . \tag{9}$$

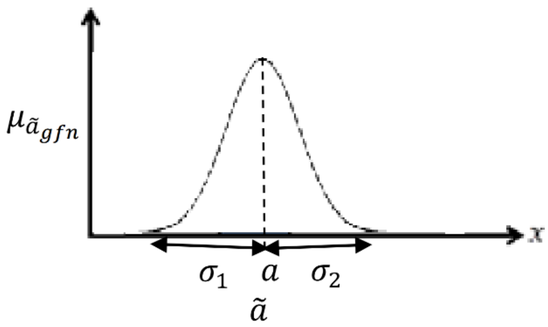


Fig. 3. Gaussian fuzzy number

Interval: A closed interval $[x]$ is $[\underline{x}, \overline{x}]$= $[\inf([x]), \sup([x])] \in \mathbb{IR}$ (set of closed intervals on $\mathbb{R}$) where $\inf([x])$ is the infimum (lower bound) and $\sup([x])$ is the supremum (upper bound).

Degenerate interval: A closed interval is referred as degenerate interval $\{x\}$ if the lower and upper bounds are same ($\underline{x} = \overline{x}$).

***r*-cut:** r-cut of a fuzzy set $\tilde{A}$ is the crisp set

$$A_r = \{x \in \mathbb{X} | \mu_{\tilde{A}}(x) \geq r\}. \tag{10}$$

The construction of fuzzy set may be through r-cuts using the membership function, $\mu_{\tilde{a}}(x) = \sup r \cdot \chi_A\,(x)$ for $r \in [0,1]$. In case of TFN, the r-cut is obtained as

$$a_r = [a + (b-a)r, c - (c-b)r], \forall\, r \in [0,1]\ . \tag{11}$$

Based on r-cut decomposition, a fuzzy number $\tilde{a} = tfn(a, b, c)$ results to an interval $[a, c]$ if $r = 0$ and results to a degenerate interval $\{b\}$ if $r = 1$. Further, in case of GFN the r-cut is obtained as

$$a_r = \left[a - \sigma_1\sqrt{-2\log r}, a + \sigma_2\sqrt{-2\log r}\,\right], \forall\, r \in [0,1]. \tag{12}$$

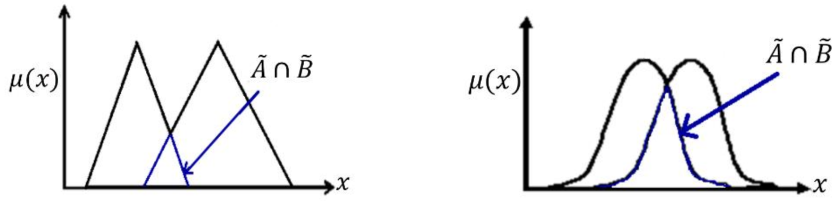
Fuzzy interval: (Dubois et al. [29]; Mazeika et al. [30]) A fuzzy interval is a fuzzy number in the real line whose r-cuts are intervals such that

$$\forall\, r \in (0,1],\ \ \mu_{a_r}^{-1}([r,1]) \in \mathbb{R}. \tag{13}$$

The membership functions given in Eqs. (7) and (9) may be reconstructed in terms of fuzzy intervals using r-cuts.

Intersection of fuzzy sets: The intersection $\tilde{C} = \tilde{A} \cap \tilde{B}$ of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{C}}(x)\right) \middle| \mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall x \in \mathbb{X}\right\}. \tag{14}$$

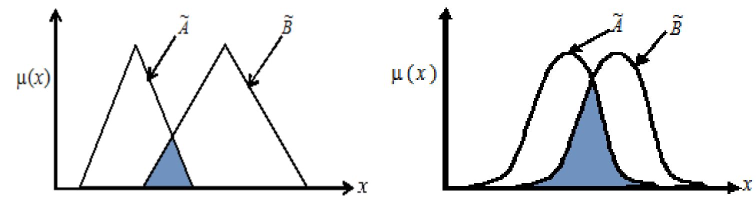


(a) (b)

Fig. 4. Intersection of two (a) TFNs and (b) GFNs

Hypograph of fuzzy intersection: The hypograph associated with intersection (14) of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$hyp\tilde{C} = \{(x,y)|y \in (0, \mu_{\tilde{C}}(x)), \mu_{\tilde{C}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}\, \forall x \in \mathbb{X}\}. \tag{15}$$



(a) (b)

Fig. 5. Hypographs of intersection of two (a) TFNs and (b) GFNs

Union of fuzzy sets: The union $\tilde{C} = \tilde{A} \cup \tilde{B}$ of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{C} = \left\{\left(x, \mu_{\tilde{C}}(x)\right) \middle| \mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall x \in \mathbb{X}\right\}. \tag{16}$$

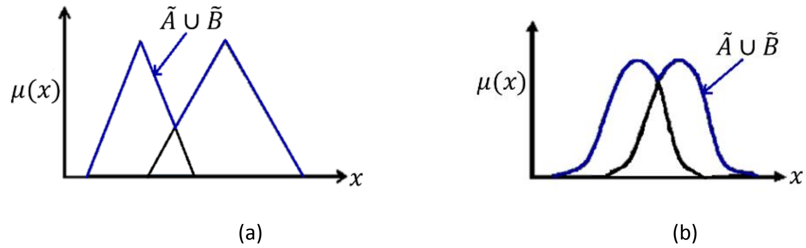


Fig. 6. Union of two (a) TFNs and (b) GFNs

Hypograph of fuzzy union: The hypograph associated with union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as

$$hyp\tilde{C}_{\cup} = \{(x,y) | y \in (0, \mu_{\tilde{C}}(x)), \mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall x \in \mathbb{X}\}. \tag{17}$$

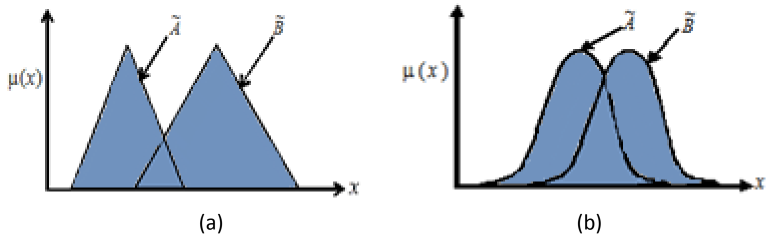


Fig. 7. Hypographs of union of two (a) TFNs and (b) GFNs

Fuzzy inclusion: (Hanss [10]) A fuzzy set $\tilde{A}$ is considered as inclusion (containment) of fuzzy set $\tilde{B}$ if the membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ of the sets satisfy:

$$\tilde{A} \subseteq \tilde{B} \iff \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall x \in \mathbb{X}\,. \tag{18}$$

2.2 Fuzzy lattice

A set $\mathbb{L}$ induced with a partial order binary relation '$\leq$' is referred as a partial ordered set (poset) if it satisfies following relations (Grätzer [31]):

- Reflexive: $a \leq a, \forall a \in \mathbb{L}$,
- Anti-symmetric: If $a \leq b$ and $b \leq a$, then $a = b, \forall a, b \in \mathbb{L}$,

- Transitive: If $a \leq b$ and $b \leq c$, then $a \leq c, \forall a, b, c \in \mathbb{L}$.

A poset $(\mathbb{L}; \leq)$ is referred as a lattice if $\exists$ infimum $(\inf(a, b))$ and supremum $(\sup(a, b))$ $\forall a, b \in \mathbb{L}$. An equivalent definition (Grätzer [31]) is that $(\mathbb{L}; \leq)$ is a lattice $\Leftrightarrow \exists \inf(\mathbb{H})$ and $\sup(\mathbb{H})$ for any finite nonempty set $\mathbb{H}$ such that $\mathbb{H} \subset \mathbb{L}$. As such, $\mathbb{R}$ is a partial ordered set and forms a lattice $(\mathbb{R}, \leq)$ and an interval $[x]$ forms a sub-lattice associated to each variable $x \in \mathbb{R}$ having infimum and supremum.

A fuzzy number may be obtained as union of fuzzy intervals over $r \in [0,1]$ that forms a convex cover. So, in case of fuzzy sets each crisp variable $x \in \mathbb{R}$ is associated with fuzzy number $\tilde{a} \in \mathbb{FR}$ (set of fuzzy numbers) that forms a sub-lattice having fuzzy inclusion associated with membership functions.

3 Contractors

Constraints or mathematical relations (equations, in-equations etc.) having fuzzy domains (or interval domains) may be solved by associating with contractors. In this section, initially a Constraint Satisfaction Problem (CSP) is discussed and then building of contractors based on specific constraints has been given.

3.1 Constraint Satisfaction Problem (CSP)

The localization problem of group of robots discussed in Section 4 involves the estimation of pose for group of robots $\boldsymbol{R_1}; \boldsymbol{R_2}; \dots; \boldsymbol{R_n}$, linked by a set of constraints.

Constraint satisfaction problem: A CSP (Jaulin et al. [32]; Araya [33]) is a triplet $(V, E, \mathbb{D})$ associated with a set of variables $V = \{x_1; x_2; \dots; x_n\}$ along with set of constraints $E = \{e_1; e_2; \dots; e_n\}$ over fuzzy domains $\mathbb{D} = \{\tilde{x}_1; \tilde{x}_2; \dots; \tilde{x}_n\}$ or interval domains $\mathbb{D} = \{[x_1]; [x_2]; \dots; [x_n]\}$.

3.2 Fuzzy contractors

Contractor: (Chabert and Jaulin [34; 35]) A (classical) contractor $\mathcal{C}$ as depicted in Fig. 8 is an operator associated with a set $\mathbb{S}$ over a domain $\mathbb{D}$, such that $\mathbb{S} \subset \mathbb{D}$

$$\mathcal{C}: \mathbb{IR}^n \to \mathbb{IR}^n \tag{19}$$

$$[x] \mapsto \mathcal{C}([x])$$

where, $\mathbb{IR}^n$ is the set of all closed intervals of $\mathbb{R}^n$ satisfying the following properties:

- Contraction: $\mathcal{C}([x]) \subseteq [x], \forall [x] \in \mathbb{IR}^n$, (20a)
- Completeness: $\mathcal{C}([x]) \cap \mathbb{S} = [x] \cap \mathbb{S}, \forall [x] \in \mathbb{IR}^n$. (20b)

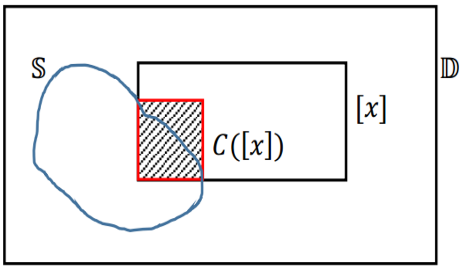


Fig. 8. Contraction of $[x]$

Based on the classical contractor defined in (19), fuzzy contractor has been defined below:

Fuzzy contractor: Fuzzy contractor associated with fuzzy set $\tilde{\mathbb{S}} \subset \mathbb{D}$ is an operator

$$\tilde{\mathcal{C}}: \mathbb{IR}^n \times [0,1] \to \mathbb{IR}^n \times [0,1]$$

$$\tilde{x} \mapsto \mathcal{C}(\tilde{x}) \tag{21}$$

where, $\tilde{x} = ([x], \mu)$ such that $[x] \in \mathbb{IR}^n$, having memberships μ and $\mu_{\tilde{\mathcal{C}}}$ before and after contractions respectively. Further, the fuzzy contractor satisfies the following properties $\forall [x] \in \mathbb{IR}^n$:

- Contraction: $\tilde{\mathcal{C}}(\tilde{x}) \subset \tilde{x}$ where $\mu_{\tilde{\mathcal{C}}}(x) \leq \mu(x), \forall x \in [x]$ (22a)

- Completeness: $\tilde{\mathcal{C}}(\tilde{x}) \cap \widetilde{\mathbb{S}} = \tilde{x} \cap \widetilde{\mathbb{S}}$. (22b)

A fuzzy contractor helps in over estimation of uncertain expressions due to ambiguity in interval computations in terms of r-cut for fuzzy computations. Accordingly, a r-cut fuzzy contractor $\mathcal{C}_r$ associated with set $\widetilde{\mathbb{S}}$ for $r \in [0,1]$ is an operator,

$$\begin{aligned} \mathcal{C}_r &: \mathbb{IR}^n \to \mathbb{IR}^n \\ [x(r)] &\mapsto \mathcal{C}_r([x(r)]) . \end{aligned} \tag{23}$$

Minimal contractor: $\tilde{\mathcal{C}}_{ml}$ is referred as a minimal contractor associated with a set $\widetilde{\mathbb{S}}$ if $\exists$ a contractor $\tilde{\mathcal{C}}$ such that $\tilde{\mathcal{C}}(\tilde{x}) \subset \tilde{\mathcal{C}}_{ml}(\tilde{x})$, then $\tilde{\mathcal{C}}(\tilde{x}) = \tilde{\mathcal{C}}_{ml}(\tilde{x}) = \tilde{x} \cap \widetilde{\mathbb{S}}$.

Composition of contractors: Composition of two contractors $\tilde{\mathcal{C}}_a$ and $\tilde{\mathcal{C}}_b$ is defined as

$$\tilde{\mathcal{C}}_a \left(\tilde{\mathcal{C}}_b(\tilde{x})\right) = (\tilde{\mathcal{C}}_a \circ \tilde{\mathcal{C}}_b)(\tilde{x}). \tag{24}$$

n^{th} iterative composition of contractor $\tilde{\mathcal{C}}$, $\forall [x] \in \mathbb{IR}^n$ is defined as

$$\tilde{\mathcal{C}}^n(\tilde{x}) = \left(\tilde{\mathcal{C}} \circ \tilde{\mathcal{C}} \circ \ldots \circ \tilde{\mathcal{C}}\right)(\tilde{x}). \tag{25}$$

The convergence of iterative composition of contractors have been stated by the Proposition 1 using Knaster-Tarski theorem as given below:

Knaster–Tarski Theorem: (Tarski [36]; Garg [37]) If $L = (\mathbb{X}, \leq)$ be a complete lattice and $f: \mathbb{X} \to \mathbb{X}$ be a monotone function on L. Then,

- $\exists$ a least fixed point z of f such that $z = \inf\ \{x | f(x) \leq x\}$ and
- $\exists$ a greatest fixed point z of f such that $z = \sup\ \{x | x \leq f(x)\}$.

Now, according to the contraction property given in Eq. (20a), the iterated composition of contractors on $\mathbb{IR}^n$ are monotonic with respect to inclusion '$\subseteq$' as,

$$\left(\tilde{\mathcal{C}} \circ \tilde{\mathcal{C}} \circ \ldots \circ \tilde{\mathcal{C}}\right)(\tilde{x}) \subseteq \cdots \subseteq \left(\tilde{\mathcal{C}} \circ \tilde{\mathcal{C}}\right)(\tilde{x}) \subseteq \tilde{\mathcal{C}}(\tilde{x}) \subseteq \tilde{x}. \tag{26}$$

Proposition 1: If $\tilde{\mathcal{C}}$ be a contractor associated with fuzzy set $\widetilde{\mathbb{S}}$, then $\tilde{\mathcal{C}} \circ \tilde{\mathcal{C}} \circ \ldots \circ \tilde{\mathcal{C}}(\tilde{x})$ will converge to the largest fixed fuzzy set $\tilde{a} = ([a], \mu_{\tilde{a}}(x))$, $\forall x \in [a]$ satisfying the following property:

- $\mu_{\tilde{a}}(x)$ is the membership function such that $\mu_{\tilde{a}}(x) = \mu_{\tilde{C}\circ\tilde{C}\circ\ldots\circ\tilde{C}}(x) = \min\{\mu_{\tilde{C}\circ\tilde{C}\circ\ldots\circ\tilde{C}}(x), \ldots, \mu_{\tilde{C}\circ\tilde{C}}(x), \mu_{\tilde{C}}(x)\}, \forall x \in [a]$.

If the CSP given in Subsection 3.1 is linked with set of variables as elements of matrices in terms of constraints (matrix equations), then the associated fuzzy contractors may be referred as *fuzzy matrix contractors.* As such, the fuzzy contractor associated with t matrices is defined as below:

Fuzzy t^{th} matrix contractor: A t^{th} matrix contractor for a set of matrices $V = \{M_1; M_2; \ldots; M_t\}$ associated with constraints $E = \{e_1; e_2; \ldots; e_n\}$ is an operator

$$\mathcal{C}_r: \mathbb{IR}^{m_1 \times n_1} \times \ldots \times \mathbb{IR}^{m_t \times n_t} \to \mathbb{IR}^{m_1 \times n_1} \times \ldots \times \mathbb{IR}^{m_t \times n_t} \qquad (27)$$

such that $[M_1^*] \subset [M_1], [M_2^*] \subset [M_2], \ldots, [M_t^*] \subset [M_t]$ where, M_i's are matrices having dimensions $m_i \times n_i$, for $i = 1,2, \ldots, t$ and $\{M_1; M_2; \ldots; M_t\}$ satisfy the constraints mentioned above.

Example 1: Consider symmetric matrix constraint '$E_{sym}: S = S^T$', where S is a $n \times n$ symmetric matrix contained in ($S \subset \mathbb{S}$) the set of all $n \times n$ symmetric matrices $\mathbb{S}$. Then, the minimal fuzzy contractor $\tilde{\mathcal{C}}_{sym}$ associated with E_{sym} is:

$$\tilde{\mathcal{C}}_{sym_r}([S]) = [S] \cap [S]^T. \qquad (28)$$

The fuzzy matrix $\tilde{S} = \begin{pmatrix} tfn(-1,2,3) & tfn(1,3,4) \\ tfn(-1,3,5) & tfn(-1,2,7) \end{pmatrix}$ may be written in term of r-cut as

$$S_r = \begin{pmatrix} [-1+3r, 3-r] & [1+2r, 4-r] \\ [-1+4r, 5-2r] & [-1+3r, 7-5r] \end{pmatrix} \qquad (29)$$

Using r-cut contractor $\mathcal{C}_{sym_r}$, S_r gets contracted to symmetric matrix for $r \in [0,1]$. Accordingly, the contractions with respect to $r = 0, 0.2, \ldots, 1$ of $\tilde{S}_{21}$ are given in Table 1.

Table 1. Contraction of $\tilde{S}_{21}$ based on contractor $\mathcal{C}_{sym_r}$.

r	$[S_{21_r}]$	$\mathcal{C}_{az}([S_{21_r}])$
0	[-1,5]	[1,4]

0.2	[-0.2,4.6]	[1.4,3.8]
0.4	[0.6,4.2]	[1.8,3.6]
0.6	[1.4,3.8]	[2.2,3.4]
0.8	[2.2,3.4]	[2.6,3.2]
1	[3,3]	---

Further, the fuzzy plot for the contraction of $tfn(-1,3,5)$ to $tfn(1,3,4)$ has been depicted in Fig. 9.

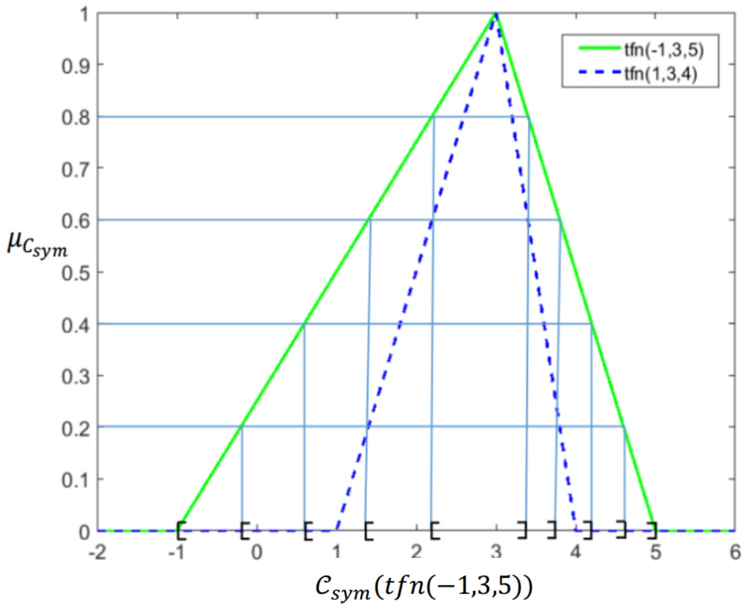


Fig. 9. Contraction of $tfn(-1,3,5)$ to $tfn(1,3,4)$ based on E_{sym}

As such, using r-cut contractor $\mathcal{C}_{sym_r}$, $\tilde{S}$ gets contracted as given below,

$$\tilde{\mathcal{C}}_{sym}\begin{pmatrix} tfn(-1,2,3) & tfn(1,3,4) \\ tfn(-1,3,5) & tfn(-1,2,7) \end{pmatrix} = \begin{pmatrix} tfn(-1,2,3) & tfn(1,3,4) \\ tfn(1,3,4) & tfn(-1,2,7) \end{pmatrix}.$$

4 Localization of group of robots

Placement of landmarks helps a robot to navigate within its frame of reference or environment. But in absence of landmarks, the absolute positioning cannot be determined. In such case, relative localization of robots has to be taken in consideration. Subsection 4.1 gives the absolute localization whereas Subsection 4.2 discusses the proposed relative localization.

4.1 Absolute localization

The absolute localization deals with estimation of instantaneous poses (x_i, y_i, θ_i) of $\boldsymbol{R}_i$ robots for $i = 1,2, \dots, n$ with respect to given landmarks within an environment. Goniometric localization is an absolute localization approach in terms of measured angles between the robots and landmark. A localization technique based on measurements for difference of distances between robot and landmark is referred as multilateration, which is generally used when the clocks between landmarks and robots are not synchronized.

4.2 Relative localization

This head presents the relative localization of group of robots on a planar surface subject to matrix constraints in terms of geometric measures between the robots. The geometric measures are considered in terms of azimuth angles and distances. Further in absence of compass, the azimuth angle cannot be determined. As such, the localization is considered in terms of bearing angles and distances.

4.2.1 Azimuth-Distance localization

4.2.1.1 Azimuth angle

Azimuth angle is measured from north (reference direction) that may be obtained using compass and goniometric sensors. In a planar surface, azimuth angle α between two points (x_1, y_1) and (x_2, y_2) is calculated using arctangent function as

$$\alpha = \operatorname{atan} 2(x_2 - x_1, y_2 - y_1). \tag{30}$$

In the absence of information regarding crisp position of robots, the azimuth angle may be considered from north with respect to other robots. So, in case of localization of group of robots, the azimuth angle is the inclination from north onto other robot as given in Fig. 10.

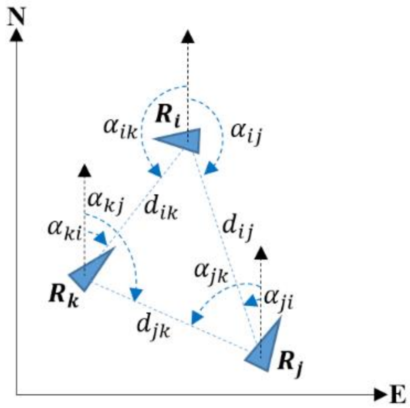


Fig. 10. Azimuth angles α for three robots $\boldsymbol{R}_i, \boldsymbol{R}_j$ and $\boldsymbol{R}_k$

Accordingly, the geometric constraints in terms of azimuth angles among three robots $\boldsymbol{R}_i, \boldsymbol{R}_j$ and $\boldsymbol{R}_k$ is written as,

$$az(A) \Rightarrow \begin{cases} \alpha_{ij} - \alpha_{ji} \sim \pi \\ (\alpha_{ij} - \alpha_{ik}) + (\alpha_{jk} - \alpha_{ji}) + (\alpha_{ki} - \alpha_{kj}) \sim \pi \\ (\alpha_{ij} - \alpha_{ki}) \sim (\alpha_{ji} - \alpha_{jk}) + (\alpha_{kj} - \alpha_{ki}) \end{cases} \tag{31}$$

$\forall\, i, j, k, i \neq j \neq k$, where the relation '$\sim$' is an equivalence relation $\alpha \sim \beta$ between two angles. In $\mathbb{R}^2$, the equivalence angle constraints is given by $\alpha \sim \beta \Leftrightarrow \alpha \equiv \beta (\mathrm{mod}\ 2\pi)$. It may further be verified using constraint,

$$\alpha \sim \beta \Leftrightarrow \frac{\beta - \alpha}{2\pi} \in \mathbb{Z} \Leftrightarrow \cos(\alpha - \beta) = 1 \quad \text{or} \tag{32}$$

$$\alpha \sim \beta \Leftrightarrow \sin(\alpha - \beta) = 0. \tag{33}$$

In case of n robots, a matrix may be considered as an azimuth matrix A associated with n robots, consisting of azimuth angles α_{ij}'s between robots $\boldsymbol{R}_i$ and $\boldsymbol{R}_j$ for $i,j = 1,2,\dots,n$ as,

$$A = \begin{pmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & 0 & & \vdots \\ \vdots & & \ddots & \alpha_{(n-1)n} \\ \alpha_{n1} & \cdots & \alpha_{n(n-1)} & 0 \end{pmatrix}.$$

Here by convention, the azimuth angles are considered as $\alpha_{ii} = 0$, for $i = 1,2,\dots,n$. It may be noted that the north measured using compass may be uncertain. As such, the uncertainty may be handled using fuzzy numbers or intervals varying over $r \in [0,1]$. Further, the uncertainty yields to fuzzy azimuth matrix given as,

$$\tilde{A} = \begin{pmatrix} 0 & \tilde{\alpha}_{12} & \cdots & \tilde{\alpha}_{1n} \\ \tilde{\alpha}_{21} & 0 & & \vdots \\ \vdots & & \ddots & \tilde{\alpha}_{(n-1)n} \\ \tilde{\alpha}_{n1} & \cdots & \tilde{\alpha}_{n(n-1)} & 0 \end{pmatrix}. \tag{34}$$

Based on the constraints given in Eqs. (31) and (32), a minimal contractor $\tilde{C}_{az}(\tilde{A})$ is built for localization of group of robots with respect to azimuth angle contractions. Accordingly, localization of four robots has been considered in Example 2.

Example 2: Let us consider a fuzzy azimuth matrix with respect to four robots as,

$$\tilde{A} = \begin{pmatrix} 0 & gfn(0.69,0.29,0.01) & gfn(1.5,0.2,0.1) & gfn(2.23,0.13,0.07) \\ gfn(-2.45,0.15,0.15) & 0 & gfn(2.09,0.29,0.11) & gfn(-2.97,0.03,0.27) \\ gfn(-1.64,0.16,0.44) & gfn(-1.05,0.15,0.15) & 0 & gfn(-1.92,0.08,42) \\ gfn(-0.91,0.09,0.11) & gfn(0.17,0.17,0.52) & gfn(1.22,0.22,0.28) & 0 \end{pmatrix}.$$

Now using PyIbex on Python environment, a minimal contractor $\tilde{C}_{az}(\tilde{A})$ is built based on constraint '$A\ is\ an\ azimuth\ matrix$' given by Eq. (31). The minimal contractor is built using the forward-backward and fixed point contractors discussed in [32]. Accordingly, the Gaussian fuzzy contraction of azimuth angle $\tilde{\alpha}_{13}$ between the robots $\boldsymbol{R_1}$ to $\boldsymbol{R_3}$ for $r \in (0,1]$ is clearly shown in Fig. 11.

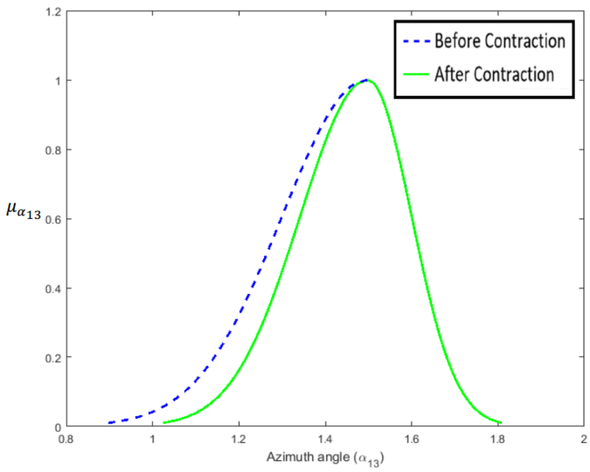


Fig. 11. Contraction of Azimuth angle ($\tilde{\alpha}_{13}$) based on contractor C_{az}

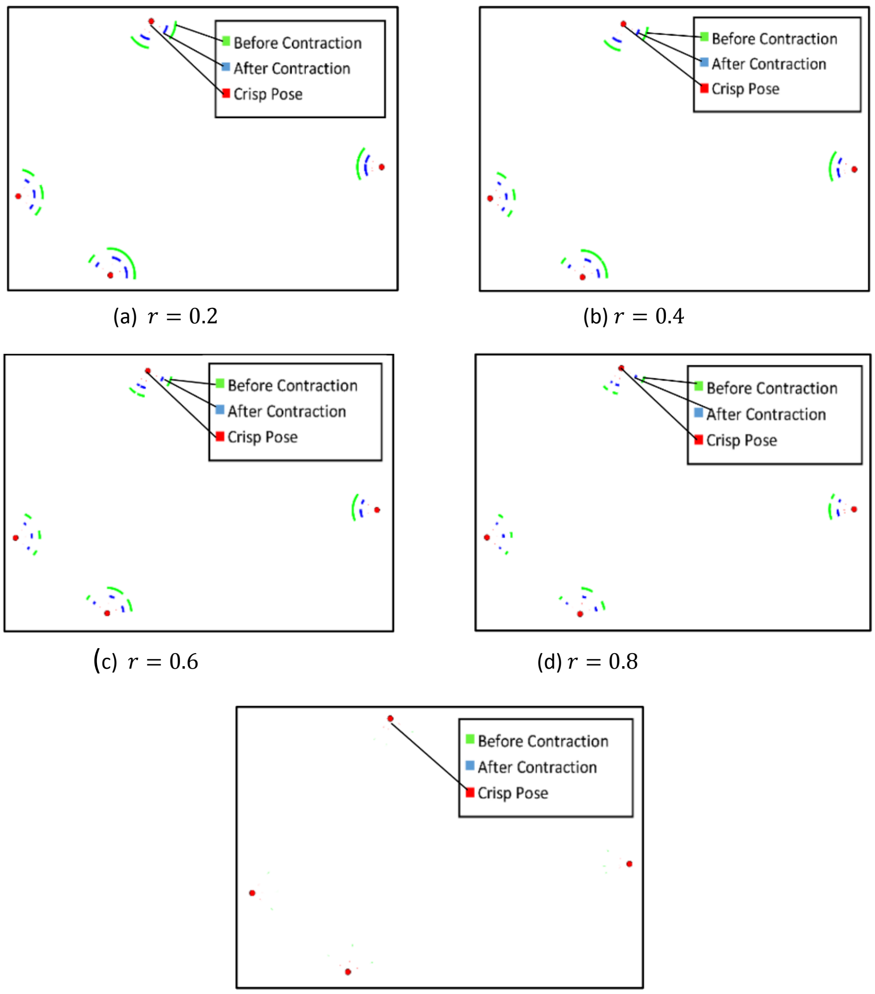
It may be seen from Fig. 11 that the initially assumed Gaussian azimuth angle results to guaranteed (retains the crisp azimuth) contracted Gaussian azimuth angle. The iterative forward-backward contractor helps in propagation of uncertainty with respect to angle contraction by obtaining the minimal contraction given in Fig 11.

The contracted r-cut intervals for $r = 0.2,0.4,\dots,1$ of the azimuth angle $\tilde{\alpha}_{13}$ based on contractor C_{az} correct to four decimals are given in Table 2.

Table 2. Contraction of $\tilde{\alpha}_{13}$ based on contractor C_{az}.

r	$[\alpha_{13_r}]$	$C_{az}([\alpha_{13_r}])$
0.2	[1.1430,1.6812]	[1.2176,1.6812]
0.4	[1.2301,1.6362]	[1.2864,1.6362]
0.6	[1.2979,1.6011]	[1.3399,1.6011]
0.8	[1.3656,1.5661]	[1.3934,1.5661]
1	[1.4977,1.4977]	---

It may be seen from Table 2 that, the uncertainty propagation with respect to membership function for r near to 1 is comparatively less than for r near to 0. Also, the crisp value of azimuth angle for r=1 is always retained within the contracted azimuth and for $r = 1$, no contraction is performed as the initial azimuth angle is a degenerate interval. Accordingly, the contractions based on contractor $C_{az_r}([A])$ is computed for $r \in (0,1]$ and have been depicted in Fig. 12 for $r = 0.2,0.4,0.6,0.8$ and 1.



(a) $r = 0.2$ (b) $r = 0.4$

(c) $r = 0.6$ (d) $r = 0.8$

(e) $r = 1$

Fig. 12. Localization of 4 robots for $r = 0.2,0.4,0.6,0.8$ and 1 based on $\tilde{C}_{az}([A])$

It may be seen from Fig. 12 that contractors help in reduction of uncertainty based on geometric constraints. The azimuth angle uncertainty before contraction as given in Fig. 12 contracts up to the fixed point using contractor $C_{az_r}([A])$. Further, the case when $r = 1$ represents the crisp pose. So, the crisp pose always lies within the contracted uncertainty.

It may further be noted that set of all angles $\mathbb{IA}$ (arcs) is not a Moore family [7;20]. But, in practice the uncertainty due to measurements is less. As such, the initial uncertain angles (arcs as given in Fig. 12) are considered less than equal to π, so that $\mathbb{IA}$ forms a Moore family (satisfies closure property with respect to binary operation '∩').

4.2.1.2 Distance

Distance is another geometrical measure used for self-localization. As given in Fig. 10, the distances d_{ij} between the robots $\boldsymbol{R_i}$ and $\boldsymbol{R_j}$ may be associated with constraints,

$$dist(D) \Rightarrow \begin{cases} d_{ij} = d_{ji} \\ d_{ij} \leq d_{ik} + d_{kj} \end{cases}, \forall\, i, j, k, i \neq j \neq k. \tag{35}$$

Now, the distance matrix D associated with n robots is the matrix consisting of distances d_{ij}'s for $i, j = 1,2, \dots, n$,

$$D = \begin{pmatrix} 0 & d_{12} & \cdots & d_{1n} \\ d_{21} & 0 & & \vdots \\ \vdots & & \ddots & d_{(n-1)n} \\ d_{n1} & \cdots & d_{n(n-1)} & 0 \end{pmatrix}.$$

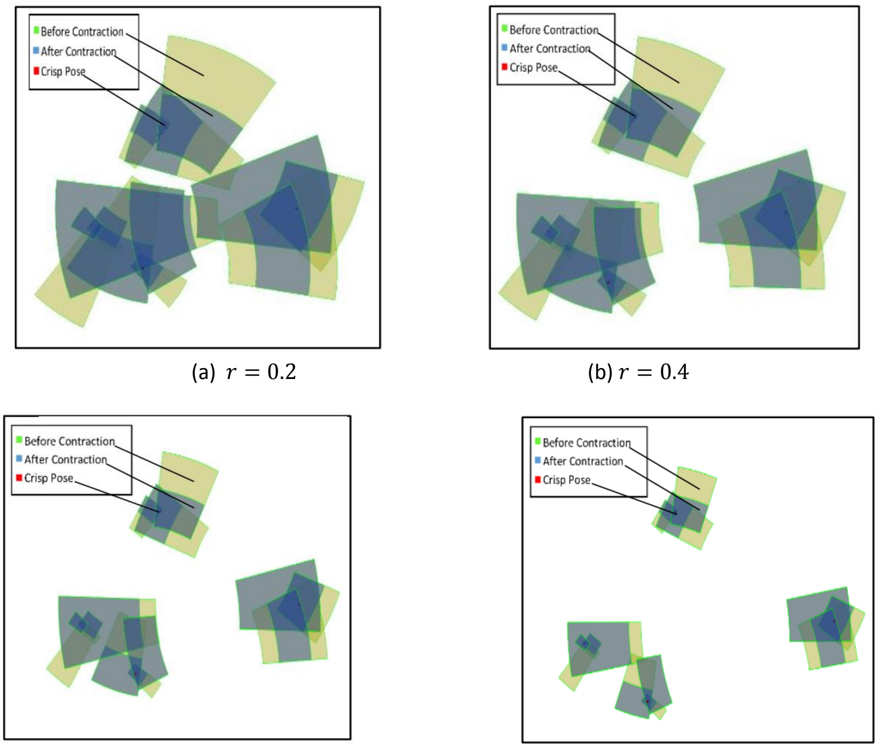
Also, the distances measured using camera sensors may not be accurate and hence uncertain in nature resulting to fuzzy distance matrix or intervals varying over $r \in [0,1]$,

$$\widetilde{D} = \begin{pmatrix} 0 & \tilde{d}_{12} & \cdots & \tilde{d}_{1n} \\ \tilde{d}_{21} & 0 & & \vdots \\ \vdots & & \ddots & \tilde{d}_{(n-1)n} \\ \tilde{d}_{n1} & \cdots & \tilde{d}_{n(n-1)} & 0 \end{pmatrix}. \tag{36}$$

Example 3: Let us consider four robots linked to each other with fuzzy distance matrix as,

$$\widetilde{D} = \begin{pmatrix} 0 & gfn(7.81,1.81,0.19) & gfn(13.7,3.7,1.3) & gfn(4.4,0.4,1.6) \\ gfn(7.81,0.81,3.19) & 0 & gfn(10,1,2) & gfn(8.85,2.85,1.15) \\ gfn(13.7,4.7,1.3) & gfn(10,3,1) & 0 & gfn(10.85,1.85,0.15) \\ gfn(4.4,1.4,0.6) & gfn(8.85,0.85,3.15) & gfn(10.85,2.85,1.15) & 0 \end{pmatrix}.$$

Using PyIbex, a minimal forward-backward contractor $\tilde{\mathcal{C}}_{dist}(\widetilde{D})$ is built based on constraint '$D\ is\ a\ distance\ matrix$' given by Eq. (35). Further, the localization of 4 robots is computed using distance contractor $\mathcal{C}_{dist_r}([D])$ for $r \in (0,1]$. Accordingly, the contractions for $r = 0.2,0.4,0.6$ and 0.8 using fixed point contractor has been depicted in Fig. 13.



(a) $r = 0.2$ (b) $r = 0.4$

(c) $r = 0.6$ (d) $r = 0.8$

Fig. 13. Localization of 4 robots for $r = 0.2,0.4,0.6$ and 0.8 based on contractor $C_{dist_r}([D])$

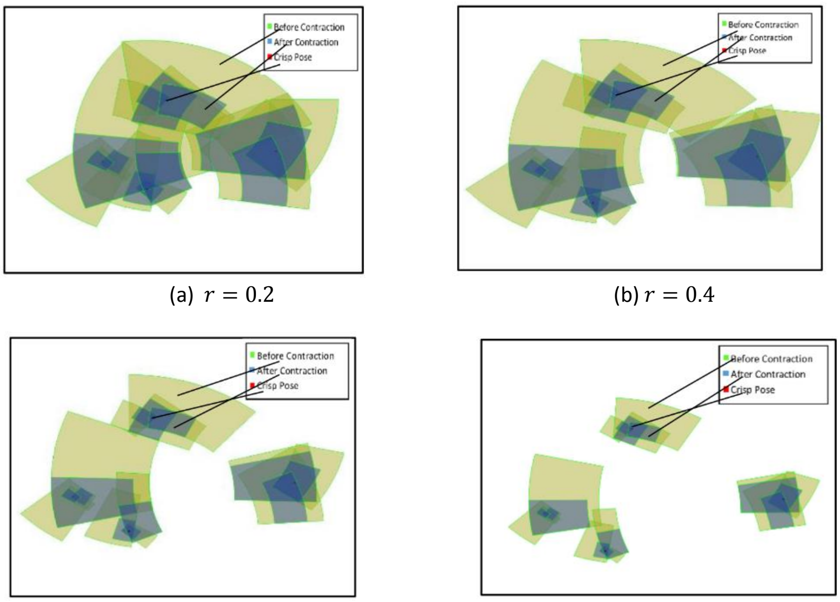
It may be seen from Fig. 13 that contractors help in reduction to symmetric uncertainty using distance constraints. The constraints with respect to r=0.2 approaches (contracts) to crisp pose as $r \to 1$.

4.2.1.3 Azimuth-Distance

Constraints associated with variables may contain dependency of variables with respect to each other. In such cases, mixed constraints helps in construction of mixed contractors. The mixed constraint based on azimuth angles and distances between the robots is given by

$$azdist(A, D) \Rightarrow \sin(\alpha_{ik} - \alpha_{ij}) \cdot d_{ij} = \sin(\alpha_{ki} - \alpha_{kj}) \cdot d_{kj} \quad (37)$$

$\forall\, i, j, k, i \neq j \neq k$. Now, the contractor associated with constraint (37), '$A\ is\ azimuth\ matrix$ and $D\ is\ distance\ matrix$' is built as $\tilde{C}_{azdist}(\tilde{A}, \tilde{D})$. Accordingly, the localization of 4 robots having fuzzy azimuth and distance matrices as given in Examples 2 and 3 has been performed and the contractions with respect to $r = 0.2, 0.4, 0.6$ and 0.8 are depicted in Fig. 14.



(a) $r = 0.2$ (b) $r = 0.4$

(c) $r = 0.6$ (d) $r = 0.8$

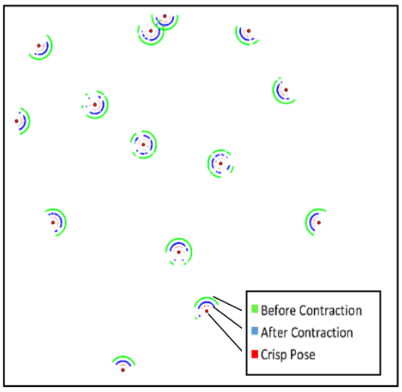
Fig. 14. Localization of 4 robots based on $\tilde{C}_{azdist}([A], [D])$

The uncertainty propagation for $r \in (0,1]$ helps in better estimation of pose for distant robots using the precise azimuths and distances known for nearby robots.

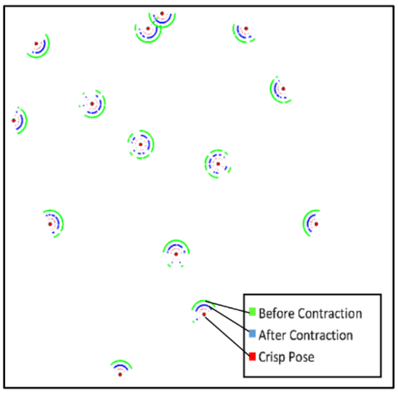
Further, efficiency of the above mentioned procedure is verified by implementing the method on 14 robots in terms of azimuth and distance constraints given by Eqs. (31), (35) and (37).

Example 4: Let us consider the azimuth and distance matrices contraction with respect to 14 robots.

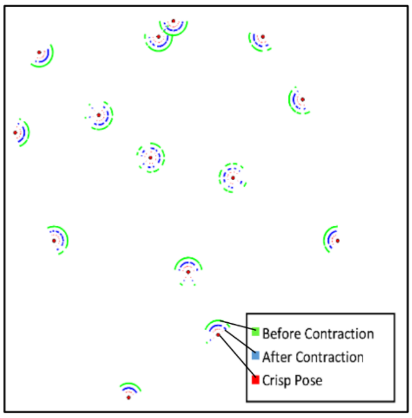
The contractor $\tilde{C}_{az}([A])$ associated with constraint Eq. (31) is used and the localization of 14 robots having fuzzy azimuth matrix is given in Fig. 15.



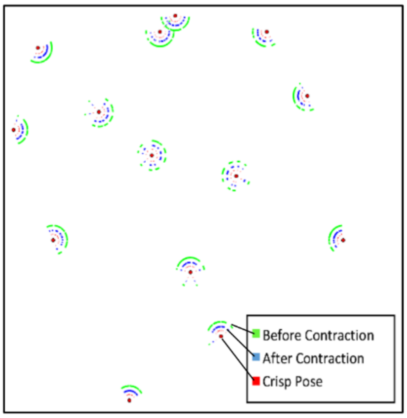
(a) $r = 0.2$



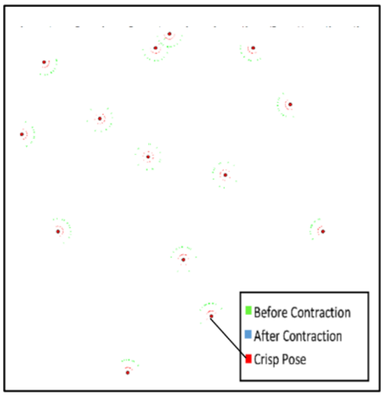
(b) $r = 0.4$



(c) $r = 0.6$



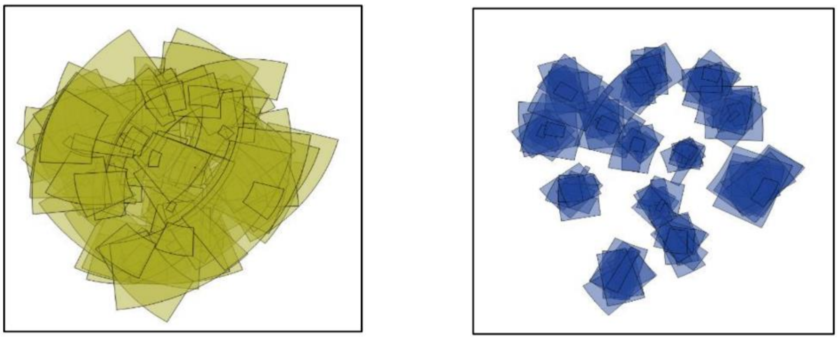
(d) $r = 0.8$



(e) $r = 1$

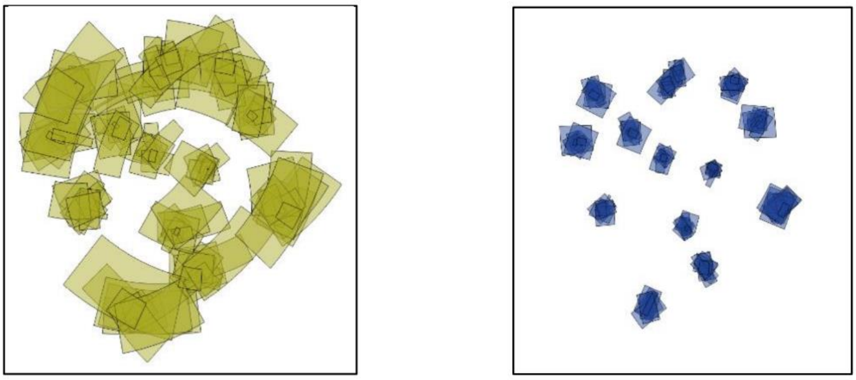
Fig. 15. Localization of 14 robots based on contractors $\tilde{C}_{az}([A])$

The localization as given in Fig. 15 based on contracted azimuth and distance matrices associated with 14 robots is used for further localization. The optimal contractors $\tilde{C}_{dist}([D])$ and $\tilde{C}_{azdist}([A],[D])$ for $r \in (0,1]$, associated with constraints (35) and (37) is used and the localization of 14 robots is illustrated in Figs. 16 and 17.



(a) Before contraction (b) After contraction

Fig. 16. Localization of 14 robots based on $\tilde{C}_{dist}([D])$ and $\tilde{C}_{azdist}([A],[D])$ for r=0.4



(a) Before contraction (b) After contraction

Fig. 17. Localization of 14 robots based on $\tilde{C}_{dist}([D])$ and $\tilde{C}_{azdist}([A],[D])$ for r=0.8

It may be seen that the large uncertainty between distant robots contracted with respect to known uncertainties of near robots.

In case of underwater robots, the determination of north is quite difficult due to absence of goniometric sensors and compass. Also, the measurements due to compass on a planar surface leads to more uncertainty. As such, next section discusses the localization in terms of bearing angles.

4.2.1.3 Bearing-Distance

In navigation, bearing angle β_{ij} as depicted in Fig. 18 is the angle measured between the heading axis (as given in Fig. 1) of robot $\boldsymbol{R_i}$ to the vector pointing towards other robot $\boldsymbol{R_j}$.

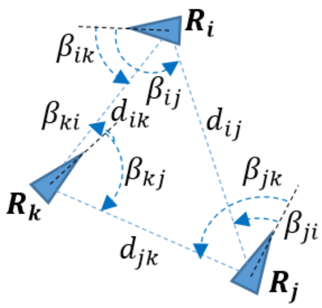


Fig. 18. Bearing angles β of three robots $\boldsymbol{R}_i, \boldsymbol{R}_j$ and $\boldsymbol{R}_k$

Now, the uncertain bearing matrix $\tilde{B}$ associated with n robots is the matrix consisting of bearing angles $\tilde{\beta}_{ij}$'s for $i, j = 1,2, \dots, n$,

$$\tilde{B} = \begin{pmatrix} 0 & \tilde{\beta}_{12} & \cdots & \tilde{\beta}_{1n} \\ \tilde{\beta}_{21} & 0 & & \vdots \\ \vdots & & \ddots & \tilde{\beta}_{(n-1)n} \\ \tilde{\beta}_{n1} & \cdots & \tilde{\beta}_{n(n-1)} & 0 \end{pmatrix}. \tag{38}$$

Accordingly, the bearing constraints $br(B)$ associated with n robots is given by

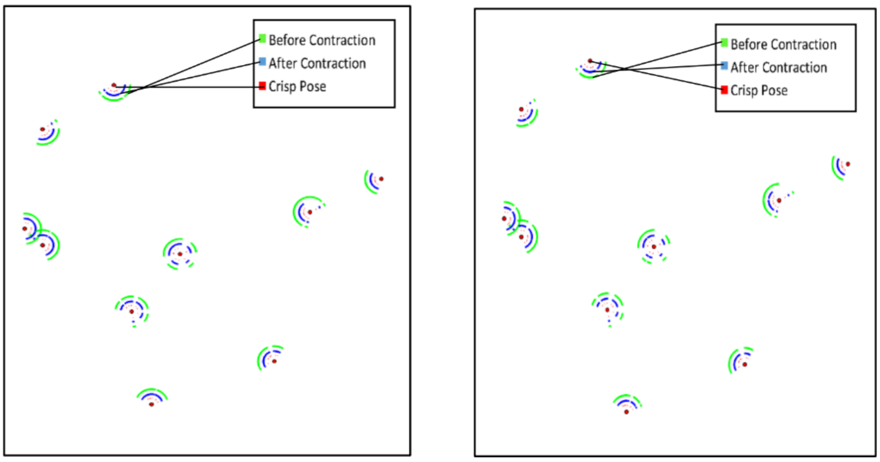
$$br(B) \Rightarrow \begin{cases} (\beta_{ij} - \beta_{ik}) + (\beta_{jk} - \beta_{ji}) + (\beta_{ki} - \beta_{kj}) \sim \pi \\ (\beta_{ij} - \beta_{ki}) \sim (\beta_{ji} - \beta_{jk}) + (\beta_{kj} - \beta_{ki}) \end{cases} \tag{39}$$

$\forall\, i, j, k, i \neq j \neq k$. Then, similar to mixed constraint given in Eq. (37), mixed bearing-distance constraint $brdist(B, D)$ is given by

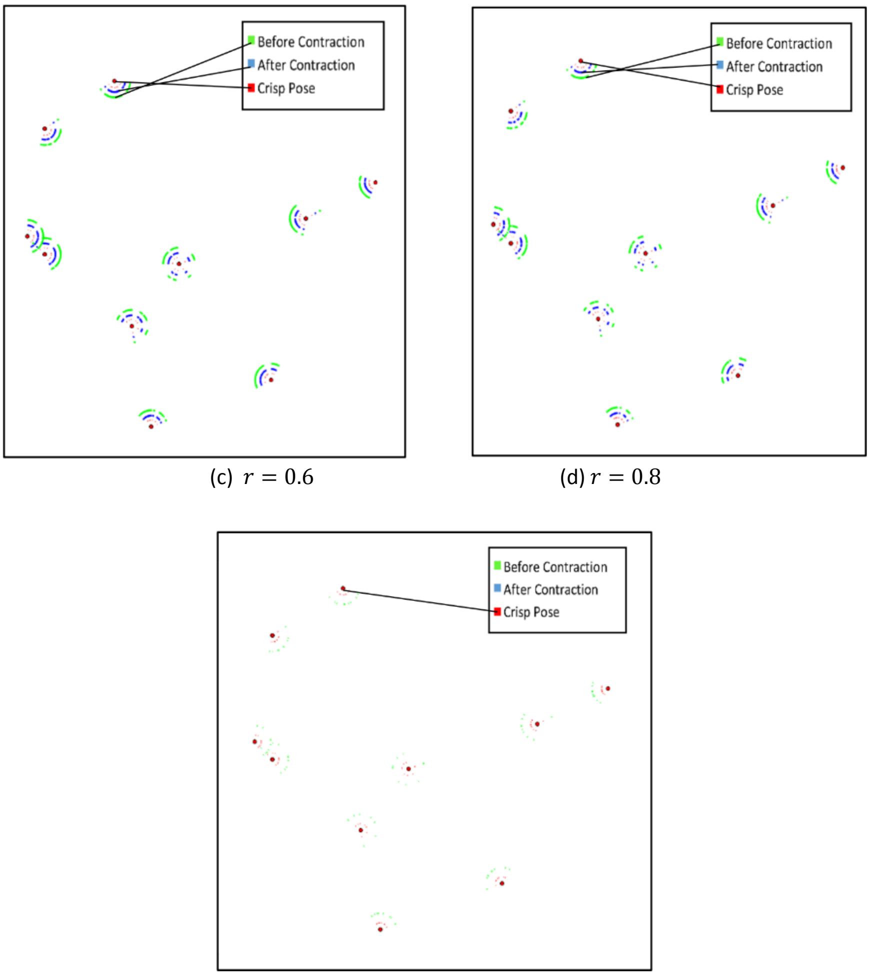
$$brdist(B, D) \Rightarrow \{\sin(\beta_{ik} - \beta_{ij}) \cdot d_{ij} = \sin(\beta_{ki} - \beta_{kj}) \cdot d_{kj} \tag{40}$$

Example 5: Consider the localization problem of 10 robots on a plane having fuzzy bearing and distances matrices.

Using PyIbex, bearing contractor $\tilde{C}_{br_r}([A])$ associated with constraint $br(B)$ given in Eq. (39) is built. Accordingly, the contracted r-cut bearing angles is computed for $r \in (0,1]$ and the localization of 10 robots for $r = 0.2, 0.4, \dots, 1$ is depicted in Fig. 19.



(a) $r = 0.2$ (b) $r = 0.4$

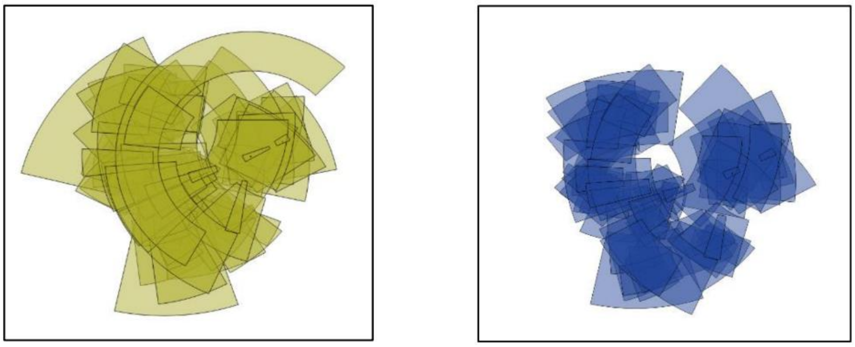


(c) $r = 0.6$ (d) $r = 0.8$

(e) $r = 1$

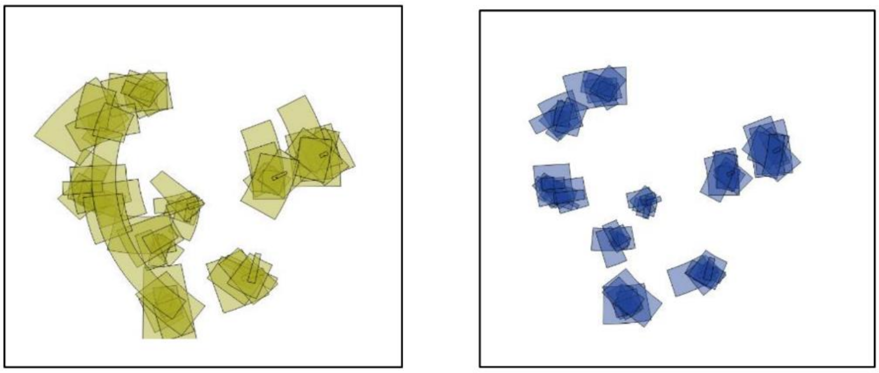
Fig. 19. Localization of 10 robots for $r = 0.2, 0.4, \ldots, 1$ based on $\tilde{\mathcal{C}}_{br_r}([A])$

Then, the localization is given for 10 robots in Figs. 20 and 21, based on optimal contractors $\tilde{\mathcal{C}}_{dist}([D])$ and $\tilde{\mathcal{C}}_{brdist}([B], [D])$ for $r \in (0,1]$, associated with constraints (39) and (40).



(a) Before contraction (b) After contraction

Fig. 20. Localization of 10 robots based on $\tilde{C}_{dist}([D])$ and $\tilde{C}_{brdist}([B],[D])$ for r=0.2



(a) Before contraction (b) After contraction

Fig. 21. Localization of 10 robots based on $\tilde{C}_{dist}([D])$ and $\tilde{C}_{brdist}([B],[D])$ for r=0.8

It may again be seen that the fuzzy contractors help in uncertainty propagation for $r = 0$ to 1. Also, the initial assumed uncertainty reduces to minimal contraction (containing crisp pose).

6 Conclusion

The localization in terms of optimal contractors based on azimuth and distance matrices for multiple robots have been considered. Further, the case of absence of compass is solved in terms of contractors built based on bearing and distance constraints. The usage of contractors helped in uncertainty propagation for pose estimation based on given angle and distance constraints. Further, the fuzzy contractors yield minimal contraction resulting to guaranteed pose estimation (localization) on a planar surface.

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