Optimal separator for an hyperbola Application to localization

Luc Jaulin

Lab-Sticc, ENSTA-Bretagne

Abstract. This paper proposes a minimal contractor and a minimal separator for an area delimited by an hyperbola of the plane. The task is facilitated using actions induced by the hyperoctahedral group of symmetries. An application related to the localization of an object using a TDoA (Time Differential of Arrival) technique is proposed.

1 Introduction

Consider the quadratic function

$$f(\mathbf{q},\mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \qquad (1)$$

where $\mathbf{q} = (q_0, \dots, q_5)$ is the parameter vector and $\mathbf{x} = (x_1, x_2)$ is the vector of variables. Equivalently, we can write the function in a matrix form:

$$f(\mathbf{q},\mathbf{x}) = \mathbf{x}^{\mathrm{T}} \cdot \underbrace{\begin{pmatrix} q_3 & \frac{1}{2}q_4 \\ \frac{1}{2}q_4 & q_5 \end{pmatrix}}_{\mathbf{Q}} \cdot \mathbf{x} + \begin{pmatrix} q_1 & q_2 \end{pmatrix} \cdot \mathbf{x} + q_0. \qquad (2)$$

The zeros of $f(\mathbf{q},\mathbf{x})$ is a conic section (a circle or other ellipse, a parabola, or a hyperbola). The characteristic polynomial of the matrix $\mathbf{Q}$ is

$$\begin{array}{rcl} P(s) & = & (s-q_3)(s-q_5) - \frac{1}{4}q_4^2 \\ & = & s^2 - (q_3+q_5)s + q_3 q_5 - \frac{1}{4}q_4^2 \end{array}$$

Its discriminant is

$$\begin{array}{rcl} \Delta & = & (q_3+q_5)^2 - 4q_3q_5 + q_4^2 \\ & = & q_3^2 + q_5^2 - 2q_3q_5 + q_4^2 \\ & = & (q_3-q_5)^2 + q_4^2 \end{array}$$

which is always positive. Which means that the matrix $\mathbf{Q}$ has two real values (this is not a surprise since $\mathbf{Q}$ is symmetric). We will assume here that $\mathbf{Q}$ has eigen values with different signs. It means that

$$\begin{array}{cc} & \left(q_3+q_5-\sqrt{\Delta}\right)\left(q_3+q_5+\sqrt{\Delta}\right)<0 \\ \Leftrightarrow & (q_3+q_5)^2-\Delta<0 \\ \Leftrightarrow & (q_3+q_5)^2-((q_3-q_5)^2+q_4^2)<0 \\ \Leftrightarrow & q_3^2+q_5^2+2q_3q_5-(q_3^2+q_5^2-2q_3q_5+q_4^2)<0 \\ \Leftrightarrow & 4q_3q_5-q_4^2<0 \\ \Leftrightarrow & \det\mathbf{Q}<0 \end{array}$$

Define the set

$$\mathbb{X}=\{(x_1,x_2|f(\mathbf{q},\mathbf{x})\leq 0\}. \tag{3}$$

In our case $\mathbb{X}$ has a boundary which is an hyperbola and will be called an *hyperbolic area*. In this paper, we propose an interval-based method [14] to generate an optimal separator [11] for the set $\mathbb{X}$. The technique is similar to that proposed in [10] for ellipses. This separator will be used to generate an inner and an outer approximations for $\mathbb{X}$.

As an application, we will consider the problem of the localization of an object using a TDoA (Time Difference of Arrival) technique. TDoA is a classical positioning methodology that determines the difference between the time-of-arrival of signals. TDoA is often used in a real-time to accurately calculate the location of some tracked entities.

This paper is organized as follows. Section 2 introduces the notion of symmetries that will be used in the construction of the separators. Section 3 defines the concept of cardinal function associated with a set. Section 4 builds the separator for the hyperbolic area using cardinal functions and symmetries. Section 5 illustrates the use of the separator to approximate the set of position for an object from the measure of pseudo-distances. Section 6 concludes the paper.

2 Symmetries

The methodology presented in this paper is based on symmetries of the equation of $f(\mathbf{q},\mathbf{x})=0$. This section defines the main concepts related to symmetries that will be used.

2.1 Conjugate pair

Consider an equation of the form

$$f(\mathbf{q},\mathbf{x})=0. \tag{4}$$

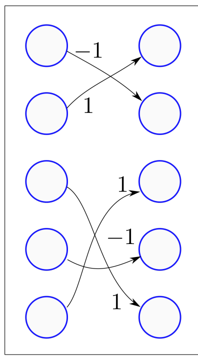
The pair of transformations (σ,γ) is *conjugate* with respect to f if

$$f(\gamma(\mathbf{q}),\sigma(\mathbf{x}))=0\Leftrightarrow f(\mathbf{q},\mathbf{x})=0. \tag{5}$$

2.2 Hyperoctahedral group

Transformations that will be consider are limited to the *hyperoctahedral group* B_n [3] which is the group of symmetries of the hypercube $[-1,1]^n$ of $\mathbb{R}^n$. The group B_n corresponds to the group of $n \times n$ orthogonal matrices whose entries are integers. Each line and each column of a matrix should contain one and only one non zero entry which should be either 1 or -1. Figure 1 shows different notations usually considered to represent a symmetry σ of B_5. We will prefer the Cauchy one line notation [20] which is shorter. We should understand the symmetry σ of the figure as the function:

$$\sigma(x_1, x_2, x_3, x_4, x_5) = (-x_2, x_1, x_5, -x_4, x_3). \tag{6}$$



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 5 & -4 & 3 \end{pmatrix}$$

Fig. 1: Different representations of an element σ of B_5. Left: graph; Top right: Matrix notation; Bottom right: Cauchy one line notation

In the plane, the group B_2 has eight elements. If we use the matrix form, the elements of B_2 are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{7}$$

Equivalently with the Cauchy notation, these 8 elements of B_2 are respectively

$$\{(1,2), (-1,2), (2,1), (-1,-2), (1,-2), (-2,1), (2,-1), (-2,-1)\}. \tag{8}$$

A symmetry of B_2 in a matrix form, satisfies

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \tag{9}$$

with $\sigma_{ij}^2 \in \{0,1\}, \sigma_{i1}^2 + \sigma_{i2}^2 = 1, \sigma_{1j}^2 + \sigma_{1j}^2 = 1$. The Cauchy form is obtained from the matrix form by left multiplying by the line vector $(1,2)$:

$$\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = (\sigma_{11} + 2\sigma_{21}, \sigma_{12} + 2\sigma_{22}). \tag{10}$$

2.3 Hyperbolic symmetries

The following theorem gives the symmetries of the hyperbola.

Proposition 1. *Take a point* $\mathbf{x} = (x_1, x_2)$ *such*

$$f(\mathbf{q}, \mathbf{x}) \overset{(1)}{=} q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 = 0 \tag{11}$$

and a symmetry

$$\sigma = (\sigma_{11} + 2\sigma_{21}, \sigma_{12} + 2\sigma_{22}) \in B_2. \tag{12}$$

Define

$$\begin{aligned} \gamma \quad = \quad & (q_0, \sigma_{11} q_1 + \sigma_{21} q_2, \sigma_{12} q_1 + \sigma_{22} q_2, \sigma_{11}^2 q_3 \\ & + \sigma_{21}^2 q_5, \sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}) q_4, \sigma_{12}^2 q_3 + \sigma_{22}^2 q_5) \end{aligned} \tag{13}$$

The pair (σ^{-1}, γ) *is conjugate with respect to* $f(\mathbf{q}, \mathbf{x})$.

Proof. Define

$$\begin{aligned} x_1 &= \sigma_{11} \cdot y_1 + \sigma_{12} \cdot y_2 \\ x_2 &= \sigma_{21} \cdot y_1 + \sigma_{22} \cdot y_2 \end{aligned} \tag{14}$$

We have

$$\begin{aligned} f(\mathbf{q}, \mathbf{x}) &= q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \\ &= q_0 + q_1(\sigma_{11} y_1 + \sigma_{12} y_2) + q_2(\sigma_{21} y_1 + \sigma_{22} y_2) + q_3(\sigma_{11} y_1 + \sigma_{12} y_2)^2 \\ & \quad + q_4(\sigma_{11} y_1 + \sigma_{12} y_2)(\sigma_{21} y_1 + \sigma_{22} y_2) + q_5(\sigma_{21} y_1 + \sigma_{22} y_2)^2 \\ &= q_0 + (\sigma_{11} q_1 + \sigma_{21} q_2)\, y_1 + (\sigma_{12} q_1 + \sigma_{22} q_2)\, y_2 + (\sigma_{11}^2 q_3 + \sigma_{21}^2 q_5) y_1^2 \\ & \quad + (\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}) q_4 y_1 y_2 + (\sigma_{12}^2 q_3 + \sigma_{22}^2 q_5) y_2^2 \end{aligned}$$

Thus

$$\begin{aligned} & f(\mathbf{q}, \mathbf{x}) = 0 \\ \Leftrightarrow \quad & f(\gamma(\mathbf{q}), \mathbf{y}) = 0 \\ \Leftrightarrow \quad & f(\gamma(\mathbf{q}), \sigma^{-1}(\mathbf{x})) = 0. \end{aligned} \tag{15}$$

□

2.4 Choice function

Considering Proposition 1, we get the choice function ψ [9]:

$$\begin{aligned} \psi_\sigma(\mathbf{q}) \quad = \quad & (q_0, \alpha_{11} q_1 + \alpha_{21} q_2, \alpha_{12} q_1 + \alpha_{22} q_2, \\ & \alpha_{11}^2 q_3 + \alpha_{21}^2 q_5, \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}) q_4, \alpha_{12}^2 q_3 + \alpha_{22}^2 q_5) \end{aligned} \tag{16}$$

where $\alpha = \sigma^{-1}$. Given a symmetry σ, this choice function allows us to get a symmetry γ such that (σ, γ) is a conjugate pair.

3 Cardinal functions

For a given $\mathbf{q}$, the solution set of the equation $f(\mathbf{q}, \mathbf{x}) = 0$ (hyperbola or not) can be decomposed into functions partially defined. A possible decomposition which works for the hyperbola is based on cardinal functions to be introduced in this section.

3.1 Some definitions

Definition 1. A cardinal vector of $\mathbb{R}^n$ is a vector

$$\mathbf{e} = (e_1, \ldots, e_n)^{\mathrm{T}} \tag{17}$$

such that $\|\mathbf{e}\| = 1$ and $e_i \in \{-1, 0, 1\}$.

For instance $\mathbf{e}_3 = (0,0,1,0)^{\mathrm{T}}$ and $\mathbf{e}_{-2} = (0,-1,0,0)$ are two cardinal vectors of $\mathbb{R}^4$. We use the notation $\mathbf{e}_i$ where $i \in I = \{-n, \ldots, -1, 1, \ldots, n\}$ to specify the cardinal vector. For instance $\mathbf{e}_{-2}$ is the vector parallel to the 2 axis with a negative direction.

Definition 2. Given a closed set $\mathbb{X}$ of $\mathbb{R}^n$. A cardinal function φ_i with $i \in \{-n, \ldots, -1, 1, \ldots, n\}$ is defined by

$$\begin{array}{l} \varphi_i(x_1, \ldots, x_{|i|-1}, x_{|i|+1}, \ldots, x_n) \\ = \ \max \left\{ \mathbf{x}^{\mathrm{T}} \cdot \mathbf{e}_i \,|\, \mathbf{x} = (x_1, \ldots, x_{|i|-1}, x_i, x_{|i|+1}, \ldots, x_n) \in \mathbb{X} \right\} \end{array} \tag{18}$$

Figure 2 shows in case of $n = 2$, a representation of the functions $\varphi_1(x_2)$ (red) and $\varphi_{-1}(x_2)$ (blue). The small squares correspond to cardinal points (East in red and West in blue). Here, we have two Easts and two Wests.

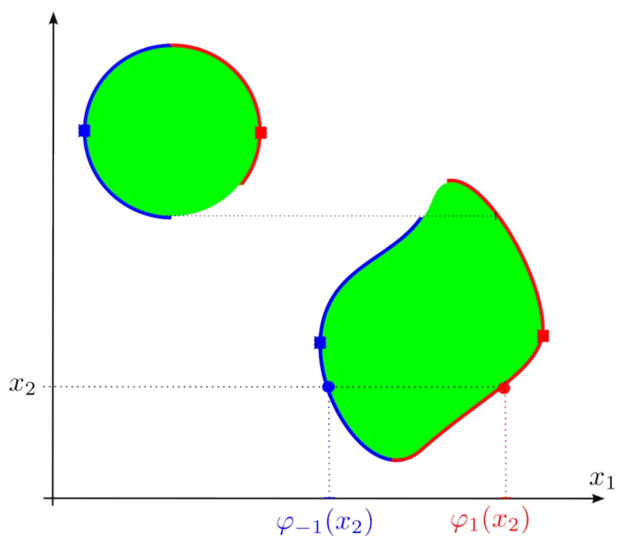


Fig. 2: Graphs of the functions $\varphi_1(x_2)$ (red) $\varphi_{-1}(x_2)$ (blue)

Figure 3 is a representation of the functions $\varphi_2(x_1)$ (black) and $\varphi_{-2}(x_1)$ (orange). The small squares correspond to cardinal points (North in black and South in orange).

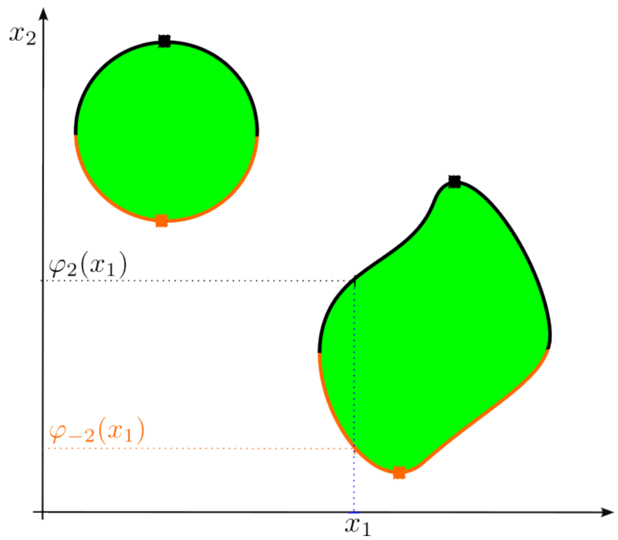


Fig. 3: Graphs of the functions $\varphi_2(x_1)$ (black) $\varphi_{-1}(x_2)$ (orange)

In Figure 2, we observe that graphs of the function φ_1 and φ_{-1} do not cover the boundary of $\mathbb{X}$. This is due to the fact that $\mathbb{X}$ is not row convex. We define the notion of row convexity (similar to the definition in [19])

Definition 3. A set $\mathbb{X} \subset \mathbb{R}^n$ is said to be *row convex* if the boundary $\partial\mathbb{X}$ of $\mathbb{X}$ corresponds to the union of the graphs of its cardinal functions, *i.e.*,

$$\partial\mathbb{X} = \cup_i \text{graph}(\varphi_i). \tag{19}$$

3.2 Case of the hyperbola

For the hyperbola defined by

$$f(\mathbf{q}, \mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 = 0. \tag{20}$$

We have four cardinal functions $\varphi_i, i \in \{-2, -1, 1, 2\}$. As it will be shown, the hyperbola is row convex and its cardinal functions will be sufficient to completely represent the equation.

To find φ_1, we fix x_2 and we search for the maximal value for x_1. The procedure leads to the following proposition. Other cardinal functions will be obtained by symmetries.

Proposition 2. *Take a point $x = (x_1, x_2)$ such that $f(\mathbf{q}, \mathbf{x}) = 0$. Given x_2, the largest x_1 such that $f(\mathbf{x}) = 0$ is given by*

$$\begin{aligned} x_1 &= \varphi_1(\mathbf{q}, x_2) \\ &= \frac{-(q_1+q_4x_2)+sign(q_3)\cdot\sqrt{(q_1+q_4x_2)^2-4q_1(q_0+q_2x_2+q_5x_2^2)}}{2q_3} \end{aligned} \tag{21}$$

Proof. Given x_2, let us compute the largest possible value for x_1. Since

$$f(\mathbf{q},\mathbf{x}) = q_3 x_1^2 + (q_1 + q_4 x_2)\, x_1 + q_2 x_2 + q_0 + q_5 x_2^2, \tag{22}$$

we get the following discriminant:

$$\Delta_1 = b_1^2 - 4a_1 c_1 \tag{23}$$

where

$$a_1 = q_3,\ b_1 = q_1 + q_4 x_2,\ c_1 = q_0 + q_2 x_2 + q_5 x_2^2 \tag{24}$$

The largest solution is

$$x_1 = \frac{-b_1 + \operatorname{sign}(a_1) \cdot \sqrt{\Delta_1}}{2a_1}. \tag{25}$$

which corresponds to (21). □

Definition 4. The cardinal points are the (x_1, x_2) which belong to the graph of at least three cardinal functions φ_i, $i \in \{-2, -1, 1, 2\}$.

For instance a North belongs to the graphs of $\varphi_1, \varphi_{-1}, \varphi_2$ and a East belongs to the graphs of $\varphi_2, \varphi_{-2}, \varphi_1$. For our hyperbola we easily find that there exist four cardinal points. Of course, the cardinal points depend on $\mathbf{q}$.

Proposition 3. *Consider the hyperbola $f(\mathbf{q},\mathbf{x}) = 0$. Define the interval function*

$$\rho(\mathbf{q}) = \frac{-2q_3 q_2 + q_1 q_4 + [-1,1] \cdot \sqrt{(2q_3 q_2 - q_1 q_4)^2 - (4q_3 q_5 - q_4^2)(4q_3 q_0 - q_1^2)}}{4q_3 q_5 - q_4^2} \tag{26}$$

If we set $[x_2] = [x_2^-, x_2^+] = \rho(\mathbf{q})$, then the North of the hyperbola is $(\varphi_1(x_2^-), x_2^-)$ and the South is $(\varphi_1(x_2^+), x_2^+)$.

Note that if the square root is not defined, then there is no cardinal points.

Proof. A value for x_2 yields a feasible x_1 if $\Delta_1 \geq 0$ (see (3)), *i.e.*,

$$\begin{array}{lll} & b_1^2 - 4a_1 c_1 & \geq 0 \\ \Leftrightarrow & -(q_1 + q_4 x_2)^2 + 4q_3(q_0 + q_2 x_2 + q_5 x_2^2) & \geq 0 \\ \Leftrightarrow & (4q_3 q_5 - q_4^2) x_2^2 + (4q_3 q_2 - 2q_1 q_4) x_2 + 4q_3 q_0 - q_1^2 & \leq 0 \end{array}$$

which is quadratic in x_2. The discriminant is

$$\Delta_2 = b_2^2 - 4a_2 c_2 \tag{27}$$

where

$$\begin{array}{lll} a_2 & = & 4q_3 q_5 - q_4^2 \\ b_2 & = & 4q_3 q_2 - 2q_1 q_4 \\ c_2 & = & 4q_3 q_0 - q_1^2 \end{array} \tag{28}$$

The corresponding values for x_2 is

$$x_2 = \frac{-b_2 \pm \sqrt{(\Delta_2}}{2a_2}.$$

and the North corresponds to the smallest one and the South to the largest. □

Corollary. Consider the hyperbola $f(\mathbf{q},\mathbf{x}) = 0$. Take the symmetry $\sigma = (1,3,2,6,5,4)$ and set $[x_1] = \rho(\sigma(\mathbf{q}))$, the East of the hyperbola is $(x_1^-, \varphi_2(x_1^-))$ and the West is $(x_1^+, \varphi_2(x_1^+))$.

Proof. The symmetry σ permutes x_1 and x_2. Then, we apply Proposition 3. The East becomes the North and the West becomes the South. □

4 Separator for the hyperbola

4.1 Interval extension of the cardinal function

Let us assume that $\mathbf{q}$ is fixed. The dependency with respect to the parameter vector $\mathbf{q}$ will omitted for simplicity. As defined in the book of Moore [14], the interval extension function of $\varphi_1(x_2)$ is

$$[\varphi_1]([x_2]) = [\{x_1 | \exists x_2 \in [x_2], x_1 = \varphi_1(x_2)\}]$$

which returns the smallest interval which contains the set $\varphi_1([x_2])$. The same definition applies for other cardinal directions to get $[\varphi_{-1}]([x_2]), [\varphi_2]([x_1])$ and $[\varphi_{-2}]([x_1])$.

Due to the monotonicity of φ_1 between the cardinal points, we have

$$[\varphi_1]([x_2]) = [\varphi_1(\{x_2^-, x_2^+, c_2(1), c_2(2), \dots\})]$$

where $\mathbf{c}(1), \mathbf{c}(2), \dots$ are the cardinal points inside the box $[-\infty, \infty] \times [x_2]$.

Take for instance $\mathbf{q} = (-1, 5, 2, -2, 30, -2)$, *i.e.*,

$$f(x_1, x_2) = -1 + 5x_1 + 2x_2 - 2x_1^2 + 30x_1x_2 - 2x_2^2.$$

For an interval sampling $[x_2] = \frac{1}{5} \cdot [k, k+1]$, $k \in \mathbb{N}$, the function $[\varphi_1]([x_2])$ generates the red boxes of Figure 4. If we do the same for $[\varphi_{-1}]([x_2])$, we get the blue boxes. The small black square corresponds to the North and the small orange square corresponds to the South.

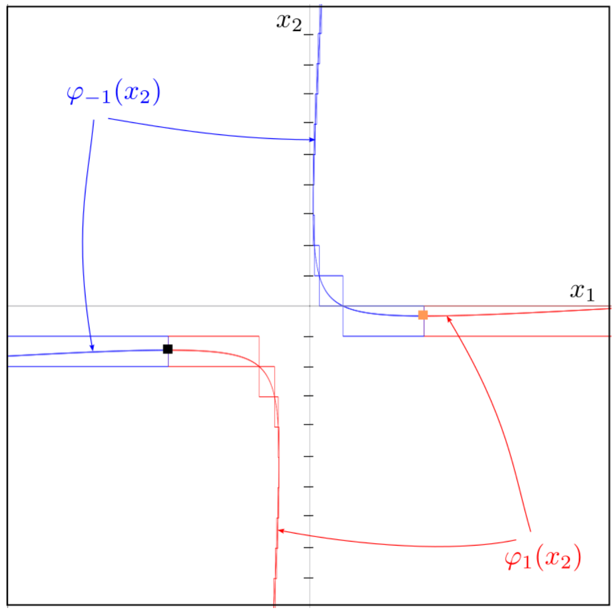


Fig. 4: Minimal interval extension for $\varphi_1([x_2])$ (red) and $\varphi_{-1}([x_2])$ (blue). The frame box is $[-2,2]^2$

For a similar sampling along x_1, Figure 5 represents $[\varphi_2]([x_1])$ (black) and $[\varphi_{-2}]([x_1])$ (orange). The small blue square corresponds to the East and the small red square corresponds to the West. Note that here, the West is on the left of the East. This is never the case for an ellipse.

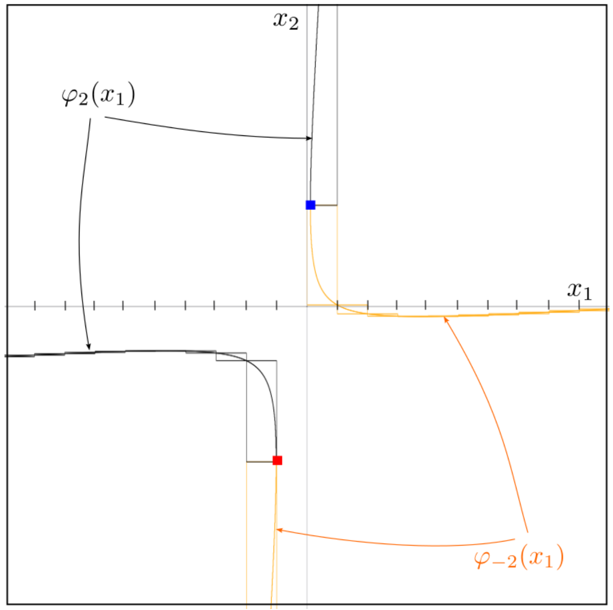


Fig. 5: Minimal inclusion for $\varphi_2([x_1])$ (black) and $\varphi_{-2}([x_1])$ (orange)

4.2 Seed contractor

From the interval evaluation, we can build a contractor for the set $x_1 = \varphi_1(x_2)$. It is is given by

$$C_0 : [\mathbf{x}] \to [x_1] \times [\varphi_1]([x_2]). \tag{29}$$

This contractor will be called a *seed contractor* because it will be used to construct all other contractors using symmetries. The contractor (29) is not minimal. It is only minimal with respect to x_1. Since this contractor depends on $\mathbf{q}$, we will write $C_0^{\mathbf{q}}$.

We understand that $C_0^{\mathbf{q}}$ corresponds to a small portion of the hyperbola. The main challenge is now to build the the separator for the whole hyperbola using the single parametric contractor $C_0^{\mathbf{q}}$ and symmetries. Of course, we could add some other seed contractors, but our idea is to factorize the implementation as much as possible to avoid bugs and make the code adaptable to other types of sets.

4.3 Contractor for the hyperbola

We have a contractor $C_0^{\mathbf{q}}$ which is minimal in the direction of x_2. Recall that $C_0^{\mathbf{q}}([\mathbf{x}])$ contracts the box $[\mathbf{x}]$ with respect to a small portion of the hyperbola. Using the notion of contractor action [7], we show how we can extend this contractor $C_0^{\mathbf{q}}$ to other portions. We recall that the action of a symmetry σ to the contractor C is defined by

$$\sigma \bullet C([\mathbf{x}]) = \sigma \circ C \circ \sigma^{-1}([\mathbf{x}]).$$

This means that $\sigma \bullet C$ is a contractor that has been built from the contractor C as follows:

- Apply to the box $[\mathbf{x}]$ the symmetry σ^{-1}
- Apply the contractor C
- Apply to the resulting box $C \circ \sigma^{-1}([\mathbf{x}])$ the symmetry σ.

For the hyperbola, we can make a partition of the curve into four portions :

- North-East : $\mathbb{X}^{(1,2)} = \{(x_1, x_2 | x_1 = \varphi_1(x_2) \text{ and } x_2 = \varphi_2(x_1)\}$
- North-West : $\mathbb{X}^{(1,-2)} = \{(x_1, x_2 | x_1 = \varphi_1(x_2) \text{ and } x_2 = \varphi_{-2}(x_1)\}$
- South-East : $\mathbb{X}^{(-1,2)} = \{(x_1, x_2 | x_1 = \varphi_{-1}(x_2) \text{ and } x_2 = \varphi_2(x_1)\}$
- South-West : $\mathbb{X}^{(-1,2)} = \{(x_1, x_2 | x_1 = \varphi_{-1}(x_2) \text{ and } x_2 = \varphi_{-2}(x_1)\}$

If we consider the pair (σ, γ) conjugate with respect to the hyperbola, the contractor $\sigma \bullet C_0^{\psi_\sigma(\mathbf{q})}$ is associated to another portion of the hyperbola. For a given σ, the selection of the symmetries γ such that (σ, γ) is conjugate is made using the choice function (16). These symmetries can be understood geometrically but can also be computed automatically as shown in [7].

To understand the construction, consider the symmetry $\sigma = (2, 1) \in B_2$. The contractor associated to $\mathbb{X}^{(1,2)}$:

$$C_1^{\mathbf{q}}([\mathbf{x}]) = \left(\sigma \bullet C_0^{\psi_\sigma(\mathbf{q})} \cap C_0^{\mathbf{q}}\right)([\mathbf{x}]).$$

It is minimal with respect to both directions x_1 and x_2 as illustrated by Figure 6. Note that the North-East portion is delimited by the two cardinal points North (black square) and East (red square). This is consistent with the fact that $\mathbb{X}^{(1,2)}$ corresponds to the North-East portion.

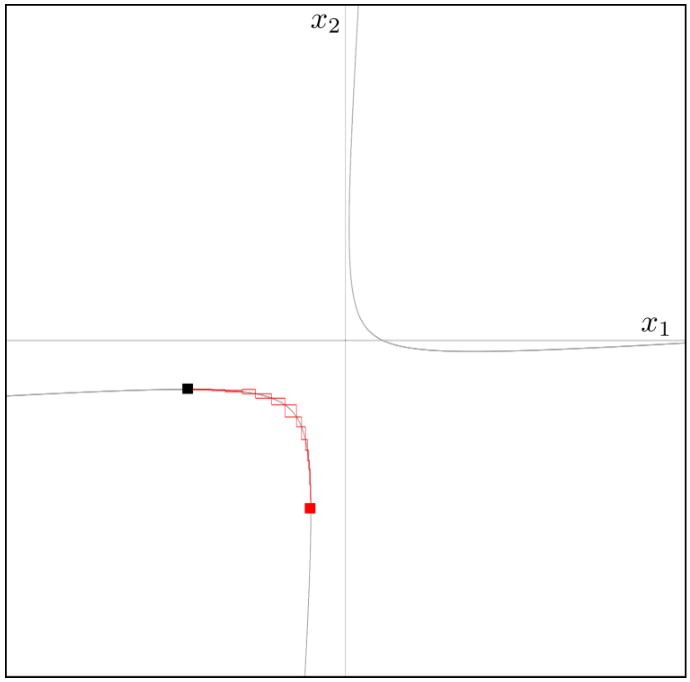


Fig. 6: Approximation of the North-East portion of the hyperbola using $(2,1) \bullet C_0^{\psi_{(2,1)}(\mathbf{q})} \cap C_0^{\mathbf{q}}$

The following proposition shows that the contractor for the hyperbola can be expressed by a simple formula involving symmetries and the unique seed contractor $C_0^{\mathbf{q}}$. Getting such a formula will ease the implementation of the contractor.

Proposition 4. *Consider an hyperbola set $\mathbb{X}$ defined by $f(\mathbf{q},\mathbf{x}) = 0$ as given by 1. A minimal contractor associated to $\mathbb{X}$ is*

$$\bigcup_{\sigma \in \{(1,2),(1,-2),(-1,2),(-1,-2)\}} \sigma \bullet \left((2,1) \bullet C_0^{\psi_{(2,1)} \cdot \psi_\sigma(\mathbf{q})} \cap C_0^{\psi_\sigma(\mathbf{q})}\right). \qquad (30)$$

where $\psi_\sigma(\mathbf{q})$ is the choice function defined by (16).

Proof. The minimal contractor for the North-East portion $\mathbb{X}^{(1,2)}$ is

$$C_1^{\mathbf{q}} = (2,1) \bullet C_0^{\psi_{(2,1)}(\mathbf{q})} \cap C_0^{\mathbf{q}}. \qquad (31)$$

The three other portions can be defined by applying symmetries in

$$\{(1,-2),(-1,2),(-1,-2)\}.$$

Define the contractor

$$C = \bigcup_{\sigma \in \{(1,2),(1,-2),(-1,2),(-1,-2)\}} \sigma \bullet C_1^{\psi_\sigma(\mathbf{q})}$$

Since, the hyperbola is row convex, C is a contractor for $\mathbf{f}(\mathbf{q},\mathbf{x}) = 0$ (*i.e.* no solution is lost). Moreover, since the union of contractors is minimal, $C$is minimal. Combining with (31), we get that the minimal contractor with respect to the seed contractor $C_0^{\mathbf{q}}$ is given by (30). □

Figure 7 illustrates the minimality of the contractor for the hyperbola.

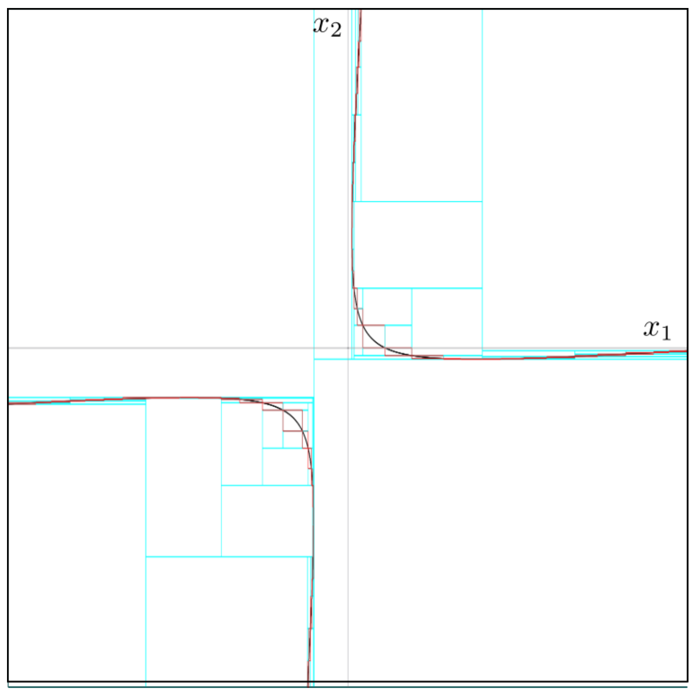


Fig. 7: Illustration of the minimality of the contractor for the hyperbola

4.4 Minimal separator for the hyperbola area

This section proposes an optimal separator for an hyperbola area defined by

$$\mathbb{X} = \{\mathbf{x} | q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \leq 0\}. \quad (32)$$

This separator is then used by a paver to compute boxes that are inside or outside the solution set.

As shown in [10], from a contractor on the boundary of a set $\mathbb{X}$ and a test for $\mathbb{X}$, we can obtain a separator. As a consequence, we can get an inner and an outer approximations for $\mathbb{X}$ as illustrated by Figure 8 for $\mathbf{q} = (-1, 5, 2, -2, 30, -2)^T$. The magenta boxes are proved to be inside $\mathbb{X}$ and the blue boxes are outside $\mathbb{X}$. The accuracy is taken as $\varepsilon = 0.1$ and corresponds to the size of the small uncertain boxes (yellow). The cardinal points (North, South, West, East) are represented by the small squares (black, orange, blue, red).

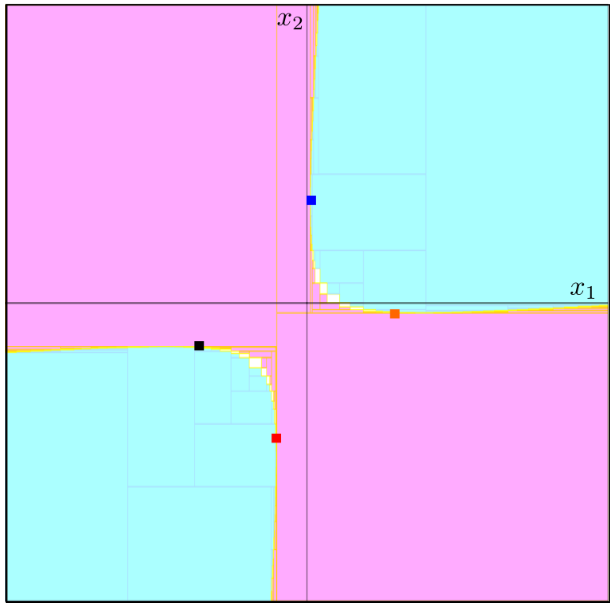


Fig. 8: Approximation of the hyperbola area obtained by our minimal separator for the hyperbola set

Figure 9 corresponds to the approximation obtained with the same accuracy with a classical forward-backward contractor. The benefice of our method seems small, but we will see later, that the improvement can become significant when the components of $\mathbf{q}$ are larger.

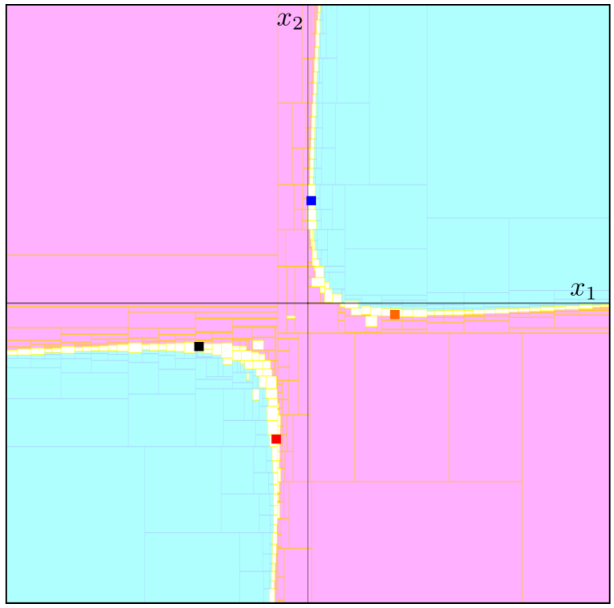


Fig. 9: Hyperbola area computed using a classical forward-backward contractor

For $\mathbf{q} = (-1, 1, 1, 3, 30, -2)$ we only have two cardinal points (West and East). The formula provided by Proposition 4 is still valid and we are able to generate Figure 10. This shows the ability of the symmetries to consider different situations easily, elegantly and safely.

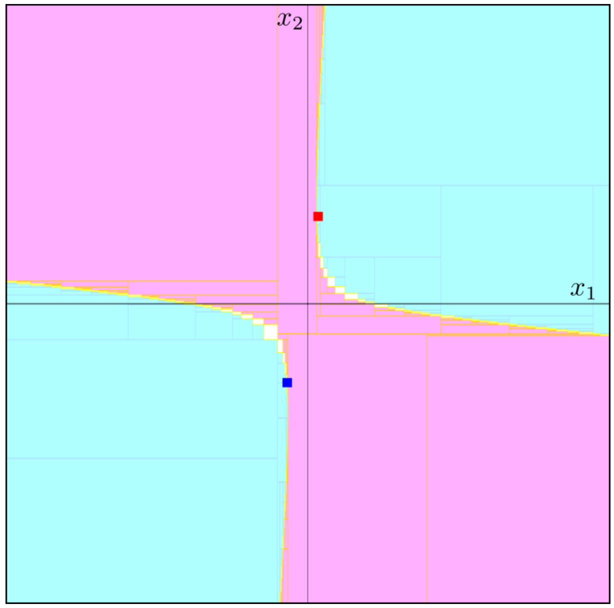


Fig. 10: Illustration of the application of the separator for an hyperbola set with two cardinal points only

5 Application

Interval methods have been used for localization of robots for several decades [12][18][2][4]. This section proposes to deal with a specific localization problem where pseudo distances are measured.

5.1 Hyperbola from foci

In this subsection, we show that an equation involving pseudo-distances corresponds to an hyperbola.

Proposition 5. *Consider two points* $\mathbf{a}, \mathbf{b}$ *of the plane. The set* $\mathbb{X}$ *of all points such that*

$$\|\mathbf{x} - \mathbf{a}\| - \|\mathbf{x} - \mathbf{b}\| \leq \ell \tag{33}$$

is an hyperbola area with foci $\mathbf{a}, \mathbf{b}$*. The set* $\mathbb{X}$ *is defined by the inequality*

$$\mathbf{f}_{\mathbf{a},\mathbf{b},\ell}(\mathbf{x}) \leq 0 \tag{34}$$

where

$$\mathbf{f}_{\mathbf{a},\mathbf{b},\ell}(\mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \tag{35}$$

with

$$\begin{array}{lll} q_0 & = - & a_1^4 - 2a_1^2a_2^2 + 2a_1^2b_1^2 + 2a_1^2b_2^2 + 2a_1^2\ell^2 \\ & & -a_2^4 + 2a_2^2b_1^2 + 2a_2^2b_2^2 \\ & & +2a_2^2\ell^2 - b_1^4 - 2b_1^2b_2^2 + 2b_1^2\ell^2 - b_2^4 + 2b_2^2\ell^2 - \ell^4 \\ q_1 & = & 4a_1^3 - 4a_1^2b_1 + 4a_1a_2^2 - 4a_1b_1^2 - 4a_1b_2^2 \\ & & -4a_1\ell^2 - 4a_2^2b_1 + 4b_1^3 + 4b_1b_2^2 - 4b_1\ell^2 \\ q_2 & = & 4a_1^2a_2 - 4a_1^2b_2 + 4a_2^3 - 4a_2^2b_2 - 4a_2b_1^2 \\ & & -4a_2b_2^2 - 4a_2\ell^2 + 4b_1^2b_2 + 4b_2^3 - 4b_2\ell^2 \\ q_3 & = & -4a_1^2 + 8a_1b_1 - 4b_1^2 + 4\ell^2 \\ q_4 & = & -8a_1a_2 + 8a_1b_2 + 8a_2b_1 - 8b_1b_2 \\ q_5 & = & -4a_2^2 + 8a_2b_2 - 4b_2^2 + 4\ell^2 \end{array}$$

Proof. We have

$$\begin{array}{ll} & \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| = \ell \\ \Rightarrow & \left(\|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\|\right)^2 = \ell^2 \\ \Leftrightarrow & \|\mathbf{x}-\mathbf{a}\|^2 + \|\mathbf{x}-\mathbf{b}\|^2 - 2\|\mathbf{x}-\mathbf{a}\| \cdot \|\mathbf{x}-\mathbf{b}\| = \ell^2 \\ \Leftrightarrow & \|\mathbf{x}-\mathbf{a}\|^2 + \|\mathbf{x}-\mathbf{b}\|^2 - \ell^2 = 2\|\mathbf{x}-\mathbf{a}\| \cdot \|\mathbf{x}-\mathbf{b}\| \\ \Leftrightarrow & \left(\|\mathbf{x}-\mathbf{a}\|^2 + \|\mathbf{x}-\mathbf{b}\|^2 - \ell^2\right)^2 - 4\|\mathbf{x}-\mathbf{a}\|^2 \cdot \|\mathbf{x}-\mathbf{b}\|^2 = 0 \end{array} \tag{36}$$

i.e.,

$$\begin{array}{ll} \left((x_1-a_1)^2 + (x_2-a_2)^2 + (x_1-b_1)^2 + (x_2-b_2)^2 - \ell^2\right)^2 & \\ -4\left((x_1-a_1)^2 + (x_2-a_2)^2\right)\left((x_1-b_1)^2 + (x_2-b_2)^2\right) & = \quad 0 \end{array} \tag{37}$$

We can develop the expression to get the coefficients of the proposition. □

5.2 Localization

We consider an example taken from [6] related to localization which can be seen as special case of interval data fitting problem [13]. Consider a robot which emits a sound at an unknown time t_0. This sound is received with a delay by three microphones located points $\mathbf{a} : (13, 7), \mathbf{b} : (4, 6), \mathbf{c} : (16, 10)$ of the plane (see Figure 11). Taking into account the time of flight of the sound we want to estimate the position of the object.

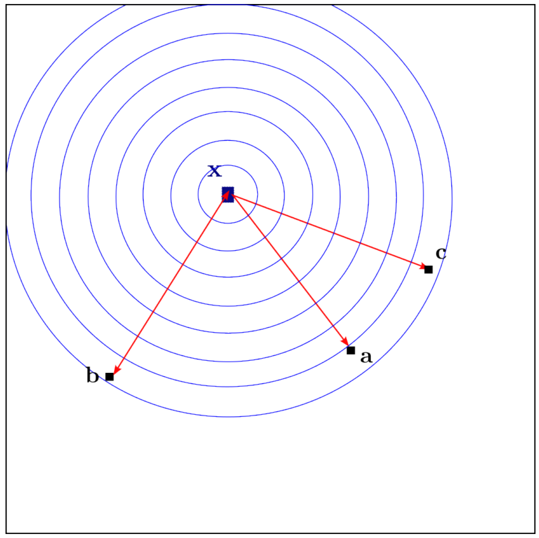


Fig. 11: The robot at position $\mathbf{x}$ emits a sound received later by three microphones $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

We have

$$\begin{array}{rcl} \|\mathbf{x}-\mathbf{a}\| & = & c\cdot(t_a-t_0) \\ \|\mathbf{x}-\mathbf{b}\| & = & c\cdot(t_b-t_0) \\ \|\mathbf{x}-\mathbf{c}\| & = & c\cdot(t_c-t_0) \end{array} \tag{38}$$

where c is the sound speed and t_a, t_b, t_c is the detection time for microphones $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We eliminate t_0 which is unknown to get

$$\begin{array}{rcl} \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{b}\| & = & c\cdot(t_a-t_b)=\ell_{ab} \\ \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{c}\| & = & c\cdot(t_a-t_c)=\ell_{ac} \end{array} \tag{39}$$

The quantity ℓ_{ab}, ℓ_{bc} are called *pseudo-distances*. We assume that we were able to measure the two pseudo distances to get $\ell_{ab} \in [7.9, 8.1]$ and $\ell_{ac} \in [3.9, 4.1]$. The set $\mathbb{X}$ of all feasible locations is defined by

$$\begin{array}{lrcl} \text{(i)} & \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{b}\| & \in & [7.9, 8.1] \\ \text{(ii)} & \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{c}\| & \in & [3.9, 4.1] \end{array} \tag{40}$$

From Proposition 5, we get that $\mathbb{X}$ is defined by

$$\mathbb{X} = \mathbb{X}_{ab} \cap \mathbb{X}_{ac} \tag{41}$$

where (see (35)):

$$\mathbb{X}_{ab} : \left\{ \begin{array}{l} \mathbf{f}_{\mathbf{a},\mathbf{b},8.1}(\mathbf{x}) \leq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{b},7.9}(\mathbf{x}) \geq 0 \end{array} \right. \tag{42}$$

and

$$\mathbb{X}_{ac} : \left\{ \begin{array}{l} \mathbf{f}_{\mathbf{a},\mathbf{c},4.1}(\mathbf{x}) \leq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{c},3.9}(\mathbf{x}) \geq 0 \end{array} \right. \tag{43}$$

Using a paver, we get an inner and an outer approximations for the set of $\mathbb{X}_{ab}, \mathbb{X}_{ac}$ and $\mathbb{X}$. Figures 12, 13, 14 have been generated with a classical forward-backward contractor [1]. We observe a strong clustering effect with many uncertain boxes that the separator is not able to classify.

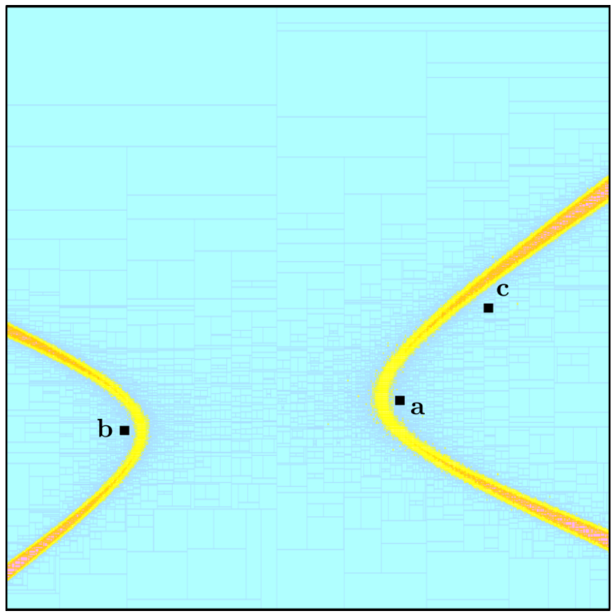


Fig. 12: Set $\mathbb{X}_{ab}$ of positions consistent with microphones $\mathbf{a}, \mathbf{b}$ (classic)

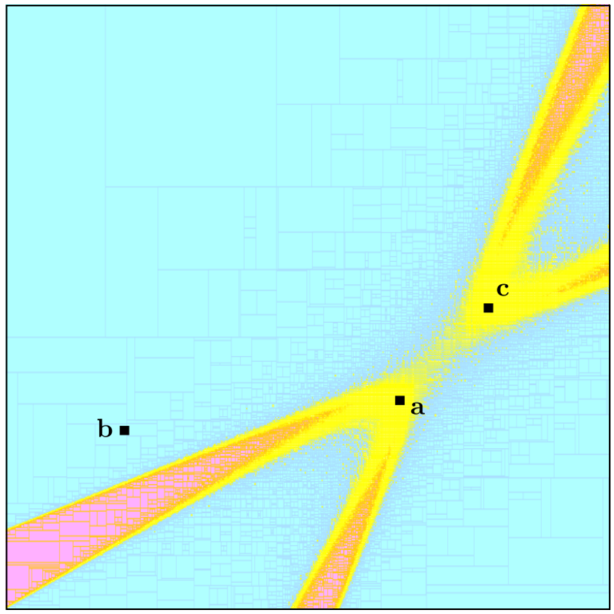


Fig. 13: Set $\mathbb{X}_{ac}$ of positions consistent with $\mathbf{a}, \mathbf{c}$ (classic)

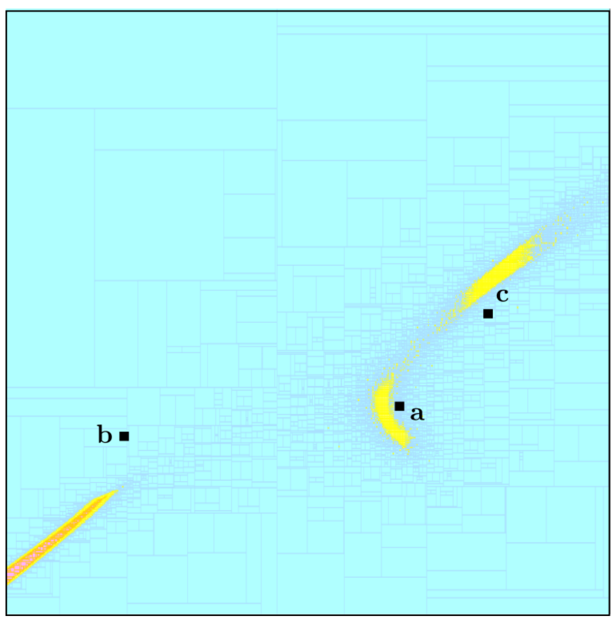


Fig. 14: Set $\mathbb{X}$ of positions using the three microphones (classic)

Figures 15, 16, 17 have been generated using the minimal contractor (30).

For all figures, the frame box is $[0, 20] \times [0, 20]$ and the accuracy is the same ($\varepsilon = 0.05$). All results are guaranteed since outward rounding is implemented [16][15]. The clustering effect almost disappeared.

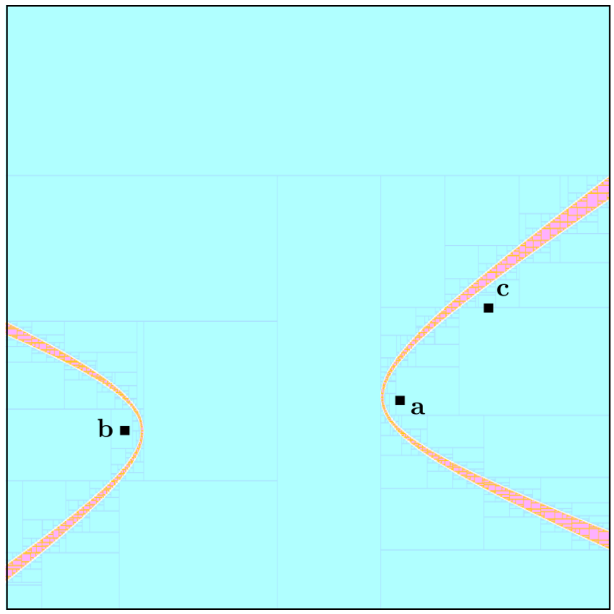


Fig. 15: Set $\mathbb{X}_{ac}$ of positions consistent with microphones $\mathbf{a}, \mathbf{c}$ (with the hyperbola separator)

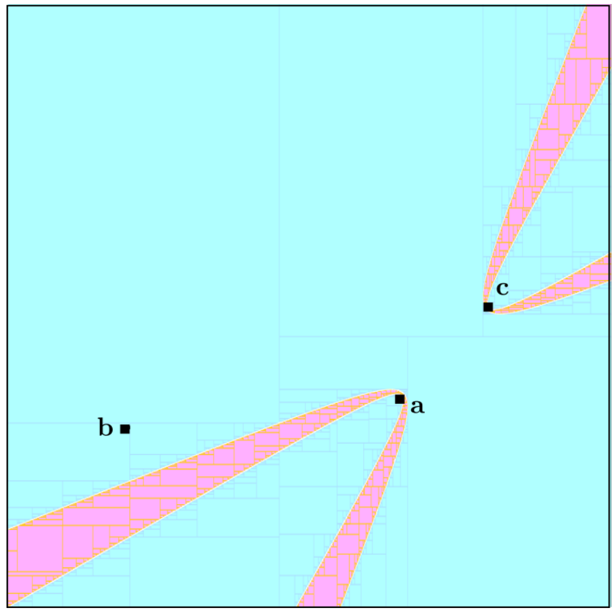


Fig. 16: Set $\mathbb{X}_{ac}$ of positions consistent with microphones $\mathbf{a}, \mathbf{c}$ (with the hyperbola separator)

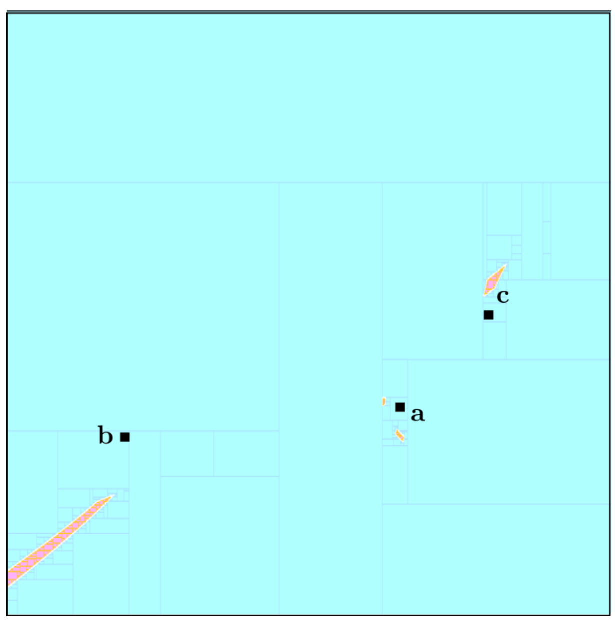


Fig. 17: Set $\mathbb{X}$ of positions using the three microphones (with the hyperbola separator)

6 Conclusion

This paper has proposed a minimal contractor and a minimal separator for an hyperbola area of the plane. The notion of actions derived from hyperoctahedral symmetries allowed us to limit the analysis to one portion of the constraint where the piece-wize monotonicity can be assumed. The symmetries was used to extend the analysis to the whole plane.

The goal of this paper was also to provide a simple example which illustrates the use of hyperoctahedral symmetries in order to build minimal separators. Now, as shown in [9], the use of these symmetries is more interesting when we deal with projection problems where quantifier elimination is needed. This type of projection problem is indeed much more difficult to solve with classical interval approaches [5].

When we build an optimal contractor for a set $\mathbb{X}$ using symmetries, the main difficulty is to find the portion of the set that can be used to reconstruct $\mathbb{X}$ using the copy-paste process allowed by the actions of the symmetries. For the hyperbola, the pattern is a cardinal function and for the ellipse, it was a quarter of the ellipse. But there is no general procedure to find the right pattern.

The Python code based on Codac [17] is given in [8].

References

[1] F. Benhamou, F. Goualard, L. Granvilliers, and J. F. Puget. Revising hull and box consistency. In *Proceedings of the International Conference on Logic Programming*, pages 230–244, Las Cruces, NM, 1999. 5.2

[2] E. Colle and S. Galerne. Mobile robot localization by multiangulation using set inversion. *Robotics and Autonomous Systems*, 61(1):39–48, 2013. 5

[3] H. Coxeter. *The Beauty of Geometry: Twelve Essays.* Dover Books on Mathematics, 1999. 2.2

[4] V. Drevelle and P. Bonnifait. High integrity gnss location zone characterization using interval analysis. In *ION GNSS*, 2009. 5

[5] M. Hladík and S. Ratschan. Efficient Solution of a Class of Quantified Constraints with Quantifier Prefix Exists-Forall. *Mathematics in Computer Science*, 8(3-4):329–340, July 2014. 6

[6] L. Jaulin. A boundary approach for set inversion. *Engineering Applications of Artificial Intelligence*, 100:104184, 2021. 5.2

[7] L. Jaulin. Actions of the hyperoctahedral group to compute minimal contractors. *Artif. Intell.*, 313:103790, 2022. 4.3

[8] L. Jaulin. Codes associated with the paper entitled: Optimal separator for the hyperbola; Application to localization. `www.ensta-bretagne.fr/jaulin/ctchyperbola.html`, 2023. 6

[9] L. Jaulin. Inner and outer characterization of the projection of polynomial equations using symmetries, quotients and intervals. *International Journal of Approximate Reasoning*, 159:108928, 2023. 2.4, 6

[10] L. Jaulin. Optimal separator for an ellipse; application to localization. *arXiv:2305.10842, math.NA*, 2023. 1, 4.4

[11] L. Jaulin and B. Desrochers. Introduction to the algebra of separators with application to path planning. *Engineering Applications of Artificial Intelligence*, 33:141–147, 2014. 1

[12] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*. Springer-Verlag, London, 2001. 5

[13] V. Kreinovich and S. Shary. Interval methods for data fitting under uncertainty: A probabilistic treatment. *Reliable Computing*, 23:105–140, 2016. 5.2

[14] R. Moore. *Methods and Applications of Interval Analysis*. Society for Industrial and Applied Mathematics, jan 1979. 1, 4.1

[15] N. Revol. Introduction to the IEEE 1788-2015 Standard for Interval Arithmetic. *10th International Workshop on Numerical Software Verification - NSV 2017*. 5.2

[16] N. Revol, L. Benet, L. Ferranti, and S. Zhilin. Testing interval arithmetic libraries, including their ieee-1788 compliance. *arXiv:2205.11837, math.NA*, 2022. 5.2

[17] S. Rohou. *Codac (Catalog Of Domains And Contractors), available at* `http://codac.io/`. Robex, Lab-STICC, ENSTA-Bretagne, 2021. 6

[18] S. Rohou, L. Jaulin, L. Mihaylova, F. Le Bars, and S. Veres. *Reliable Robot Localization*. Wiley, dec 2019. 5

[19] D.J. Sam-Haroud and B. Faltings. Consistency techniques for continuous constraints. *Constraints*, 1(1-2):85–118, 1996. 3.1

[20] H. Wussing. *The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory*. Dover Publications, 2007. 2.2