Robust Localisation Using Separators

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Abstract. Contractor algebra is a numerical tool based on interval analysis often used to solve localization problems. This paper proposes to use the *separators* which is a pair of complementary contractors and recalls the corresponding algebra. Separator algebra inside a paver will allow us to get an inner and an outer approximation of the solution set in a very easy way. An application to robust localization is presented in order to illustrate the principle of the approach.

1 Introduction

Many problems in engineering amount to characterizing a set of $\mathbb{R}^n$ defined by constraints (see, e.g., [2]). For instance, the solution set may correspond (i) to the set of all parameters that are consistent with a some interval measurements [15] [14] [3], (ii) to the set of all configuration vectors such that robot does not meet any obstacles [7], (iii) to the set of all parameter vectors of a controller such that the closed loop system is stable [17], ... More formally, the problem to be considered in this paper is to bracket a set $\mathbb{X}$ defined by constraints between two subpavings (i.e. union of non overlapping boxes) $\mathbb{X}^-$ and $\mathbb{X}^+$ such that $\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+$. Interval analysis [16] [12] combined with contractors [4] has been shown to be able to solve a large class of set estimation problem. For the inner subpaving, the *De Morgan* rules can be used to express the complementary set $\overline{\mathbb{X}}$ of $\mathbb{X}$. Then basic contractor techniques can be used to get an inner characterization $\mathbb{X}^-$. Now, the task is not so easy and the role of *separators*, recently introduced [8], is to make it automatic. A separator is composed of two contractors and an algebra, similar to the algebra developed for contractors [4], can be developed for in order to be able to deal with complex sets. Combined with a paver, separators are able to bracket the solution set $\mathbb{X}$ with an efficiency similar to that of contractors.

The paper is organized as follows. Section 2 defines separators and shows how they can be used inside a paver to characterize subsets of $\mathbb{R}^n$. Section 3 explains how to extend all basic operations on sets (such as union, intersection, difference, complementary, inversion) to separators. An application related to localization is considered in Section 4. Section 5 concludes the paper.

2 Separators

In this section, we first present the notion of contractors that will be needed to define separators. Then, we present separators and show how they can be used by a paver in order to bracket the solution set $\mathbb{X}$.

Contractors. An *interval* of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$. A box $[\mathbf{x}]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. A *contractor* $\mathcal{C}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that $\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$ and $[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$. We define the inclusion between two contractors $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall\, [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]). \tag{1}$$

A set $\mathbb{S}$ is *consistent* with the contractor $\mathcal{C}$ (we will write $\mathbb{S} \sim \mathcal{C}$) if for all $[\mathbf{x}]$, we have $\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}$. Two contractors $\mathcal{C}$ and $\mathcal{C}_1$ are *consistent* each other (we will write $\mathcal{C} \sim \mathcal{C}_1$) if for any set $\mathbb{S}$, we have $\mathbb{S} \sim \mathcal{C} \Leftrightarrow \mathbb{S} \sim \mathcal{C}_1$. We define the *negation* $\neg\mathcal{C}$ of a contractor $\mathcal{C}$ as follows $\neg\mathcal{C}\left([\mathbf{x}]\right) = \left\{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{x} \notin \mathcal{C}\left([\mathbf{x}]\right)\right\}$. Note that $\neg\mathcal{C}\left([\mathbf{x}]\right)$ is not a box in general, but a union of boxes. If $\mathbf{f}$ is a function and if $\mathcal{C}_{\mathbb{Y}}$ is a contractor for $\mathbb{Y}$, a contractor $\mathcal{C}_{\mathbb{X}}$ for $\mathbb{X} = \mathbf{f}^{-1}\left(\mathbb{Y}\right)$ can be defined using a generalization of the forward-backward contractor. The contractor $\mathcal{C}_{\mathbb{X}}$ is called the inverse of $\mathcal{C}_{\mathbb{Y}}$ by $\mathbf{f}$ and we write $\mathcal{C}_{\mathbb{X}} = \mathbf{f}^{-1}\left(\mathcal{C}_{\mathbb{Y}}\right)$.

Separators. A *separator* $\mathcal{S}$ is pair of contractors $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$, we have $\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}]$. A set $\mathbb{S}$ is *consistent* with the separator $\mathcal{S}$ (we write $\mathbb{S} \sim \mathcal{S}$), if $\mathbb{S} \sim \mathcal{S}^{\text{out}}$ and $\overline{\mathbb{S}} \sim \mathcal{S}^{\text{in}}$. where $\overline{\mathbb{S}} = \{\mathbf{x} \mid \mathbf{x} \notin \mathbb{S}\}$. We define the *remainder* of a separator $\mathcal{S}$ as$\partial\mathcal{S}([\mathbf{x}]) = \mathcal{S}^{\text{in}}([\mathbf{x}]) \cap \mathcal{S}^{\text{out}}([\mathbf{x}])$. Note that the remainder is a contractor and not a separator. For a given box $[\mathbf{x}]$, it is trivial to show that $\neg\mathcal{S}^{\text{in}}([\mathbf{x}])$, $\neg\mathcal{S}^{\text{out}}([\mathbf{x}])$ and $\partial\mathcal{S}([\mathbf{x}])$ cover $[\mathbf{x}]$, i.e., $\neg\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \neg\mathcal{S}^{\text{out}}([\mathbf{x}]) \cup \partial\mathcal{S}([\mathbf{x}]) = [\mathbf{x}]$. Moreover, they do not overlap each other. We define the inclusion between separators $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows

$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow {\mathcal{S}_1}^{\text{in}} \subset \mathcal{S}_2^{\text{in}} \text{ and } \mathcal{S}_1^{\text{out}} \subset \mathcal{S}_2^{\text{out}}. \tag{2}$$

Paver. A *paver* is a branch-and-bound algorithm which calls the separator $\mathcal{S}$ to classify part of the search space inside or outside the solution set $\mathbb{X}$ associated with $\mathcal{S}$. The algorithm is given in the table below. Step 1 initializes a list $\mathcal{L}$ containing all boxes to be studied. Step 2 takes one box $[\mathbf{x}]$ in $\mathcal{L}$. At Step 3, the separator $\mathcal{S}$ is then called to contract $[\mathbf{x}]$ into two boxes $[\mathbf{x}^{\text{in}}]$ and $[\mathbf{x}^{\text{out}}]$. Step 4 stores $\neg\mathcal{S}^{\text{in}}([\mathbf{x}])$, the part of $[\mathbf{x}]$ that is proved to be inside $\mathbb{X}$, into $\mathbb{X}^-$ and also into $\mathbb{X}^+$. Step 5 computes $\partial\mathcal{S}([\mathbf{x}])$ by intersecting $[\mathbf{x}^{\text{out}}]$ and $[\mathbf{x}^{\text{in}}]$. If this box is too small (i.e. with a width smaller than ε), it is store inside $\mathbb{X}^+$ and will not be studied anymore. Otherwise, it is bisected at Step 7. After completion of the algorithm, we have the enclosure $\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+$.

Algorithm Paver(in: $[\mathbf{x}], \mathcal{S}$; out: $\mathbb{X}^-$, $\mathbb{X}^+$)
1 $\mathcal{L} := \{[\mathbf{x}]\}$;
2 Pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 $\left\{[\mathbf{x}^{\text{in}}], [\mathbf{x}^{\text{out}}]\right\} = \mathcal{S}([\mathbf{x}])$;
4 Store $[\mathbf{x}] \backslash [\mathbf{x}^{\text{in}}]$ into $\mathbb{X}^-$ and also into $\mathbb{X}^+$;
5 $[\mathbf{x}] = [\mathbf{x}^{\text{in}}] \cap [\mathbf{x}^{\text{out}}]$;
6 If $w([\mathbf{x}]) < \varepsilon$, then store $[\mathbf{x}]$ in $\mathbb{X}^+$,
7 Else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$ the two resulting boxes
8 If $\mathcal{L} \neq \emptyset$, go to 2.

3 Algebra

The algebra for separators is a direct extension of contractor algebra [4]. The main difference is that contractor algebra does not allow any decreasing operation, with respect to the inclusion. As a consequence the complementary $\overline{\mathcal{C}}$ of a contractor $\mathcal{C}$ or the restriction $\mathcal{C}_1 \backslash \mathcal{C}_2$ of two contractors $\mathcal{C}_1, \mathcal{C}_2$ (which both correspond to non monotonic operation) cannot be defined. The main advantage of separators is that it extends the operations allowed for contractors to non monotonic expressions. Let us now define some operations for separator. If $\mathcal{S} = \left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ is a separator, we define the *complement* as $\overline{\mathcal{S}} = \left\{\mathcal{S}^{\text{out}}, \mathcal{S}^{\text{in}}\right\}$. If $\mathcal{S}_i = \left\{\mathcal{S}_i^{\text{in}}, \mathcal{S}_i^{\text{out}}\right\}, i \in \{1, 2, \ldots\}$ are separators, we define

$$\begin{array}{rcll}
\mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\right\} & \text{(intersection)} \\
\mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\right\} & \text{(union)} \\
\bigcap^{\{q\}} \mathcal{S}_i & = & \left\{ \bigcap^{\{m-q-1\}} \mathcal{S}_i^{\text{in}}, \bigcap^{\{q\}} \mathcal{S}_i^{\text{out}} \right\} & \text{(relaxed intersection)} \\
\mathbf{f}^{-1}\left(\mathcal{S}_{\mathbb{Y}}\right) & = & \left\{\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}^{\text{in}}), \mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}^{\text{out}})\right\} & \text{(inverse)}
\end{array} \tag{3}$$

The q-relaxed intersection [10] of n sets corresponds to the set of all elements which belongs to at least $n-q$ of these sets. It is used for robust bounded-error estimation [13] or for certified calibration of robots [1]. If $\mathbb{S}_i$ are sets of $\mathbb{R}^n$, we have [8]

$$\begin{array}{lccc}
\text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathcal{S}_1 \cap \mathcal{S}_2 \\
\text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathcal{S}_1 \cup \mathcal{S}_2 \\
\text{(iii)} & \overline{\mathbb{S}_i} & \sim & \overline{\mathcal{S}_i} \\
\text{(iv)} & \bigcap^{\{q\}} \mathbb{S}_i & \sim & \bigcap^{\{q\}} \mathcal{S}_i \\
\text{(vi)} & \mathbf{f}^{-1}\left(\mathbb{Y}\right) & \sim & \mathbf{f}^{-1}\left(\mathcal{S}_{\mathbb{Y}}\right).
\end{array} \tag{4}$$

4 Application to localization

Localization aims at estimating the position of a robot from a set of measurements performed by the robot. This problem can be cast into a parameter estimation problem [11] where the parameters correspond to the position of the robot. Interval analysis combined with probabilistic techniques has already been considered to deal with localization problem [5, 6] . Here, we use the approach developed in this paper to obtain inner and outer approximations of all consistent positions for a robot in a probabilistic context. Consider a robot which measures its own distance to three beacons [9]. The intervals corresponding to the distances and the coordinates of the beacons are given by the following table.

beacons	x_i	y_i	$[d_i]$
1	1	3	$[1, 2]$
2	3	1	$[2, 3]$
3	-1	-1	$[3, 4]$

The collected intervals $[d_i]$ supposed to contain the true distance d_i. Define

$$\mathbb{P}^{\{q\}} = \bigcap^{\{q\}} \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5] \right\}$$

Separators can easily be obtained from the expression of $\mathbb{P}^{\{q\}}$ using the separator algebra. For $q = 0, 1, 2$, the paver provides the sets represented on Figure 1.

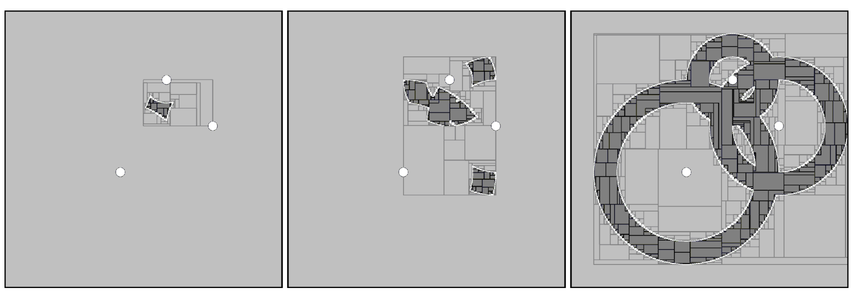

Fig. 1. Sets $\mathbb{P}^{\{q\}}$ obtained for $q = 0, 1, 2$. The frame box is $[-6, 6]^3$.

5 Conclusion

Contractor algebra is an efficient tool to compute with subsets of $\mathbb{R}^n$. To compute the union, the intersection or any other monotonic operations between sets, it suffices to apply the same operations on the contractors. Then, a paver with the corresponding contractor will provide a guaranteed approximation of the solution set. Now, contractors cannot deal with decreasing operations such as the complement or the set difference. Using separators, which is a pair of two complementary contractors, the complementary operator or any other decreasing operation are now available. This allows us to compute with sets in a much more general way. As an illustration, we have considered a robust localization problem where the inner and the outer approximations of the feasible set for all locations of a robot is computed.

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