Karush–Kuhn–Tucker conditions to build efficient contractors
Application to TDoA localization

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Abstract. This paper proposes an efficient contractor for the TDoA (Time Differential of Arrival) equation. The contractor is based on a minimal inclusion test which is built using the Karush–Kuhn–Tucker (KKT) conditions. An application related to the localization of sound sources using a TDoA technique is proposed.

1 Introduction

To solve nonlinear problems with nonlinear constraints, a classical approach is based on the Karush–Kuhn–Tucker (KKT) conditions [12] [13]. To use the KKT conditions, we first have to formulate our problem as an optimization problem in standard form:

$$\begin{array}{c} \text{minimize} f(\mathbf{x}) \\ \text{s.t. } \mathbf{g}(\mathbf{x}) \leq 0 \end{array} \qquad (1)$$

where function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathbf{g} : \mathbb{R}^n \mapsto \mathbb{R}^m$ are assumed to be differentiable. We then build the Lagrangian

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}) + \boldsymbol{\mu}^{\mathrm{T}} \cdot \mathbf{g}(\mathbf{x}). \qquad (2)$$

The necessary conditions for $\mathbf{x}$ to be an optimizer are:

$$\left\{ \begin{array}{ll} \frac{df}{d\mathbf{x}}(\mathbf{x}) + \sum_i \mu_i \frac{dg_i}{d\mathbf{x}}(\mathbf{x}) = 0 & \text{(stationarity)} \\ g_i(\mathbf{x}) \leq 0, \forall i & \text{(primal feasibility)} \\ \mu_i \geq 0 & \text{(dual feasibility)} \\ \mu_i g_i(\mathbf{x}) = 0, \forall i & \text{(complementary slackness)} \end{array} \right.$$

The problem can be interpreted as moving a particle at position $\mathbf{x}$ in the space, with two kinds of forces:

- f is a potential and the force generated by f is $-\text{grad} f$.

- The constraint $g_i(\mathbf{x}) \leq 0$ corresponds to reaction forces generated by one-sided constraint surfaces delimiting the free space for $\mathbf{x}$. The particle is allowed to move inside $g_i(\mathbf{x}) \leq 0$, but as soon as it touches the surface $g_i(\mathbf{x}) = 0$, it is pushed inwards the free space.

Stationarity states that $-\text{grad}f$ is a linear combination of the reaction forces. Dual feasibility states that reaction forces point inwards the free space for $\mathbf{x}$. Slackness states that if $g_i(\mathbf{x}) < 0$, then the corresponding reaction force must be zero, since the particle is not in contact with the surface.

Interval methods have used these KKT conditions to solve nonlinear optimization problems [15][6][17] when inequality constraints are involved.

In this paper, we propose to use the KKT conditions to build minimal inclusion tests in order to derive efficient contractors. For this, we consider a constraint of the form $y = f(\mathbf{x})$ where $f : \mathbb{R}^n \to \mathbb{R}$ and we assume that $\mathbf{x} \in [\mathbf{x}]$, where $[\mathbf{x}]$ is an axis aligned closed box of $\mathbb{R}^n$. The feasible values for y is an interval $[y] = [y^-, y^+]$ which can be obtained by solving the two minimization problems

$$y^- = \min_{\mathbf{x} \in [\mathbf{x}]} f(\mathbf{x}) \tag{3}$$

and

$$y^+ = -\min_{\mathbf{x} \in [\mathbf{x}]} (-f(\mathbf{x})) \tag{4}$$

As a consequence, the KKT conditions could be used as least for the forward contraction, *i.e.*, to contract the feasible interval $[y]$ for y. These conditions can be treated either symbolically for simple constraints or automatically with pessimism as in [6].

Section 2 defines the TDoA constraint [14] which will illustrate the benefit brought by the use of the KKT conditions. Section 3 introduces the notion of action of a contractor on a separator. This notion will allow us build complex separator using the composition with other constraints. Section 4 illustrates how the notion of action of a TDoA contractor on the separator (obtained after data treatment) can be used to localize sound sources. Section 5 concludes the paper.

2 TDoA constraint

The TDoA constraint is defined by

$$\|\mathbf{x} - \mathbf{a}\| - \|\mathbf{x} - \mathbf{b}\| = y \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^2$ and $y \in \mathbb{R}$ are the variables. The parameters $\mathbf{a} \in \mathbb{R}^2, \mathbf{b} \in \mathbb{R}^2$ are assumed to be known. Equivalently, we have

$$f(\mathbf{x}) = y \tag{6}$$

where

$$f(\mathbf{x}) = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} - \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2}. \tag{7}$$

In this section, we want to build an efficient contractor for (5) , *i.e.*, given a box $[\mathbf{x}] \ni \mathbf{x}$ and an interval $[y] \ni y$, we want to contract $[\mathbf{x}]$ and $[y]$ without removing a single pair $(\mathbf{x}, y)$ of the constraint (see [2] for a formal definition of a contractor). We will mainly focus on the forward contraction, *i.e.*, the contraction of $[y]$ which can be interpreted as an interval evaluation of f. Interval analysis has already been used to solve problems involving the TDoA constraint in [18], [4] and [9].

Notation. In what follows, $[\mathbb{A}]$ represents the smallest closed interval which contains the set $\mathbb{A} \subset \mathbb{R}$. When $\mathbb{A} \subset \mathbb{R}^n$, $[\mathbb{A}]$ denotes the smallest axis-aligned box which contains $\mathbb{A}$.

2.1 Interval evaluation

The interval evaluation of $f([\mathbf{x}])$ over a box $[\mathbf{x}]$ can be obtained using the following proposition.

Proposition 1. *Given a non-degenerated box $[\mathbf{x}] \in \mathbb{R}^2$, and the function f given by (7), we have*

$$f([\mathbf{x}]) = [f(\mathbb{P}_0 \cup \mathbb{P}_1 \cup \mathbb{P}_2)] \tag{8}$$

with

$$\begin{array}{lcl} \mathbb{P}_0 & = & \{(x_1^-, x_2^-), (x_1^-, x_2^+), (x_1^+, x_2^-), (x_1^+, x_2^+)\} \\ \mathbb{P}_1 & = & \{(x_1, x_2) \in \partial[x_1] \times [x_2] \,|\, x_2 = \varphi_1(x_1, \mathbf{a}, \mathbf{b})\} \\ \mathbb{P}_2 & = & \{(x_1, x_2) \in [x_1] \times \partial[x_2] \,|\, x_1 = \varphi_2(x_2, \mathbf{a}, \mathbf{b})\} \end{array} \tag{9}$$

with

$$\begin{array}{lcl} \varphi_1(x_1, \mathbf{a}, \mathbf{b}) & = & \frac{a_2 \cdot |x_1 - b_1| - b_2 \cdot |x_1 - a_1|}{|x_1 - b_1| - |x_1 - a_1|} \\ \varphi_2(x_2, \mathbf{a}, \mathbf{b}) & = & \frac{a_1 \cdot |x_2 - b_2| - b_1 \cdot |x_2 - a_2|}{|x_2 - b_2| - |x_2 - a_2|} \end{array} \tag{10}$$

This proposition tells us that the extrema for f are reached either on the corners of $[\mathbf{x}]$ or on the edges of $[\mathbf{x}]$ but never in the interior of $[\mathbf{x}]$. The border operator ∂ is used to get the bounds of the interval. For instance $\partial[x_1] \times [x_2] = \{x_1^-, x_1^+\} \times [x_2] = (\{x_1^-\} \times [x_2]) \cup (\{x_1^+\} \times [x_2])$. An illustration is given by Figure 1.

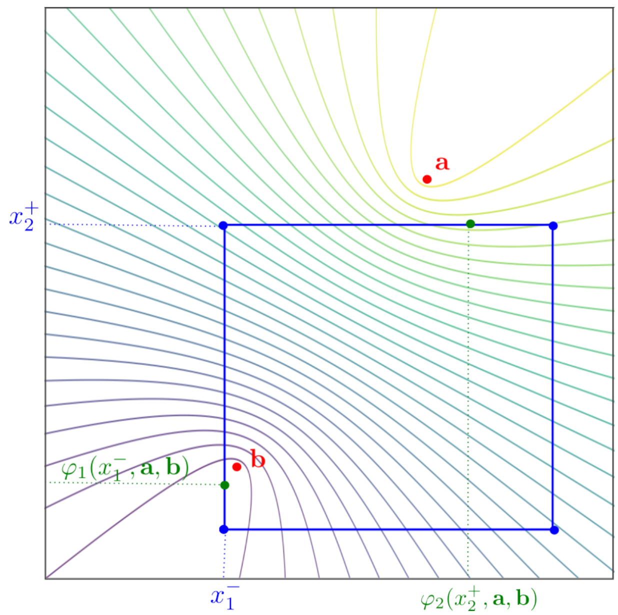

Fig. 1: Level curve of the TDoA function f. The extrema of f over a box are reached either on the corners (blue) or on some specific points of the edges (green)

Proof. We need to solve

$$\begin{array}{c} \text{Optimize} f(\mathbf{x}) \\ \text{s.t. } \mathbf{g}(\mathbf{x}) \leq 0 \end{array} \tag{11}$$

where

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_1 - x_1^+ \\ -x_1 + x_1^- \\ x_2 - x_2^+ \\ -x_2 + x_2^- \end{pmatrix}. \tag{12}$$

The Lagrangian is

$$\begin{array}{rcc} \mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) & = & f(\mathbf{x}) + \boldsymbol{\mu} \cdot \mathbf{g}(\mathbf{x}) \\ & = & \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} - \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2} \\ & & + \mu_1(x_1 - x_1^+) - \mu_2(x_1 - x_1^-) + \mu_3(x_2 - x_2^+) - \mu_4(x_2 - x_2^-) \end{array} \tag{13}$$

The necessary conditions for $\mathbf{x}$ to be an optimizer are:

$$\left\{ \begin{array}{cc} \frac{\partial f}{\partial x_1}(\mathbf{x}) + \mu_1 x_1 - \mu_2 x_1 = 0 & \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) + \mu_3 x_2 - \mu_4 x_2 = 0 & (\text{stationarity}) \\ g_i(\mathbf{x}) \leq 0, \forall i & (\text{primal feasibility}) \\ \mu_i \geq 0 & (\text{dual feasibility}) \\ \mu_i g_i(\mathbf{x}) = 0, \forall i & (\text{complementary slackness}) \end{array} \right. \tag{14}$$

where

$$\begin{array}{rcl} \frac{\partial f}{\partial x_1}(\mathbf{x}) & = & \frac{x_1-a_1}{\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}} - \frac{x_1-b_1}{\sqrt{(x_1-b_1)^2+(x_2-b_2)^2}} \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) & = & \frac{x_2-a_2}{\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}} - \frac{x_2-b_2}{\sqrt{(x_1-b_1)^2+(x_2-b_2)^2}} \end{array} \quad (15)$$

□

Since the box $[\mathbf{x}]$ is non degenerated (*i.e.*, it has a non empty interior), and from the complementary slackness condition, an optimizer should correspond to one of the following situations:

- $\mathbf{x}$ is a corner, $(\mu_1, \mu_3) = (0,0)$, $(\mu_1, \mu_4) = (0,0)$, $(\mu_2, \mu_3) = (0,0)$ or $(\mu_2, \mu_4) = (0,0)$,
- $\mathbf{x}$ is an edge, $(\mu_1, \mu_2, \mu_3) = (0,0,0)$, $(\mu_1, \mu_2, \mu_4) = (0,0,0)$, $(\mu_1, \mu_3, \mu_4) = (0,0,0)$ or $(\mu_2, \mu_3, \mu_4) = (0,0,0)$.
- $\mathbf{x}$ is in the interior of $[\mathbf{x}]$, *i.e.*, $(\mu_1, \mu_2, \mu_3, \mu_4) = (0,0,0,0)$.

Case 1. $\mathbf{x}$ is a corner. The optimizers are inside the set

$$\mathbb{P}_0 = \{(x_1^-, x_2^-), (x_1^-, x_2^+), (x_1^+, x_2^-), (x_1^+, x_2^+)\}. \quad (16)$$

Case 2. $\mathbf{x}$ is an edge. Take first $\mu_2 = \mu_3 = \mu_4 = 0$ which means that we consider the right edge of $[\mathbf{x}]$: $x_1 - x_1^+ = 0$. The other edges are deduced by symmetry. We get

$$\left\{ \begin{array}{c} \frac{\partial f}{\partial x_1}(\mathbf{x}) + \mu_1 x_1 = 0 \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) = 0 \end{array} \right.$$

$$\begin{array}{ll} \Rightarrow & \frac{\partial f}{\partial x_2}(\mathbf{x}) = 0 \\ \Leftrightarrow & \frac{x_2-a_2}{\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}} - \frac{x_2-b_2}{\sqrt{(x_1-b_1)^2+(x_2-b_2)^2}} = 0 \\ \Leftrightarrow & \left\{ \begin{array}{c} (x_2-a_2)^2(x_1-b_1)^2 = (x_2-b_2)^2(x_1-a_1)^2 \\ (x_2-a_2)(x_2-b_2) \geq 0 \end{array} \right. \\ \Leftrightarrow & (x_2-a_2)\sqrt{(x_1-b_1)^2} = (x_2-b_2)\sqrt{(x_1-a_1)^2} \\ \Leftrightarrow & (x_2-a_2)\cdot|x_1-b_1| = (x_2-b_2)\cdot|x_1-a_1| \\ \Leftrightarrow & x_2\cdot|x_1-b_1| - a_2\cdot|x_1-b_1| = x_2\cdot|x_1-a_1| - b_2\cdot|x_1-a_1| \\ \Leftrightarrow & x_2\cdot(|x_1-b_1| - |x_1-a_1|) = a_2\cdot|x_1-b_1| - b_2\cdot|x_1-a_1| \\ \Leftrightarrow & x_2 = \frac{a_2\cdot|x_1-b_1| - b_2\cdot|x_1-a_1|}{|x_1-b_1| - |x_1-a_1|} \end{array}$$

Since $x_1 = x_1^+$, we conclude that $x_2 = \varphi_1(x_1^+, \mathbf{a}, \mathbf{b})$. It means that if an optimizer is in the interior of the right edge, it is the point $(x_1^+, \varphi_1(x_1^+, \mathbf{a}, \mathbf{b}))$.

Case 3. $\mathbf{x}$ is in the interior of $[\mathbf{x}]$. We have

$$\left\{\begin{array}{lll} \frac{\partial f}{\partial x_1}(\mathbf{x}) & = & 0 \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) & = & 0 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{lll} \frac{x_1-a_1}{\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}} & = & \frac{x_1-b_1}{\sqrt{(x_1-b_1)^2+(x_2-b_2)^2}} \\ \frac{x_2-a_2}{\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}} & = & \frac{x_2-b_2}{\sqrt{(x_1-b_1)^2+(x_2-b_2)^2}}) \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{lll} \frac{(x_1-a_1)^2}{(x_1-a_1)^2+(x_2-a_2)^2} & = & \frac{(x_1-b_1)^2}{(x_1-b_1)^2+(x_2-b_2)^2} \\ \frac{(x_2-a_2)^2}{(x_1-a_1)^2+(x_2-a_2)^2} & = & \frac{(x_2-b_2)^2}{(x_1-b_1)^2+(x_2-b_2)^2} \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (x_1-a_1)^2(x_2-b_2)^2 = (x_2-a_2)^2(x_1-b_1)^2 \\ (x_1-a_1)(x_1-b_1) \geq 0 \\ (x_2-a_2)(x_2-b_2) \geq 0 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (x_1-a_1)(x_2-b_2) = (x_2-a_2)(x_1-b_1) \\ (x_1-a_1)(x_1-b_1) \geq 0 \end{array}\right.$$

It means that $\mathbf{x}$ belongs to one of the two exterior half line delimited by $\mathbf{a}, \mathbf{b}$ which crosses the boundary of $[\mathbf{x}]$. The extremum correspond to $\pm\|\mathbf{a}-\mathbf{b}\|$, which is also reached by one element of the boundary.

2.2 Illustration

Consider the set

$$\mathbb{X} = \left\{\mathbf{x} \in \mathbb{R}^2 | \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| \in [y]\right\} \tag{17}$$

where $\mathbf{a} = (-1, -2)$, $\mathbf{b} = (2, 3)$ and $[y] = [3, 5]$. Equivalently, we have

$$\mathbb{X} = f^{-1}([y]) \tag{18}$$

where

$$f(\mathbf{x}) = \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| \tag{19}$$

We have the following tests

$$\begin{array}{lll} f([\mathbf{x}]) \cap [y] = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset \\ f([\mathbf{x}]) \subset [y] = \emptyset & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \end{array} \tag{20}$$

These tests can be used by a paver to approximate $\mathbb{X}$. Now, only an outer approximation of $f([\mathbf{x}])$ can be computed. If we use the inclusion test based on the KKT conditions, we get Figure 2. We observe that only boxes that intersect the boundary of $\mathbb{X}$ are bisected by the paver. This is due to the fact that we have a minimal inclusion test and that f is scalar. Equivalently, we can say that we have no *clustering effect*. Using a classical interval extension [15], we get Figure 3 which contains more boxes (238853 instead of 52779) for the same accuracy ($\varepsilon = 0.01$). The clustering effect is now visible. The computing time approximately 10 times smaller (less than 0.05 sec) with the KKT approach. The frame box is $[-15, 15]^2$.

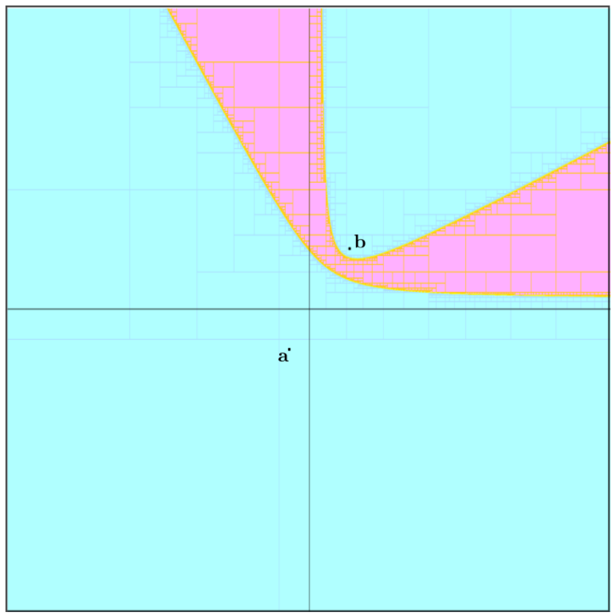

Fig. 2: Set $\mathbb{X}$ obtained by the paver using the KKT conditions

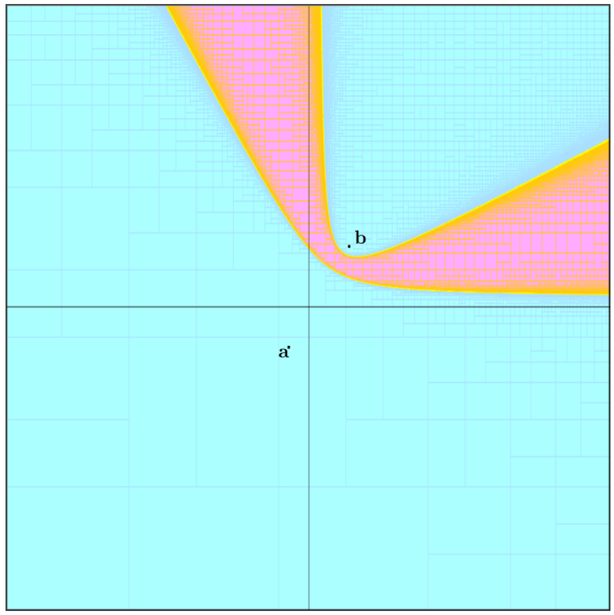

Fig. 3: Set $\mathbb{X}$ obtained by the paver using a natural interval extension for f

2.3 Contractor from the inclusion test

Consider a set $\mathbb{X}$ for which we have an inclusion test $[t]$ (see [11]). Recall that an inclusion test returns a Boolean interval, *i.e.*, an element of $\mathbb{IB} = \{[0,0],[0,1],[1,1]\}$ such that

$$\begin{array}{lll} [t]([\mathbf{x}]) = [0,0] & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset \\ [t]([\mathbf{x}]) = [1,1] & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} = \emptyset. \end{array} \tag{21}$$

From an inclusion test $[t]$ for $\mathbb{X}$, we can define the contractor $\mathcal{C}_{\mathbb{X}}$ for $\mathbb{X}$ as

$$\begin{array}{ll} \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) = \emptyset & \text{if } [t]([\mathbf{x}]) = [0,0] \\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) = [\mathbf{x}] & \text{otherwize} \end{array} \tag{22}$$

Such a contractor is said to be *binary* since it contracts a box either to the empty set or not at all. If the test $[t]$ is minimal, the contractor $\mathcal{C}_{\mathbb{X}}$ will not yield a clustering effect. This shows why when we want to build an efficient contractor, it is important to focus mostly on the forward part.

Casting an inclusion test into a contractor allows us to use the contractor algebra and the composition. This will be illustrated by the following section.

3 Action of a contractor on a separator

For contractors as well for separators, classical operations of sets, such as $\cap, \cup, \ldots$ can be used. We propose here a new operation combining contractors and separators. We will first introduce the classical notion of correspondence (or multivalued mapping) which can be seen as a generalization of functions. This leads us to the notion of *directed contractors* defined in [7].

3.1 Correspondence

A *correspondence* [1](or *binary relation*) between two sets $\mathbb{A}$ and $\mathbb{B}$ is any subset $\mathbb{F}$ of the Cartesian product $\mathbb{A} \times \mathbb{B}$. The domain of $\mathbb{F}$ is

$$\mathrm{dom}\mathbb{F} = \{a \in \mathbb{A} \,|\, \exists b, (a, b) \in \mathbb{F}\}. \tag{23}$$

The range of $\mathbb{F}$ is

$$\mathrm{range}\mathbb{F} = \{b \in \mathbb{B} \,|\, \exists a, (a, b) \in \mathbb{F}\}. \tag{24}$$

The image of $a \in \mathbb{A}$ by $\mathbb{F}$ is

$$\mathbb{F}(a) = \{b, (a, b) \in \mathbb{F}\} \tag{25}$$

and the co-image of b by $\mathbb{F}$ is

$$\mathbb{F}^{-1}(b) = \{a, (a, b) \in \mathbb{F}\} \tag{26}$$

The *inverse* of $\mathbb{F}$ is the correspondence defined by

$$\mathbb{F}^{\#} = \{(b, a) | (a, b) \in \mathbb{F}\}. \tag{27}$$

3.2 Contractor for a correspondence

Consider a contractor $\mathcal{C}_{\mathbb{F}}$ for the correspondence $\mathbb{F} \subset \mathbb{R}^n \times \mathbb{R}^p$. We define the *forward contractor* as

$$\overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]}([\mathbf{x}]) = \pi_{\mathbf{y}} \circ \mathcal{C}_{\mathbb{F}}([\mathbf{x}], [\mathbf{y}]) \tag{28}$$

where $\pi_{\mathbf{y}}$ represents the projection onto $\mathbb{R}^p$ parallel to $\mathbb{R}^n$. The *backward contractor* is defined by

$$\overleftarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{x}]}([\mathbf{y}]) = \pi_{\mathbf{x}} \circ \mathcal{C}_{\mathbb{F}}([\mathbf{x}], [\mathbf{y}]) \tag{29}$$

Often, in our applications, $\mathbb{F}$ corresponds to a function $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^p$ or more precisely to the graph of a function: $\mathbb{F} = \{(\mathbf{x}, \mathbf{y}) | \mathbf{y} = \mathbf{f}(\mathbf{x})\}$.

3.3 Action

Consider the correspondence $\mathbb{F} \subset \mathbb{R}^n \times \mathbb{R}^p$ and the set $\mathbb{Y} \subset \mathbb{R}^p$. We define the action of $\mathbb{F}$ on $\mathbb{Y}$ as

$$\mathbb{F} \bullet \mathbb{X} = \{\mathbf{y} \in \mathbb{R}^p | \exists \mathbf{x} \in \mathbb{X}, (\mathbf{x}, \mathbf{y}) \in \mathbb{F}\}. \tag{30}$$

As a consequence,

$$\mathbb{F}^{\#} \bullet \mathbb{Y} = \{\mathbf{x} \in \mathbb{R}^n | \exists \mathbf{y} \in \mathbb{Y}, (\mathbf{x}, \mathbf{y}) \in \mathbb{F}\}. \tag{31}$$

Note that the action used here has some similarities with the operators used for group action [16], even if the group structure does not exist here.

Proposition 2. *Consider a separator* $\mathcal{S}_{\mathbb{X}} = \left\{\mathcal{S}_{\mathbb{X}}^{in}, \mathcal{S}_{\mathbb{X}}^{out}\right\}$ *for* $\mathbb{X}$ *and a contractor* $\mathcal{C}_{\mathbb{F}}$ *for* $\mathbb{F} \subset \mathbb{R}^n \times \mathbb{R}^p$. *A separator* $\mathcal{S}_{\mathbb{Y}}$ *for the set* $\mathbb{Y} = \mathbb{F} \bullet \mathbb{X}$, *denoted by* $\mathcal{S}_{\mathbb{Y}} = \mathcal{C}_{\mathbb{F}} \bullet \mathcal{S}_{\mathbb{X}}$ *is:*

$$\begin{array}{rcl} \mathcal{S}_{\mathbb{Y}}([\mathbf{y}]) & = & \mathcal{C}_{\mathbb{F}} \bullet \mathcal{S}_{\mathbb{X}}([\mathbf{y}]) \\ & = & \left\{\mathcal{S}_{\mathbb{Y}}^{in}([\mathbf{y}]), \mathcal{S}_{\mathbb{Y}}^{out}([\mathbf{y}])\right\} \\ & = & \left\{([[\mathbf{y}] \backslash \mathbb{F} \bullet \mathbb{R}^n]) \sqcup \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{in} \circ \overleftarrow{\mathcal{C}}_{\mathbb{F}}^{\mathbb{R}^n}([\mathbf{y}]), \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{out} \circ \overleftarrow{\mathcal{C}}_{\mathbb{F}}^{\mathbb{R}^n}([\mathbf{y}])\right\} \end{array}$$

In this formula, $[[\mathbf{y}] \backslash \mathbb{F} \bullet \mathbb{R}^n]$ represents the smallest box which encloses the set, where

$$[\mathbf{y}] \backslash \mathbb{F} \bullet \mathbb{R}^n = \{\mathbf{y} \in [\mathbf{y}] \mid \mathbf{y} \notin \mathbb{F} \bullet \mathbb{R}^n\}.$$

An illustration is provided by Figures 4 and 5.

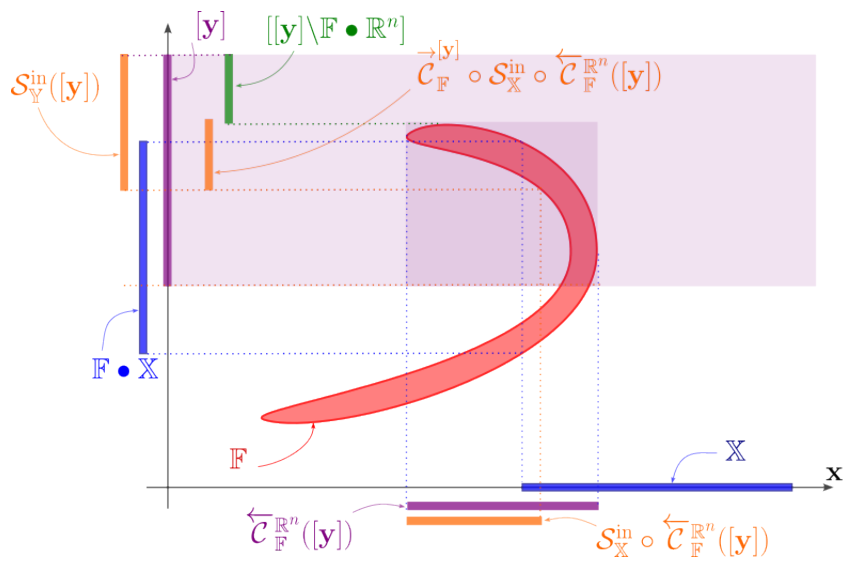

Fig. 4: Inner contractor for the set $\mathbb{F} \bullet \mathbb{X}$

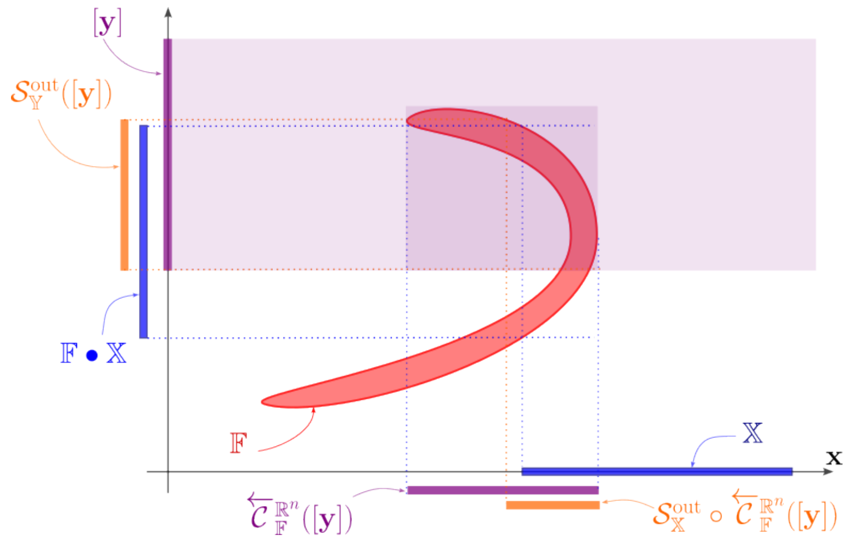

Fig. 5: Outer contractor for the set $\mathbb{F} \bullet \mathbb{X}$

Proof. We have

$$\begin{aligned}
\mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}]) &= [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \sqcup \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \overleftarrow{\mathcal{C}}_{\mathbb{F}}^{\mathbb{R}^n}([\mathbf{y}]) \\
&= [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \sqcup \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \pi_{\mathbf{x}}\left(\mathcal{C}_{\mathbb{F}}(\mathbb{R}^n, [\mathbf{y}])\right) \\
&\supset [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \sqcup \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \pi_{\mathbf{x}}\left([(\mathbf{x},\mathbf{y})|(\mathbf{x},\mathbf{y}) \in \mathbb{F}]\right) \\
&= [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \sqcup \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{in}}\left([\mathbf{x}|\exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x},\mathbf{y}) \in \mathbb{F}]\right) \\
&\supset [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \sqcup \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]}\left([\mathbf{x} \notin \mathbb{X}|\exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x},\mathbf{y}) \in \mathbb{F}]\right) \\
&= [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \sqcup \pi_{\mathbf{y}} \circ \mathcal{C}_{\mathbb{F}}([\mathbf{x} \notin \mathbb{X}|\exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x},\mathbf{y}) \in \mathbb{F}], [\mathbf{y}]) \\
&\supset [[\mathbf{y}]\backslash\mathbb{F} \bullet \mathbb{R}^n] \cap \{\mathbf{y} \in [\mathbf{y}]|\exists \mathbf{x} \notin \mathbb{X}, (\mathbf{x},\mathbf{y}) \in \mathbb{F}\} \\
&= [\mathbf{y}] \cap (\mathbb{R}^p\backslash\mathbb{F} \bullet \mathbb{X})
\end{aligned} \tag{32}$$

Moreover

$$\begin{aligned}
\mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) &= \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \overleftarrow{\mathcal{C}}_{\mathbb{F}}^{\mathbb{R}^n}([\mathbf{y}]) \\
&= \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \pi_{\mathbf{x}}\left(\mathcal{C}_{\mathbb{F}}(\mathbb{R}^n, [\mathbf{y}])\right) \\
&\supset \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \pi_{\mathbf{x}}\left([(\mathbf{x},\mathbf{y})|(\mathbf{x},\mathbf{y}) \in \mathbb{F}]\right) \\
&= \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]} \circ \mathcal{S}_{\mathbb{X}}^{\text{out}}\left([\mathbf{x}|\exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x},\mathbf{y}) \in \mathbb{F}]\right) \\
&\supset \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{y}]}\left([\mathbf{x} \in \mathbb{X}|\exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x},\mathbf{y}) \in \mathbb{F}]\right) \\
&= \pi_{\mathbf{y}} \circ \mathcal{C}_{\mathbb{F}}([\mathbf{x} \in \mathbb{X}|\exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x},\mathbf{y}) \in \mathbb{F}], [\mathbf{y}]) \\
&\supset \{\mathbf{y} \in [\mathbf{y}]|\exists \mathbf{x} \in \mathbb{X}, (\mathbf{x},\mathbf{y}) \in \mathbb{F}\} \\
&= [\mathbf{y}] \cap \mathbb{F} \bullet \mathbb{X}
\end{aligned} \tag{33}$$

□

Proposition 3. *Consider a separator* $\mathcal{S}_{\mathbb{Y}} = \left\{\mathcal{S}_{\mathbb{Y}}^{in}, \mathcal{S}_{\mathbb{Y}}^{out}\right\}$ *for* $\mathbb{Y} \subset \mathbb{R}^p$ *and a contractor* $\mathcal{C}_{\mathbb{F}}$ *for* $\mathbb{F} \subset \mathbb{R}^n \times \mathbb{R}^p$*. A separator* $\mathcal{S}_{\mathbb{X}}$ *for the set* $\mathbb{X} = \mathbb{F}^{\#} \bullet \mathbb{Y}$ *is:*

$$\begin{array}{rcl} \mathcal{S}_{\mathbb{X}}([\mathbf{x}]) & = & \mathcal{C}_{\mathbb{F}^{\#}} \bullet \mathcal{S}_{\mathbb{X}}([\mathbf{x}]) \\ & = & \left\{\mathcal{S}_{\mathbb{X}}^{in}([\mathbf{x}]), \mathcal{S}_{\mathbb{X}}^{out}([\mathbf{x}])\right\} \\ & = & \left\{\left([[\mathbf{x}] \backslash \mathbb{F}^{\#} \bullet \mathbb{R}^p]\right) \sqcup \overleftarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{x}]} \circ \mathcal{S}_{\mathbb{Y}}^{in} \circ \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{\mathbb{R}^p}([\mathbf{x}]), \overleftarrow{\mathcal{C}}_{\mathbb{F}}^{[\mathbf{x}]} \circ \mathcal{S}_{\mathbb{Y}}^{out} \circ \overrightarrow{\mathcal{C}}_{\mathbb{F}}^{\mathbb{R}^p}([\mathbf{x}])\right\} \end{array}$$

Proof. It is a direct consequence of Proposition 2. □

3.4 Illustration

Consider the two disks

$$\begin{array}{rcl} \mathbb{Y}_1 & = & \left\{(y_1, y_2) | \, (y_1 - 2)^2 + (y_2 - 1)^2 - 1 \leq 0\right\} \\ \mathbb{Y}_2 & = & \left\{(y_1, y_2) | \, (y_1 + 1)^2 + (y_2 + 2)^2 - 1 \leq 0\right\} \end{array} \quad (34)$$

Consider the set $\mathbb{F}$ of all $(\mathbf{x}, \mathbf{y}) \subset \mathbb{R}^n$which satisfy:

$$\mathbb{F} : \left\{ \begin{array}{rcl} \|\mathbf{x} - \mathbf{m}(1)\| - \|\mathbf{x} - \mathbf{m}(2)\| - y_1 & = & 0 \\ \|\mathbf{x} - \mathbf{m}(2)\| - \|\mathbf{x} - \mathbf{m}(3)\| - y_2 & = & 0 \end{array} \right. \quad (35)$$

where

$$\mathbf{m}(1) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \mathbf{m}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{m}(3) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \quad (36)$$

We want to characterize the set

$$\mathbb{X} = \{\mathbf{x} | \exists \mathbf{y} \in \mathbb{Y}_1 \cup \mathbb{Y}_2, (\mathbf{x}, \mathbf{y}) \in \mathbb{F}\}. \quad (37)$$

Since

$$\mathbb{X} = \mathbb{F}^{\#} \bullet (\mathbb{Y}_1 \cup \mathbb{Y}_2), \quad (38)$$

we get the following separator for $\mathbb{X}$:

$$\mathcal{S}_{\mathbb{X}} = \mathcal{C}_{\mathbb{F}^{\#}} \bullet (\mathcal{S}_{\mathbb{Y}_1} \cup \mathcal{S}_{\mathbb{Y}_2}) \quad (39)$$

where $\mathcal{C}_{\mathbb{F}^{\#}}$ is contractor for $\mathbb{F}$ and $\mathcal{S}_{\mathbb{Y}_1}, \mathcal{S}_{\mathbb{Y}_2}$ are separators for $\mathbb{Y}_1, \mathbb{Y}_2$. Using a paver, we get the approximation of $\mathbb{X}$ depicted in Figure 6.

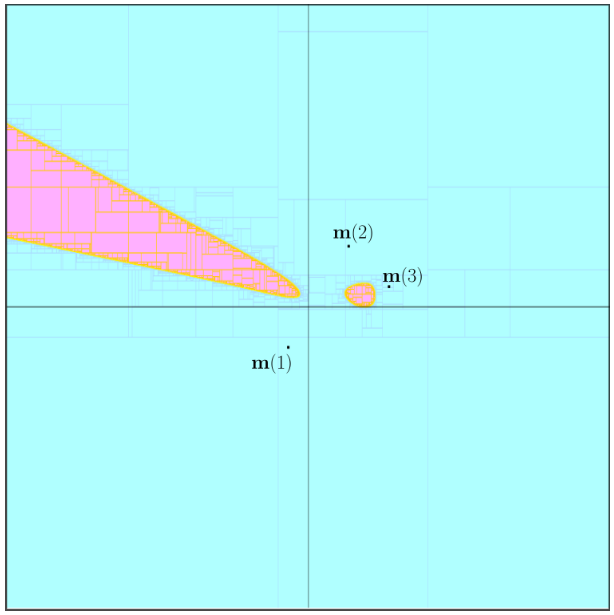

Fig. 6: Approximation of the set $\mathbb{X}$ of all $\mathbf{x}$ consistent with the two disks $\mathbb{Y}_1, \mathbb{Y}_2$

4 Application

Consider three microphones at positions $\mathbf{m}(1), \mathbf{m}(2), \mathbf{m}(3)$ of the plane (see (36)). They record sounds $s_1(t), s_2(t), s_3(t)$ of the noisy environment for a short time window. If for τ_1, τ_2,we observe that $s_1(t), s_2(t+\tau_1), s_3(t+\tau_1+\tau_2)$ are correlated, then we can guess the noise has possibly been emitted from a position $\mathbf{x}$ which satisfies

$$\left\{ \begin{array}{lcl} \|\mathbf{x}-\mathbf{m}(1)\| - \|\mathbf{x}-\mathbf{m}(2)\| - c\tau_1 & = & 0 \\ \|\mathbf{x}-\mathbf{m}(2)\| - \|\mathbf{x}-\mathbf{m}(3)\| - c\tau_2 & = & 0 \end{array} \right.$$

where c is the celerity of the sound. The quantity $y_1 = c\tau_1$ and $y_2 = c\tau_2$ are called *pseudo-distances*. Using a time-frequency analysis [3], it is possible to get a possibility distribution [5] in the pseudo-distance plane (y_1, y_2).

For simplicity, assume that this possibility distribution is given by:

$$\mu(\mathbf{y}) = e^{-(y_1-2)^2-(y_2-1)^2}.$$

Figure 7 illustrates this possibility distribution for some α-cuts, where

$$\alpha_i = e^{-2^{i-1}}, i \in \{0, \ldots, 5\}.$$

Equivalently, the α-cuts are defined by

$$\begin{array}{rcl} \mathbb{Y}_{\alpha_i} & = & \left\{\mathbf{y} | e^{-(y_1-2)^2-(y_2-1)^2} \geq \alpha_i\right\} \\ & = & \left\{\mathbf{y} | (y_1-2)^2 - (y_2-1)^2 \leq 2^{i-1}\right\} \end{array}$$

The frame box is $[-10,10]^2$.

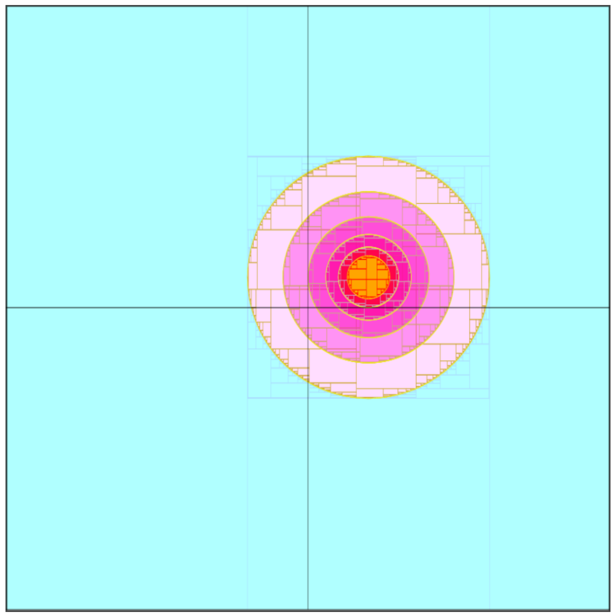

Fig. 7: Possibility distribution $\mu(\mathbf{y})$ represented by its α cuts $\mathbb{Y}_\alpha$

For real applications, the possibility distribution has no reason to be nested disks except maybe in the case where we have a unique source.

The corresponding possibility distribution for $\mathbf{x}$ is described by the α-cuts:

$$\mathbb{X}_\alpha = \mathbb{F}^\# \bullet \mathbb{Y}_\alpha$$

as represented by Figure 8. This image gives us an idea of where the sound sources can possibly be located.

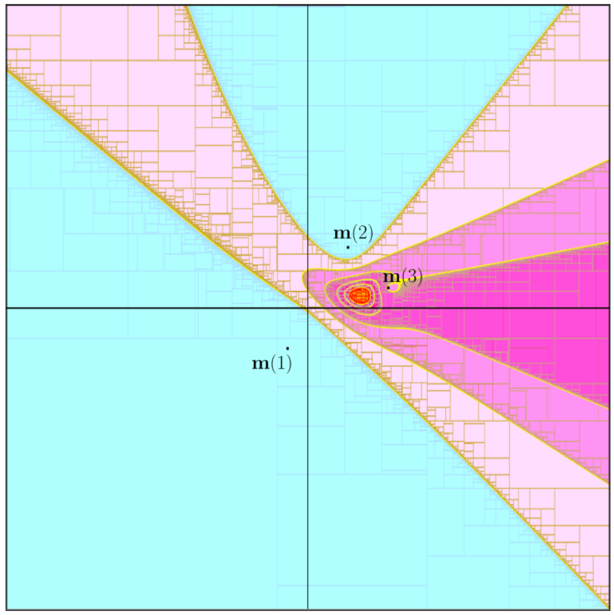

Fig. 8: Possibility distribution $\mathbb{X}_\alpha$ for the location of the sound sources

5 Conclusion

This paper has proposed to use the Karush–Kuhn–Tucker (KKT) conditions to build efficient contractors for constraints of the form $y = f(\mathbf{x})$. The motivating example that has been chosen considers the equation of TDoA (Time Difference of Arrival) where the classical interval propagation creates an unwanted pessimism due to the multi-occurences of the variables. The KKT conditions lead us to a minimal inclusion test. As a consequence, we were able to build a binary contractor for the TDoA constraint with no clustering effect.

Another contribution of the paper is the definition of the action of a contractor on a separator. This operation allowed us to build separators by composition. The separator algebra, as defined in [10], extended set operations such as the intersection, the union or the complement, to separators, but without the possibility the compose the separators. We have shown that the compositions should not be performed between separators, but between contractors and separators.

The application that has been considered illustrates that a possibility distribution can easily and efficiently be inverted through the TDoA contractor to localize sources in a noisy environment.

The Python code based on Codac [19] is given in [8].

References

[1] J.P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, Boston, 1990. 3.1

[2] G. Chabert and L. Jaulin. Contractor Programming. *Artificial Intelligence*, 173:1079–1100, 2009. 2

[3] L. Cohen. *Time–Frequency Analysis*. Prentice-Hall, New York, 1995. 4

[4] V. Drevelle. *Etude de méthodes ensemblistes robustes pour une localisation multisensorielle intègre. Application à la navigation des véhicules en milieu urbain.* PhD dissertation, Université de Technologie de Compiègne, Compiègne, France, 2011. 2

[5] D. Dubois and H. Prade. *Fussy Sets and Systems-Theory and Applications.* Academic Press, New York, NY, 1980. 4

[6] E. R. Hansen. Bounding the solution of interval linear equations. *SIAM Journal on Numerical Analysis*, 29(5):1493–1503, 1992. 1, 1

[7] L. Jaulin. A boundary approach for set inversion. *Engineering Applications of Artificial Intelligence*, 100:104184, 2021. 3

[8] L. Jaulin. Codes associated with the paper entitled: Karush–Kuhn–Tucker conditions to build efficient contractors; Application to TDoA localization. `www.ensta-bretagne.fr/jaulin/ctctdoa.html`, 2023. 5

[9] L. Jaulin. Optimal separator for an hyperbola; application to localization. *arXiv:2305.15519, math.NA*, 2023. 2

[10] L. Jaulin and B. Desrochers. Introduction to the algebra of separators with application to path planning. *Engineering Applications of Artificial Intelligence*, 33:141–147, 2014. 5

[11] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*. Springer-Verlag, London, 2001. 2.3

[12] W. Karush. *Minima of Functions of Several Variables with Inequalities as Side Constraints*. (M.Sc. thesis). Dept. of Mathematics, Univ. of Chicago, Chicago, Illinois, 1939. 1

[13] H. Kuhn and A. Tucker. Nonlinear programming. In *Proceedings of 2nd Berkeley Symposium*, pages 481–492. University of California Press, 1951. 1

[14] H. B. Lee. Accuracy limitations of hyperbolic multilateration systems. *IEEE Transactions on Aerospace and Electronic Systems*, AES-11(1):16–29, 1975. 1

[15] R. Moore. *Methods and Applications of Interval Analysis.* Society for Industrial and Applied Mathematics, jan 1979. 1, 2.2

[16] P.J. Olver. *Classical Invariant Theory.* Graduate Texts in Mathematics. Univ. Press, Cambridge, U.K., 1999. 3.3

[17] H. Ratschek and J. Rokne. *New Computer Methods for Global Optimization.* Ellis Horwood, Chichester, UK, 1988. 1

[18] O. Reynet, L. Jaulin, and G. Chabert. Robust tdoa passive location using interval analysis and contractor programming. In *Radar*, Bordeaux, France, 2009. 2

[19] S. Rohou. *Codac (Catalog Of Domains And Contractors), available at* `http://codac.io/`. Robex, Lab-STICC, ENSTA-Bretagne, 2021. 5