Attraction domain of a nonlinear system using interval analysis.

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Abstract. Consider a given dynamical system, described by $\dot{x} = f(x)$ (where f is a nonlinear function) and $[x_0]$ a subset of $\mathbb{R}^n$. We present an algorithm, based on interval analysis, able to show that there exists a unique equilibrium state $x_\infty \in [x_0]$ which is asymptotically stable. The effective method also provides a set $[x]$ (subset of $[x_0]$) which is included in the attraction domain of x_∞.
In a second time, the flow of the equation $\dot{x} = f(x)$ is discretized and inclusion methods are combined with graph theory to compute a set which is included in the attraction domain.

keywords :
interval computations, reliable algorithm, stability of nonlinear system, attraction domain.

1 Introduction

There is a considerable number of works devoted to the stability problem of dynamical systems $\dot{x} = f(x)$ using interval computations [13] [14] [15] [16]. We recall some definitions and notations related to stability.

Consider a dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable function. Let $\{\varphi^t\}$ denotes the flow associated to the vector field $x \mapsto f(x)$.

Definition 1. *A subset D of $\mathbb{R}^n$ is stable if $\varphi^{\mathbb{R}^+}(D) \subset D$, where $\varphi^{\mathbb{R}^+}(D) = \{\varphi^t(x), x \in D, t \in \mathbb{R}^+\}$*

Definition 2. *Let D and D' be two subsets of $\mathbb{R}^n$ such that $D \subset D'$. A equilibrium point x_∞ is asymptotically (D, D')-stable if $\varphi^{\mathbb{R}^+}(D) \subset D'$ and $\varphi^\infty(D) = \{x_\infty\}$, where $\varphi^\infty(D)$ denotes the set $\{x_\infty \in \mathbb{R}^n \mid x_\infty = \lim_{t\to\infty} \varphi^t(x), x \in D\}$*

When f is sufficiently regular around an equilibrium state x_∞ and $Df(x_\infty)$ is hyperbolic, the qualitative behavior of the dynamical system $\dot{x} = f(x)$ around x_∞ is the same that of $\dot{x} = Df(x_\infty)(x - x_\infty)$, the stability of which can be determined by counting the number of eigenvalues with negative real parts. Now, in practice, we are only able to compute an approximation $\tilde{x}_\infty$ of x_∞ and thus we cannot conclude to the local stability of (1) around x_∞.

Moreover, even if we were able to compute exactly x_∞ and to prove its local stability, to our knowledge, no general method seems to be available to compute a neighborhood D of x_∞ such that $\varphi^\infty(D) = \{x_\infty\}$.

The main contribution of this paper is a method to compute :

- from a set D', a domain D such that the system is (D, D')-stable.
- a domain $A \supset D$ such that A is included in the attraction domain of $\{x_\infty\}$.

The approach to be considered is based on interval analysis. Interval analysis is used to prove uniqueness of an equilibrium state. The paper provides a method and a sufficient condition to check that a real valued function is positive.
An algorithm combines interval analysis and Lyapunov to solve our stability problem is proposed.

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2 Asymptotic stability

Consider the example :

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 - (1 - x_1^2)x_2 \end{pmatrix} \tag{2}$$

where $[x_0] = [-0.6, 0.6]^2$.

The vector field associated to this dynamical system is represented on Figure 1. To prove existence and uniqueness of the equilibrium state $x_\infty \in [x]_\infty$, one uses the famous interval Newton method.
To compute a set $[x]$ included in the attraction domain of x_∞, one combines interval analysis and Lyapunov theory. In this case, the Lyapunov function created is :

$$L_{x_\infty}(x) = (x - x_\infty)^T \begin{pmatrix} -1,51 & 0,49 \\ 0,49 & -1,01 \end{pmatrix} (x - x_\infty) \tag{3}$$

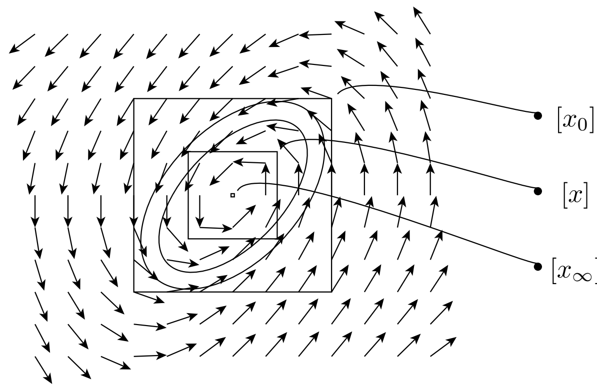

Fig. 1. Lyapunov function level curves and a box $[x_\infty]$ which contains a unique equilibrium state.

3 Attraction domain

The previously computed set $[x]$ is in the attraction domain of x_∞. The proposed method improves $[x]$ combining inclusion methods and graph theory. (Initialization $A \leftarrow [x]$). Given an ordinary differential equation, inclusion methods gives inclusion function of the flow. From $t \in \mathbb{R}$ and a cover $\{\mathbb{S}_i\}$ of D, the proposed method creates a relation (a graph) $\mathcal{R}$ with :

$$\mathbb{S}_i \mathcal{R} \mathbb{S}_j \Leftrightarrow \varphi^t(\mathbb{S}_i) \cap \mathbb{S}_j \neq \emptyset$$

If $\mathbb{S}_i \mathcal{R} \mathbb{S}_j \Rightarrow \mathbb{S}_j \subset A$ then $\mathbb{S}_i$ is added to A ($A \leftarrow A \cup \mathbb{S}_i$). This algorithm converges to a set which is included in the attraction domain of x_∞.

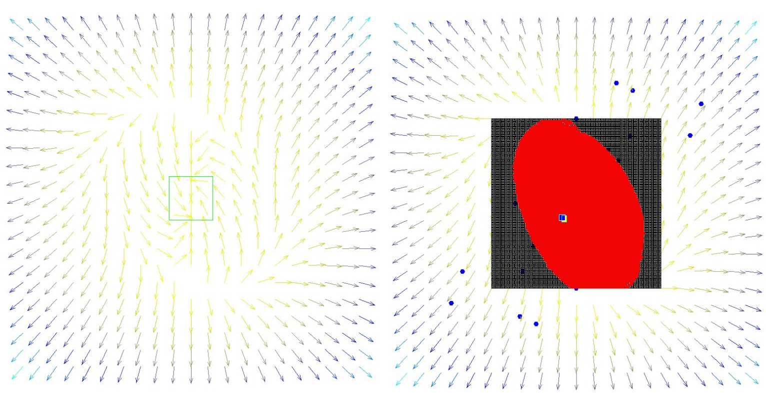

Fig. 2. The computed set A is included in the attraction domain of the equilibrium state.

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