Attraction domain of a nonlinear system using interval analysis.

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Abstract. Consider a given dynamical system, described by \( \dot{x} = f(x) \) (where \( f \) is a nonlinear function) and \([x_0]\) a subset of \( \mathbb{R}^n \). We present an algorithm, based on interval analysis, able to show that there exists a unique equilibrium state \( x_\infty \in [x_0] \) which is asymptotically stable. The effective method also provides a set \([x]\) (subset of \([x_0]\)) which is included in the attraction domain of \( x_\infty \).

In a second time, the flow of the equation \( \dot{x} = f(x) \) is discretized and inclusion methods are combined with graph theory to compute a set which is included in the attraction domain.

keywords : interval computations, reliable algorithm, stability of nonlinear system, attraction domain.

1 Introduction

There is a considerable number of works devoted to the stability problem of dynamical systems \( \dot{x} = f(x) \) using interval computations \cite{13} \cite{14} \cite{15} \cite{16}. We recall some definitions and notations related to stability.

Consider a dynamical system

\[ \dot{x} = f(x) \tag{1} \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a differentiable function. Let \( \{\varphi^t\} \) denotes the flow associated to the vector field \( x \mapsto f(x) \).

Definition 1. A subset \( D \) of \( \mathbb{R}^n \) is stable if \( \varphi^{\mathbb{R}^+}(D) \subset D \), where \( \varphi^{\mathbb{R}^+}(D) = \{\varphi^t(x), x \in D, t \in \mathbb{R}^+\} \).
Definition 2. Let $D$ and $D'$ be two subsets of $\mathbb{R}^n$ such that $D \subset D'$. A equilibrium point $x_\infty$ is asymptotically $(D, D')$-stable if $\varphi^+(D) \subset D'$ and $\varphi^\infty(D) = \{x_\infty\}$, where $\varphi^\infty(D)$ denotes the set $\{x_\infty \in \mathbb{R}^n \mid x_\infty = \lim_{t \to \infty} \varphi^t(x), x \in D\}$.

When $f$ is sufficiently regular around an equilibrium state $x_\infty$ and $Df(x_\infty)$ is hyperbolic, the qualitative behavior of the dynamical system $\dot{x} = f(x)$ around $x_\infty$ is the same that of $\dot{x} = Df(x_\infty)(x - x_\infty)$, the stability of which can be determined by counting the number of eigenvalues with negative real parts.

Now, in practice, we are only able to compute an approximation $\tilde{x}_\infty$ of $x_\infty$ and thus we cannot conclude to the local stability of (1) around $x_\infty$.

Moreover, even if we were able to compute exactly $x_\infty$ and to prove its local stability, to our knowledge, no general method seems to be available to compute a neighborhood $D$ of $x_\infty$ such that $\varphi^\infty(D) = \{x_\infty\}$.

The main contribution of this paper is a method to compute:

- from a set $D'$, a domain $D$ such that the system is $(D, D')$-stable.
- a domain $A \supset D$ such that $A$ is included in the attraction domain of $\{x_\infty\}$.

The approach to be considered is based on interval analysis. Interval analysis is used to prove uniqueness of an equilibrium state. The paper provides a method and a sufficient condition to check that a real valued function is positive. An algorithm combines interval analysis and Lyapunov to solve our stability problem is proposed.

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2 Asymptotic stability

Consider the example:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-x_2 \\
x_1 - (1 - x_1^2)x_2
\end{pmatrix} \tag{2}
\]

where $[x_0] = [-0.6, 0.6]^2$.

The vector field associated to this dynamical system is represented on Figure 1. To prove existence and uniqueness of the equilibrium state $x_\infty \in [x]_\infty$, one uses the famous interval Newton method.

To compute a set $[x]$ included in the attraction domain of $x_\infty$, one combines interval analysis and Lyapunov theory. In this case, the Lyapunov function created is:

\[
L_{x_\infty}(x) = (x - x_\infty)^T \begin{pmatrix}
-1 & 0.49 \\
0.49 & -1 & 0.1
\end{pmatrix} (x - x_\infty) \tag{3}
\]
3 Attraction domain

The previously computed set $[x]$ is in the attraction domain of $x_\infty$. The proposed method improves $[x]$ combining inclusion methods and graph theory. (Initialization $A \leftarrow [x]$). Given an ordinary differential equation, inclusion methods give inclusion function of the flow. From $t \in \mathbb{R}$ and a cover $\{S_i\}$ of $D$, the proposed method creates a relation (a graph) $\mathcal{R}$ with:

$$S_i \mathcal{R} S_j \iff \varphi^t(S_i) \cap S_j \neq \emptyset$$

If $S_i \mathcal{R} S_j \Rightarrow S_j \subset A$ then $S_i$ is added to $A$ ($A \leftarrow A \cup S_i$). This algorithm converges to a set which is included in the attraction domain of $x_\infty$. 

Fig. 2. The computed set $A$ is included in the attraction domain of the equilibrium state.
References

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