# Solving set-valued constraint satisfaction problems

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**Abstract**. In this paper, we consider the resolution of constraint satisfaction problems in the case where the variables of the problem are subsets of $\mathbb{R}^n$. In order to use a constraint propagation approach, we introduce set intervals (named i-sets), which are sets of subsets of $\mathbb{R}^n$ with a lower bound and an upper bound with respect to the inclusion. Then, we propose basic operations for i-sets. This makes possible to build contractors that are then used by the propagation to solve problem involving sets as unknown variables. In order to illustrate the principle and the efficiency of the approach, a testcase is provided.

**Keywords**. Constraint propagation, Constraint satisfaction, Contractors, Interval analysis, Set intervals.

## 1 Introduction

Constraint satisfaction problems involving subsets of $\mathbb{R}^n$ (namely *set-valued constraint satisfaction problems* or SVCSP for short) can appear in several engineering applications, typically, when arbitrary shapes (i.e. that cannot be parametrized) are involved. The reconstruction of a three dimensional object from photos [4], mapping an environment from sonar measurements ([16], [20]), SLAM (simultaneous localization and mapping) [11] or characterizing invariant sets of dynamic systems [2] can be represented by SVCSP. This paper introduces in Section 2 a new type of numbers, namely *set intervals* (or *i-sets*), which make possible to use constraint propagation methods for solving SVCSP. Some basic operators for i-sets are also proposed. These operators are then used to build contraction operators (or contractors) in Section 3. An illustrative application is provided in Section 4 where a SVCSP is solved. Section 5 concludes the paper.

## 2 Set intervals (or *i-sets*)

### 2.1 Definition

Given two sets $\mathbb{A}^-$ and $\mathbb{A}^+$ of $\mathbb{R}^n$, the pair $[\mathbb{A}^-, \mathbb{A}^+]$ which encloses all sets $\mathbb{A}$ such that

$$\mathbb{A}^- \subset \mathbb{A} \subset \mathbb{A}^+$$

is a *set interval* (or *i-set* for short) and will be denoted by $[\mathbb{A}]$ (see Figure 1). The i-set $[\emptyset, \emptyset]$ is a singleton which contains a single element: the empty set $\emptyset$.

The i-set $[\emptyset, \mathbb{R}^n]$ encloses all sets of $\mathbb{R}^n$. If $\mathbb{A}^- \not\subset \mathbb{A}^+$, then $[\mathbb{A}^-, \mathbb{A}^+]$ is empty. A i-set is a way to handle and to compute with uncertain sets (see [9], [23]). The idea that is developed in this paper follows the foundations of interval analysis that has been built to handle uncertain real numbers [17], [14], to solve real-valued nonlinear problems (see e.g. [7], [10]), to minimize nonconvex criteria (see, e.g., [12], [18]) or to provide mathematical proofs (see, e.g., [21], [8], [19], [15]).

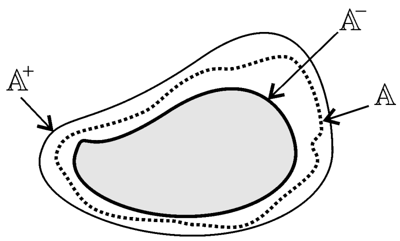


Figure 1: The set $\mathbb{A}$ can be approximated by the i-set $[\mathbb{A}^-, \mathbb{A}^+]$.

## 2.2 Operations

We shall now define some operations that can be used for i-sets. Two types of operations can be considered.

- *Specific i-set operations*. Since i-sets are sets (their elements are sets), the intersection, the union, the inclusion can be defined. In order to avoid any confusion with the operations of their elements, these operations will be denoted in a squared manner (e.g. $\sqcap, \sqcup, \sqsubset$).
- *Set extension*. All operations existing for elements of a i-set (which are sets) such as $\cap, \cup, \backslash, +$, reciprocal image , direct image, ... can be extended to i-sets [13].

Let us first start with specific i-set operations.
**Intersection**. The *i-set intersection* between two i-sets is defined by

$$[\mathbb{A}] \sqcap [\mathbb{B}] = \{\mathbb{X}, \mathbb{X} \in [\mathbb{A}] \text{ and } \mathbb{X} \in [\mathbb{B}]\}.$$

Since

$$\begin{array}{cccc} & \left\{ \begin{array}{l} \mathbb{X} \in [\mathbb{A}] \\ \mathbb{X} \in [\mathbb{B}] \end{array} \right. & \Leftrightarrow & \left\{ \begin{array}{l} \mathbb{A}^- \subset \mathbb{X} \subset \mathbb{A}^+ \\ \mathbb{B}^- \subset \mathbb{X} \subset \mathbb{B}^+ \end{array} \right. \\ \Leftrightarrow & \mathbb{A}^- \cup \mathbb{B}^- \subset \mathbb{X} \subset \mathbb{A}^+ \cap \mathbb{B}^+ & \Leftrightarrow & \mathbb{X} \in [\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+], \end{array}$$

the i-set $[\mathbb{A}] \sqcap [\mathbb{B}]$ is given by

$$\left[\mathbb{A}^-, \mathbb{A}^+\right] \sqcap \left[\mathbb{B}^-, \mathbb{B}^+\right] = \left[\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+\right]. \quad (1)$$

**Inclusion**. We define the *i-set inclusion* as follows

$$[\mathbb{A}] \sqsubset [\mathbb{B}] \quad \Leftrightarrow \quad [\mathbb{A}] \sqcap [\mathbb{B}] = [\mathbb{A}] .$$

**i-set envelope**. Consider a collection $\{\mathbb{A}_i, i \in \mathbb{I}\}$ of sets of $\mathbb{R}^n$. The i-set envelope $\square \{\mathbb{A}_i, i \in \mathbb{I}\}$ is the smallest i-set (with respect to $\sqsubset$) enclosing all $\mathbb{A}_i, i \in \mathbb{I}$. We have

$$\square \{\mathbb{A}_i, i \in \mathbb{I}\} = \left[ \bigcap_{i \in \mathbb{I}} \mathbb{A}_i, \bigcup_{i \in \mathbb{I}} \mathbb{A}_i \right] .$$

For instance,

$$\square \{[1,4], [3,7], [2,6]\} = [[3,4],[1,7]] .$$

**Union**. The *i-set union* between two i-sets $[\mathbb{A}]$ and $[\mathbb{B}]$ is the smallest i-set which encloses both $[\mathbb{A}]$ and $[\mathbb{B}]$. We have

$$[\mathbb{A}] \sqcup [\mathbb{B}] = \square \{\mathbb{X}, \mathbb{X} \in [\mathbb{A}] \text{ or } \mathbb{X} \in [\mathbb{B}]\} .$$

It can easily be proven that

$$[\mathbb{A}] \sqcup [\mathbb{B}] = \left[\mathbb{A}^- \cap \mathbb{B}^-, \mathbb{A}^+ \cup \mathbb{B}^+\right] .$$

**Extension of operators**. If $\diamond$ is a binary operator in $\mathbb{R}^n$ (such as $+, -$, the multiplication $*$ when $n = 1$ or the vector product $\wedge$ when $n = 3$) then it can be extended to subsets of $\mathbb{R}^n$ (in the Minkowski sense) as follows

$$\mathbb{A} \diamond \mathbb{B} = \{a \diamond b, a \in \mathbb{A}, b \in \mathbb{B}\} .$$

There exists a second class of binary operators such as $\diamond \in \{\cup, \cap, \times, \backslash, \dots\}$, where $\times$ is the Cartesian product, $\backslash$ is the restriction (or trim) operator, for subsets of $\mathbb{R}^n$ that do not correspond to any extension of operators in $\mathbb{R}^n$. Following the basic idea of Moore [17], it is possible to extend the operators from these two classes to i-sets as follows

$$[\mathbb{A}] \diamond [\mathbb{B}] = \square \{\mathbb{C}, \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \mathbb{C} = \mathbb{A} \diamond \mathbb{B}\} . \tag{2}$$

From the monotony of the operators, we have

$$\begin{array}{lllll} \text{(i)} & [\mathbb{A}^-, \mathbb{A}^+] \cap [\mathbb{B}^-, \mathbb{B}^+] & = & [\mathbb{A}^- \cap \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+] \\ \text{(ii)} & [\mathbb{A}^-, \mathbb{A}^+] \cup [\mathbb{B}^-, \mathbb{B}^+] & = & [\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cup \mathbb{B}^+] \\ \text{(iii)} & [\mathbb{A}^-, \mathbb{A}^+] \times [\mathbb{B}^-, \mathbb{B}^+] & = & [\mathbb{A}^- \times \mathbb{B}^-, \mathbb{A}^+ \times \mathbb{B}^+] \\ \text{(iv)} & [\mathbb{A}^-, \mathbb{A}^+] \backslash [\mathbb{B}^-, \mathbb{B}^+] & = & [\mathbb{A}^- \backslash \mathbb{B}^+, \mathbb{A}^+ \backslash \mathbb{B}^-] \\ \text{(v)} & [\mathbb{A}^-, \mathbb{A}^+] + [\mathbb{B}^-, \mathbb{B}^+] & = & [\mathbb{A}^- + \mathbb{B}^-, \mathbb{A}^+ + \mathbb{B}^+] . \end{array} \tag{3}$$

**Extension of functions**. If $f$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. It can be extended to i-sets as follows

$$f([\mathbb{A}]) = \square \left\{ f(\mathbb{A}), \mathbb{A} \in \left[\mathbb{A}^-, \mathbb{A}^+\right] \right\} = \left[ f\left(\mathbb{A}^-\right), f\left(\mathbb{A}^+\right) \right] . \tag{4}$$

For instance,

$$\sin\left(\left[[\frac{\pi}{6},\frac{\pi}{4}],[0,\pi]\right]\right)=\left[[\frac{1}{2},\frac{\sqrt{2}}{2}],[0,1]\right].$$

**Wrappingless operators or functions**

A binary operator $\diamond \in \{+,-,*,\wedge,\cap,\cup,\backslash,\dots\}$ is *wrappingless*, if

$$[\mathbb{A}]\diamond[\mathbb{B}]=\{\mathbb{C},\exists\mathbb{A}\in[\mathbb{A}],\exists\mathbb{B}\in[\mathbb{B}],\mathbb{C}=\mathbb{A}\diamond\mathbb{B}\},\tag{5}$$

i.e., if the operator $\square$ is not needed in (2). A function $f$ is wrappingless if

$$f([\mathbb{A}])=\left\{f(\mathbb{A}),\mathbb{A}\in\left[\mathbb{A}^-,\mathbb{A}^+\right]\right\}.\tag{6}$$

**Lemma 1**. The operators $\cup,\cap,\backslash$ are wrappingless.

**Proof**. If $\mathbb{C}\in[\mathbb{A}]\diamond[\mathbb{B}]$, with $\diamond\in\{\cap,\cup,\backslash\}$, we only need to find one $\mathbb{A}\in[\mathbb{A}]$ and one $\mathbb{B}\in[\mathbb{B}],\mathbb{C}=\mathbb{A}\diamond\mathbb{B}$. (i) For the intersection, take $\mathbb{A}=\mathbb{A}^-\cup\mathbb{C}$ and $\mathbb{B}=\mathbb{B}^-\cup\mathbb{C}$. We first check that $\mathbb{A}\in[\mathbb{A}]$ and $\mathbb{B}\in[\mathbb{B}]$. Moreover

$$\begin{aligned}\mathbb{A}\cap\mathbb{B} &= \left(\mathbb{A}^-\cup\mathbb{C}\right)\cap\left(\mathbb{B}^-\cup\mathbb{C}\right)\\ &= \left(\mathbb{A}^-\cap\mathbb{B}^-\right)\cup\left(\mathbb{C}\cap\mathbb{B}^-\right)\cup\left(\mathbb{A}^-\cap\mathbb{C}\right)\cup\mathbb{C}\\ &= \left(\mathbb{A}^-\cap\mathbb{B}^-\right)\cup\mathbb{C}=\mathbb{C}\text{ (since }\mathbb{C}\in[\mathbb{A}]\cap[\mathbb{B}]\text{ ).}\end{aligned}$$

(ii) For the union, we apply the same reasoning by taking $\mathbb{A}=\mathbb{A}^+\cap\mathbb{C}$ and $\mathbb{B}=\mathbb{B}^+\cap\mathbb{C}$. (iii) For the restriction, we take $\mathbb{A}=\mathbb{A}^-\cup\mathbb{C}$ and $\mathbb{B}=\mathbb{B}^-\cup(\mathbb{A}^-\setminus\mathbb{C})$. ■

**Lemma 2**. If $f:\mathbb{R}^n\to\mathbb{R}^n$ is bijective, then its extension to i-sets is wrappingless.

**Proof**. If $\mathbb{C}\in f([\mathbb{A}])$, we only need to find one $\mathbb{A}\in[\mathbb{A}],\mathbb{C}=f(\mathbb{A})$. Since $f$ is bijective, we can take $\mathbb{A}=f^{-1}(\mathbb{C})$. We easily check that $\mathbb{A}\in[\mathbb{A}]$ and that $\mathbb{C}=f(\mathbb{A})$. ■

## 2.3 Natural i-set extension

Consider a set-valued expression $f(\mathbb{X}_1,\mathbb{X}_2,\dots,\mathbb{X}_p)$ made as a finite composition of wrappingless operators (such as $\cap,\cup,\backslash$) and wrappingless functions. We define the *natural i-set extension* $[f]$ of $f$ as the i-set function whose expressions is obtained by taking that of $f$ and by replacing all sets $\mathbb{X}_i$ by i-sets $[\mathbb{X}_i]$ and all operators and elementary functions involved in $f$ by their i-set counterparts. For instance, the natural i-set extension associated with the set expression

$$f(\mathbb{X}_1,\mathbb{X}_2,\mathbb{X}_3)=\mathbb{X}_1\cup(\mathbb{X}_2\cap g(\mathbb{X}_3))$$

is

$$[f]([\mathbb{X}_1],[\mathbb{X}_2],[\mathbb{X}_3])=[\mathbb{X}_1]\cup([\mathbb{X}_2]\cap g([\mathbb{X}_3])).$$

**Theorem 1**. Consider an expression $f(\mathbb{X}_1,\dots,\mathbb{X}_p)$ composed of wrappingless operators or functions. If $\mathbb{X}_1\in[\mathbb{X}_1],\dots,\mathbb{X}_p\in[\mathbb{X}_p]$ then

$$f(\mathbb{X}_1,\dots,\mathbb{X}_p)\in[f]([\mathbb{X}_1],[\mathbb{X}_2],\dots,[\mathbb{X}_p]).$$

Moreover, if in the expression of $f$, each $\mathbb{X}_i$ occurs only once, the i-set evaluation is minimal with respect to the inclusion, i.e.,

$$\begin{aligned} [f]\left([\mathbb{X}_1],\ldots,[\mathbb{X}_n]\right) &= f\left([\mathbb{X}_1],\ldots,[\mathbb{X}_n]\right) \\ &= \left\{\mathbb{Y},\left(\exists \mathbb{X}_i\in[\mathbb{X}_i]\right)_{i\leq p},\mathbb{Y}=f\left(\mathbb{X}_1,\ldots,\mathbb{X}_p\right)\right\}. \end{aligned} \tag{7}$$

**Proof**. We shall prove by induction that the theorem is true for $f$ but also for all subexpressions of $f$, in the case when each $\mathbb{X}_i$ occurs only once in the expression of $f$. (i) First, it is trivial to check that the theorem is true for all atomic subexpressions. (ii) Assume now that the theorem is true for two subexpressions $a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right)$ and $b\left(\mathbb{X}_{i_{p+1}},\ldots,\mathbb{X}_{i_q}\right)$ of $f$ and let us show that it is also true for a subexpression of the form

$$c\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_q}\right)=a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right)\diamond b\left(\mathbb{X}_{i_{p+1}},\ldots,\mathbb{X}_{i_q}\right). \tag{8}$$

We have

$$\begin{aligned} & [a]\left([\mathbb{X}_{i_1}],\ldots,[\mathbb{X}_{i_p}]\right)\diamond[b]\left([\mathbb{X}_{i_{p+1}}],\ldots,[\mathbb{X}_{i_q}]\right) \\ \overset{(5)}{=} & \left\{\mathbb{C},\exists\mathbb{A}\in[a]\left([\mathbb{X}_{i_1}],\ldots,[\mathbb{X}_{i_p}]\right),\exists\mathbb{B}\in[b]\left([\mathbb{X}_{i_{p+1}}],\ldots,[\mathbb{X}_{i_q}]\right),\mathbb{C}=\mathbb{A}\diamond\mathbb{B}\right\} \\ \overset{(7)}{=} & \left\{\mathbb{C},\left(\exists\mathbb{X}_{i_k}\in[\mathbb{X}_{i_k}]\right)_{1\leq k\leq q},\ \mathbb{A}=a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right),\ \mathbb{B}=b\left(\mathbb{X}_{i_{p+1}},\ldots,\mathbb{X}_{i_q}\right),\mathbb{C}=\mathbb{A}\diamond\mathbb{B}\right\} \\ = & \left\{\mathbb{C},\left(\exists\mathbb{X}_{i_k}\in[\mathbb{X}_{i_k}]\right)_{1\leq k\leq q},\mathbb{C}=a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right)\diamond b\left(\mathbb{X}_{i_{p+1}},\ldots,\mathbb{X}_{i_q}\right)\right\} \\ \overset{(8)}{=} & \left\{\mathbb{C},\left(\exists\mathbb{X}_{i_k}\in[\mathbb{X}_{i_k}]\right)_{1\leq k\leq q},\mathbb{C}=c\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_q}\right)\right\}, \end{aligned}$$

where the number above the equal sign refers to an equation number. Note that the last equality becomes an inclusion $\supset$ in the multi-occurence case. (iii) Let us show that the theorem is true for a subexpression of $f$ of the form

$$c\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right)=\psi\circ a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right). \tag{9}$$

From (4), we have

$$\begin{aligned} & \psi\left([a]\left([\mathbb{X}_{i_1}],\ldots,[\mathbb{X}_{i_p}]\right)\right) \\ \overset{(6)}{=} & \left\{\mathbb{C},\exists\mathbb{A}\in[a]\left([\mathbb{X}_{i_1}],\ldots,[\mathbb{X}_{i_p}]\right),\mathbb{C}=\psi(\mathbb{A})\right\} \\ \overset{(7)}{=} & \left\{\mathbb{C},\left(\exists\mathbb{X}_{i_k}\in[\mathbb{X}_{i_k}]\right)_{k\leq p},\mathbb{A}=a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right),\mathbb{C}=\psi(\mathbb{A})\right\} \\ = & \left\{\mathbb{C},\left(\exists\mathbb{X}_{i_k}\in[\mathbb{X}_{i_k}]\right)_{k\leq p},\mathbb{C}=\psi\circ a\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right)\right\} \\ \overset{(9)}{=} & \left\{\mathbb{C},\left(\exists\mathbb{X}_{i_k}\in[\mathbb{X}_{i_k}]\right)_{k\leq p},\mathbb{C}=c\left(\mathbb{X}_{i_1},\ldots,\mathbb{X}_{i_p}\right)\right\}. \end{aligned}$$

Again, the last equality becomes an inclusion $\supset$ in the multi-occurence case.■

**Example 3**. Using Theorem 1, if $\mathbb{A}\in[\mathbb{A}],\mathbb{B}\in[\mathbb{B}]$, then

$$(\mathbb{A}\cup\mathbb{B})\setminus(\mathbb{A}\cap\mathbb{B})\in([\mathbb{A}]\cup[\mathbb{B}])\setminus([\mathbb{A}]\cap[\mathbb{B}]).$$

Take for instance

$$\begin{aligned} \left[\mathbb{A}^-,\mathbb{A}^+\right] &= \left[[1,3],[0,4]\right] \\ \left[\mathbb{B}^-,\mathbb{B}^+\right] &= \left[[2,5],[1,6]\right]. \end{aligned}$$

Since

$$\begin{aligned}([\mathbb{A}]\cup[\mathbb{B}])\setminus([\mathbb{A}]\cap[\mathbb{B}]) &= \left[\mathbb{A}^-\cup\mathbb{B}^-,\mathbb{A}^+\cup\mathbb{B}^+\right]\setminus\left[\mathbb{A}^-\cap\mathbb{B}^-,\mathbb{A}^+\cap\mathbb{B}^+\right]\\ &= \left[\left(\mathbb{A}^-\cup\mathbb{B}^-\right)\setminus\left(\mathbb{A}^+\cap\mathbb{B}^+\right),\left(\mathbb{A}^+\cup\mathbb{B}^+\right)\setminus\left(\mathbb{A}^-\cap\mathbb{B}^-\right)\right],\end{aligned}$$

we have

$$\begin{aligned}&\left[\left(\mathbb{A}^-\cup\mathbb{B}^-\right)\setminus\left(\mathbb{A}^+\cap\mathbb{B}^+\right),\left(\mathbb{A}^+\cup\mathbb{B}^+\right)\setminus\left(\mathbb{A}^-\cap\mathbb{B}^-\right)\right]\\ =\;&[([1,3]\cup[2,5])\setminus([0,4]\cap[1,6]),([0,4]\cup[1,6])\setminus([1,3]\cap[2,5])]\\ =\;&[[1,5]\setminus[1,4],[0,6]\setminus[2,3]]\\ =\;&[]4,5]\;,\;[0,2[\cup]3,6]\,.\blacksquare\end{aligned}$$

**Dependency problem**. As it is the case for interval arithmetic, the dependency problem also exists for i-sets. For instance,

$$\left[\mathbb{A}^-,\mathbb{A}^+\right]\setminus\left[\mathbb{A}^-,\mathbb{A}^+\right]=\left[\mathbb{A}^-\setminus\mathbb{A}^+,\mathbb{A}^+\setminus\mathbb{A}^-\right]=\left[\emptyset,\mathbb{A}^+\setminus\mathbb{A}^-\right].$$

Of course, we have the inclusion property

$$\left\{\mathbb{A}\setminus\mathbb{A},\mathbb{A}\in\left[\mathbb{A}^-,\mathbb{A}^+\right]\right\}=[\emptyset,\emptyset]\sqsubset\left[\emptyset,\mathbb{A}^+\setminus\mathbb{A}^-\right],$$

but the resulting i-set is not minimal.

# 3 Contractors

Contractors are powerful tools to solve efficiently CSP [3], [5], [6], [1]. They will now be considered in the context of constraints on sets.

## 3.1 Definitions

Consider a constraint on sets of the form $\mathcal{R}(\mathbb{X}_1,\dots,\mathbb{X}_p)$. A *contractor* associated with the constraint $\mathcal{R}$ is an operator

$$([\mathbb{X}_1],\dots,[\mathbb{X}_p])\overset{\mathcal{C}_\mathcal{R}}{\mapsto}([\mathbb{Y}_1],\dots,[\mathbb{Y}_p])$$

where $[\mathbb{X}_1],[\mathbb{Y}_1],\dots,[\mathbb{X}_p],[\mathbb{Y}_p]$ are i-sets, such that

$$\begin{array}{ll}\forall i\in\{1,\dots,p\},[\mathbb{Y}_i]\sqsubset[\mathbb{X}_i] & \text{(contractance)}\\ \left.\begin{array}{l}\mathcal{R}(\mathbb{X}_1,\dots,\mathbb{X}_p)\\ \forall i,\mathbb{X}_i\in[\mathbb{X}_i]\end{array}\right\}\Rightarrow\forall i,\mathbb{X}_i\in[\mathbb{Y}_i]\,. & \text{(completeness)}\end{array}$$

Given two contractors $\mathcal{C}_a$ and $\mathcal{C}_b$ operating on $p$ i-sets $[\mathbb{X}_1],\dots,[\mathbb{X}_p]$, we define the inclusion as follows

$$\mathcal{C}_a\sqsubset\mathcal{C}_b\Leftrightarrow\left(\left.\begin{array}{l}([\mathbb{A}_1],\dots,[\mathbb{A}_p])=\mathcal{C}_a([\mathbb{X}_1],\dots,[\mathbb{X}_p])\\ ([\mathbb{B}_1],\dots,[\mathbb{B}_p])=\mathcal{C}_b([\mathbb{X}_1],\dots,[\mathbb{X}_p])\end{array}\right\}\Rightarrow\forall i,[\mathbb{A}_i]\sqsubset[\mathbb{B}_i]\right),$$

and the intersection by

$$\mathcal{C}_c = \mathcal{C}_a \sqcap \mathcal{C}_b \Leftrightarrow \left( \left. \begin{array}{c} ([\mathbb{A}_1], \dots, [\mathbb{A}_p]) = \mathcal{C}_a([\mathbb{X}_1], \dots, [\mathbb{X}_p]) \\ ([\mathbb{B}_1], \dots, [\mathbb{B}_p]) = \mathcal{C}_b([\mathbb{X}_1], \dots, [\mathbb{X}_p]) \\ ([\mathbb{C}_1], \dots, [\mathbb{C}_p]) = \mathcal{C}_c([\mathbb{X}_1], \dots, [\mathbb{X}_p]) \end{array} \right\} \Rightarrow \forall i, [\mathbb{C}_i] = [\mathbb{A}_i] \sqcap [\mathbb{B}_i] \right).$$

If $\mathcal{C}_a$ and $\mathcal{C}_b$ are two contractors associated with the constraint $\mathcal{R}$, then $\mathcal{C}_a \sqcap \mathcal{C}_b$ is also a contractor for $\mathcal{R}$. As a consequence, there exists a smallest (with respect to $\sqsubset$) contractor $\mathcal{C}^*$ for $\mathcal{R}$. It corresponds to the intersection of all contractors for $\mathcal{R}$. The contractor $\mathcal{C}^*$ is the *minimal contractor* for $\mathcal{R}$ and returns the smallest i-sets $([\mathbb{A}_1], \dots, [\mathbb{A}_p])$ that are consistent with the constraint $\mathcal{R}$ and all $[\mathbb{X}_i]$'s, i.e., for all $i \in \{1, \dots, p\}$ we should have

$$[\mathbb{A}_i] = \square \left\{ \mathbb{X}_i \in [\mathbb{X}_i], (\exists \mathbb{X}_j \in [\mathbb{X}_j])_{j \neq i}, \mathcal{R}(\mathbb{X}_1, \dots, \mathbb{X}_p) \right\}.$$

The following theorem will be used to build minimal contractors.

**Theorem 2**. Consider a function $f(\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_p)$ composed of wrappingless operators or functions which returns a subset of $\mathbb{R}^n$ from $p$ subsets of $\mathbb{R}^n$. Assume that in the expression of $f$, each $\mathbb{X}_i$ occurs only once. We have

$$\left\{ \mathbb{Y} \in [\mathbb{Y}], (\exists \mathbb{X}_i \in [\mathbb{X}_i])_{i \leq p}, \mathbb{Y} = f(\mathbb{X}_1, \dots, \mathbb{X}_p) \right\} = [\mathbb{Y}] \sqcap [f]([\mathbb{X}_1], \dots, [\mathbb{X}_p]).$$

**Proof**. In the mono-occurence case, from Theorem 1, $[f]([\mathbb{X}_1], \dots, [\mathbb{X}_p]) = f([\mathbb{X}_1], \dots, [\mathbb{X}_p])$. Thus

$$\begin{array}{rcl} [\mathbb{Y}] \sqcap [f]([\mathbb{X}_1], \dots, [\mathbb{X}_p]) & = & [\mathbb{Y}] \sqcap f([\mathbb{X}_1], \dots, [\mathbb{X}_p]) \\ & = & \left\{ \mathbb{Y} \in [\mathbb{Y}], (\exists \mathbb{X}_i \in [\mathbb{X}_i])_{i \leq p}, \mathbb{Y} = f(\mathbb{X}_1, \dots, \mathbb{X}_p) \right\}. \blacksquare \end{array}$$

## 3.2 Some minimal contractors

This section presents some minimal contractors associated with specific primitive set-valued constraints. The methodology that will be used to build contractors for a set constraint $\mathcal{R}(\mathbb{X}_1, \dots, \mathbb{X}_p)$ is very similar that what is done to build contractors for constraints involving real numbers [22]. Recall for instance that the constraint $\mathcal{R}(x, y, z) : z = x + y$ yields the contractor

$$\mathcal{C}_+ \begin{pmatrix} [x] \\ [y] \\ [z] \end{pmatrix} = \begin{pmatrix} ([z] - [y]) \cap [x] \\ ([z] - [x]) \cap [y] \\ ([x] + [y]) \cap [z] \end{pmatrix}$$

To get the expression for $\mathcal{C}_+$, we first had to rewrite the constraint into three equivalent forms: $x = f_1(y, z) = z - y \Leftrightarrow y = f_2(x, z) = z - x \Leftrightarrow z = f_3(x, y) = x + y$. Then, we performed an interval evaluation of the $f_i$ and an intersection with the initial interval. The principle of the methodology to build i-set contractors is similar: the constraint $\mathcal{R}(\mathbb{X}_1, \dots, \mathbb{X}_p)$ is first rewritten as $p$ equivalent forms: $\mathbb{X}_1 = f_1(\mathbb{X}_2, \dots, \mathbb{X}_p) \Leftrightarrow \mathbb{X}_2 = f_2(\mathbb{X}_1, \mathbb{X}_3, \dots, \mathbb{X}_p) \Leftrightarrow \dots$ (in a similar way to what is done for constraints involving real numbers). The i-set arithmetic is then used to automatically generate the contractors.

**Proposition 1**. The minimal contractor associated with the constraint $\mathbb{A} \subset \mathbb{B}$ is

$$\mathcal{C}_{\subset}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}] \sqcap ([\mathbb{B}] \setminus [\emptyset, \mathbb{R}^n]) \\ [\mathbb{B}] \sqcap ([\mathbb{A}] \cup [\emptyset, \mathbb{R}^n]) \end{pmatrix} \quad (10)$$

or equivalently

$$\mathcal{C}_{\subset}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}^-, \mathbb{A}^+ \cap \mathbb{B}^+] \\ [\mathbb{B}^- \cup \mathbb{A}^-, \mathbb{B}^+] \end{pmatrix}.$$

**Proof**. By definition, the minimal contractor for the constraint $\mathbb{A} \subset \mathbb{B}$ is given by

$$\mathcal{C}_{\subset}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} \square \{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \mathbb{A} \subset \mathbb{B}\} \\ \square \{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \mathbb{A} \subset \mathbb{B}\} \end{pmatrix}.$$

Now, since

$$\mathbb{A} \subset \mathbb{B} \Leftrightarrow \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{A} = \mathbb{B} \setminus \mathbb{Z} \Leftrightarrow \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{B} = \mathbb{A} \cup \mathbb{Z},$$

we have

$$\mathcal{C}_{\subset}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} \square \{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{A} = \mathbb{B} \setminus \mathbb{Z}\} \\ \square \{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{B} = \mathbb{A} \cup \mathbb{Z}\} \end{pmatrix}.$$

From Theorem 2, we get (10). Moreover, using i-set arithmetic, we have

$$\begin{array}{rcl} [\mathbb{A}] \sqcap ([\mathbb{B}] \setminus [\emptyset, \mathbb{R}^n]) & \overset{(3,\text{iv})}{=} & [\mathbb{A}^-, \mathbb{A}^+] \sqcap [\mathbb{B}^- \setminus \mathbb{R}^n, \mathbb{B}^+ \setminus \emptyset] \\ & \overset{(1)}{=} & [\mathbb{A}^- \cup \emptyset, \mathbb{A}^+ \cap \mathbb{B}^+] \\ & = & [\mathbb{A}^-, \mathbb{A}^+ \cap \mathbb{B}^+], \end{array}$$

and

$$\begin{array}{rcl} [\mathbb{B}] \sqcap ([\mathbb{A}] \cup [\emptyset, \mathbb{R}^n]) & \overset{(3,\text{ii})}{=} & [\mathbb{B}^-, \mathbb{B}^+] \sqcap [\mathbb{A}^- \cup \emptyset, \mathbb{A}^+ \cup \mathbb{R}^n] \\ & \overset{(1)}{=} & [\mathbb{B}^- \cup \mathbb{A}^-, \mathbb{B}^+ \cap \mathbb{R}^n] \\ & = & [\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{B}^+]. \blacksquare \end{array}$$

**Proposition 2**. The minimal contractor associated with the constraint $\mathbb{A} \cap \mathbb{B} = \emptyset$ is

$$\mathcal{C}_{\neq}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}] \sqcap ([\emptyset, \mathbb{R}^n] \setminus [\mathbb{B}]) \\ [\mathbb{B}] \sqcap ([\emptyset, \mathbb{R}^n] \setminus [\mathbb{A}]) \end{pmatrix} \quad (11)$$

or equivalently

$$\mathcal{C}_{\neq}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}^-, \mathbb{A}^+ \setminus \mathbb{B}^-] \\ [\mathbb{B}^-, \mathbb{B}^+ \setminus \mathbb{A}^-] \end{pmatrix}.$$

**Proof**. By definition, the minimal contractor is given by

$$\mathcal{C}_{\neq}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} \square \{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \mathbb{A} \cap \mathbb{B} = \emptyset\} \\ \square \{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \mathbb{A} \cap \mathbb{B} = \emptyset\} \end{pmatrix}.$$

Now, since

$$\mathbb{A} \cap \mathbb{B} = \emptyset \Leftrightarrow \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{A} = \mathbb{Z} \setminus \mathbb{B} \Leftrightarrow \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{B} = \mathbb{Z} \setminus \mathbb{A}$$

we have

$$\mathcal{C}_{\neq}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} \square\{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{A} = \mathbb{Z} \setminus \mathbb{B}\} \\ \square\{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{B} = \mathbb{Z} \setminus \mathbb{A}\} \end{pmatrix}.$$

Using Theorem 2, we get (11). Using the i-set arithmetic, we get

$$\begin{array}{rcl} [\mathbb{A}] \sqcap ([\emptyset, \mathbb{R}^n] \setminus [\mathbb{B}]) & = & [\mathbb{A}^-, \mathbb{A}^+] \sqcap ([\emptyset, \mathbb{R}^n] \setminus [\mathbb{B}^-, \mathbb{B}^+]) \\ & \overset{(3,\text{iv})}{=} & [\mathbb{A}^-, \mathbb{A}^+] \sqcap ([\emptyset \setminus \mathbb{B}^+, \mathbb{R}^n \setminus \mathbb{B}^-]) \\ & \overset{(1)}{=} & [\mathbb{A}^-, \mathbb{A}^+ \cap (\mathbb{R}^n \setminus \mathbb{B}^-)] \\ & = & [\mathbb{A}^-, \mathbb{A}^+ \setminus \mathbb{B}^-]. \blacksquare \end{array}$$

**Proposition 3**. The minimal contractor associated with the constraint $\mathbb{A} \cap \mathbb{B} = \mathbb{C}$ is

$$\mathcal{C}_{\cap}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \\ [\mathbb{C}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}] \sqcap (([\emptyset, \mathbb{R}^n] \setminus [\mathbb{B}]) \cup [\mathbb{C}]) \\ [\mathbb{B}] \sqcap (([\emptyset, \mathbb{R}^n] \setminus [\mathbb{A}]) \cup [\mathbb{C}]) \\ [\mathbb{C}] \sqcap ([\mathbb{A}] \cap [\mathbb{B}]) \end{pmatrix} \quad (12)$$

or equivalently

$$\mathcal{C}_{\cap}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \\ [\mathbb{C}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}^- \cup \mathbb{C}^-, \mathbb{A}^+ \setminus (\mathbb{B}^- \setminus \mathbb{C}^+)] \\ [\mathbb{B}^- \cup \mathbb{C}^-, \mathbb{B}^+ \setminus (\mathbb{A}^- \setminus \mathbb{C}^+)]. \\ [\mathbb{C}^- \cup (\mathbb{A}^- \cap \mathbb{B}^-), \mathbb{C}^+ \cap \mathbb{A}^+ \cap \mathbb{B}^+] \end{pmatrix}. \quad (13)$$

An illustration is represented on Figure 2. Subfigure (a) represents the initial i-sets $[\mathbb{A}], [\mathbb{B}], [\mathbb{C}]$, before contraction. These i-sets can be contracted without removing any set which is consistent with the constraint and the domains for other sets. The principle of the contractions is illustrated by the Figure 2 (b),(c),(d).

**Proof**. By definition, the minimal contractor is given by

$$\mathcal{C}_{\cap}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \\ [\mathbb{C}] \end{pmatrix} = \begin{pmatrix} \square\{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \exists \mathbb{C} \in [\mathbb{C}], \mathbb{A} \cap \mathbb{B} = \mathbb{C}\} \\ \square\{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{C} \in [\mathbb{C}], \mathbb{A} \cap \mathbb{B} = \mathbb{C}\} \\ \square\{\mathbb{C} \in [\mathbb{C}], \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \mathbb{A} \cap \mathbb{B} = \mathbb{C}\} \end{pmatrix}$$

Now, since

$$\mathbb{A} \cap \mathbb{B} = \mathbb{C} \Leftrightarrow \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{A} = (\mathbb{Z} \setminus \mathbb{B}) \cup \mathbb{C} \Leftrightarrow \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{B} = (\mathbb{Z} \setminus \mathbb{A}) \cup \mathbb{C},$$

we have

$$\mathcal{C}_{\cap}\begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \\ [\mathbb{C}] \end{pmatrix} = \begin{pmatrix} \square\{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \exists \mathbb{C} \in [\mathbb{C}], \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{A} = (\mathbb{Z} \setminus \mathbb{B}) \cup \mathbb{C}\} \\ \square\{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{C} \in [\mathbb{C}], \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n], \mathbb{B} = (\mathbb{Z} \setminus \mathbb{A}) \cup \mathbb{C}\} \\ \square\{\mathbb{C} \in [\mathbb{C}], \exists \mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \mathbb{C} = \mathbb{A} \cap \mathbb{B}\} \end{pmatrix}.$$

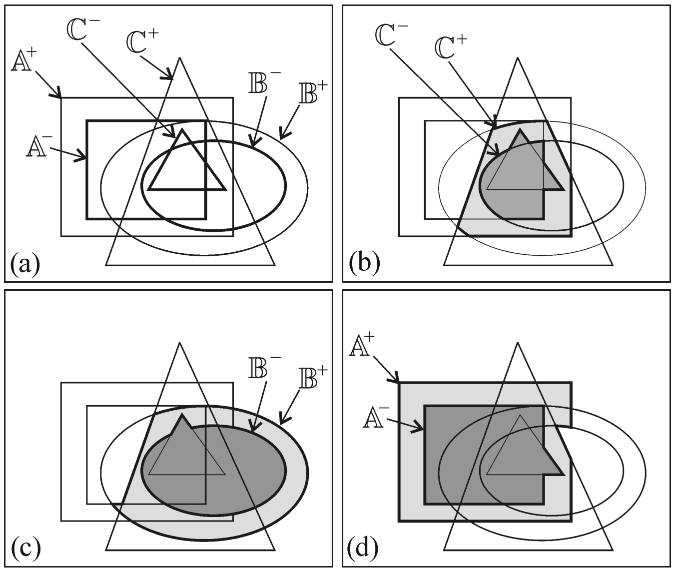


Figure 2: Minimal contractor associated with the constraint $\mathbb{A} \cap \mathbb{B} = \mathbb{C}$.

Using Theorem 2, we get (12). Using i-set arithmetic, as for the previous proofs, we get (13). ■

**Proposition 4**. The minimal contractor associated with the constraint $f(\mathbb{A}) = \mathbb{B}$ where $f$ is bijective is

$$\mathcal{C}_f \begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} [\mathbb{A}] \sqcap f^{-1}([\mathbb{B}]) \\ [\mathbb{B}] \sqcap f([\mathbb{A}]) \end{pmatrix} \tag{14}$$

or equivalently

$$\mathcal{C}_f \begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} \left[\mathbb{A}^- \cup f^{-1}(\mathbb{B}^-), \mathbb{A}^+ \cap f^{-1}(\mathbb{B}^+)\right] \\ \left[\mathbb{B}^- \cup f(\mathbb{A}^-), \mathbb{B}^+ \cap f(\mathbb{A}^+)\right] \end{pmatrix}. \tag{15}$$

**Proof**. By definition, the minimal contractor for the constraint $f(\mathbb{A}) = \mathbb{B}$ is given by

$$\mathcal{C}_f \begin{pmatrix} [\mathbb{A}] \\ [\mathbb{B}] \end{pmatrix} = \begin{pmatrix} \square \{\mathbb{A} \in [\mathbb{A}], \exists \mathbb{B} \in [\mathbb{B}], \mathbb{B} = f(\mathbb{A})\} \\ \square \left\{\mathbb{B} \in [\mathbb{B}], \exists \mathbb{A} \in [\mathbb{A}], \mathbb{A} = f^{-1}(\mathbb{B})\right\} \end{pmatrix}.$$

Using Theorem 2, we get (14) and using the i-set arithmetic, we get (15). ■

## 3.3 Propagation

Contractors can be used to solve SVCSP. The first step is to decompose all constraints of the SVCSP into constraints for which minimal contractors are available. Such constraints are called *primitive constraints.* For instance, a constraint of the form

$$\mathbb{A} + \mathbb{B} \subset f(\mathbb{A}) \cap \mathbb{C}$$

can be decomposed into

$$\begin{cases} \mathbb{A} + \mathbb{B} = \mathbb{Z}_1 \\ \mathbb{Z}_2 = f(\mathbb{A}) \\ \mathbb{Z}_2 \cap \mathbb{C} = \mathbb{Z}_3 \\ \mathbb{Z}_1 \subset \mathbb{Z}_3 \end{cases}$$

The sets $\mathbb{Z}_i$ are slack sets that have been introduced for the decomposition. Their domains should be initialized to $[\emptyset, \mathbb{R}^n]$. We assumed here that a minimal contractor for the constraint $\mathbb{A} + \mathbb{B} = \mathbb{Z}_1$ was available, even if it has not been given in this paper. In the second step, we take all minimal contractors associated with each primitive constraint and we put them into a list of contractors named the *store*. The last step, called the propagation, calls all contractors of the store several times until no more contractor is able to contract any i-set associated to each unknown set. The result of the propagation is a list of i-sets which enclose all unknown sets that satisfy all constraints of the initial SVCSP. The process will be illustrated on the following section.

# 4 Test-case

Consider the following SVCSP

$$\begin{cases} \text{(i)} & \mathbb{X} \subset \mathbb{A} \\ \text{(ii)} & \mathbb{B} \subset \mathbb{X} \\ \text{(iii)} & \mathbb{X} \cap \mathbb{C} = \emptyset \\ \text{(iv)} & f(\mathbb{X}) = \mathbb{X}, \end{cases}$$

where $\mathbb{X}$ is an unknown subset of $\mathbb{R}^2$, $f$ is a rotation of $\mathbb{R}^2$ around $\mathbf{0}$ with an angle $-\frac{\pi}{6}$, and

$$\begin{cases} \mathbb{A} &= \left\{(x_1, x_2), x_1^2 + x_2^2 \leq 3\right\} \\ \mathbb{B} &= \left\{(x_1, x_2), (x_1 - 0.5)^2 + x_2^2 \leq 0.3\right\} \\ \mathbb{C} &= \left\{(x_1, x_2), (x_1 - 1)^2 + (x_2 - 1)^2 \leq 0.15\right\}. \end{cases}$$

In our context, a constraint propagation approach consists in contracting all i-sets with respect to all constraints several times until no more significant contraction can be observed. Figure 3 illustrates the propagation process[1]. Subfigures (a), (b), (c) represent $\mathbb{A}, \mathbb{B}, \mathbb{C}$. Subfigure (d) represents the i-set $[\mathbb{X}]$ after contracting with respect to constraint (i). If we now contract with respect to constraint (ii), we get Subfigure (e) for $[\mathbb{X}]$. Constraint (iii) yields Subfigure (f). Another contraction with respect to all four constraints produces Subfigure (g). Finally, Subfigure (h) represents the fixed point that is obtained for $[\mathbb{X}]$.

[1] **Color code**. For the graphical representation of an i-set $[\mathbb{X}] = [\mathbb{X}^-, \mathbb{X}^+]$, the black boxes are inside $\mathbb{X}^-$, the grey boxes are outside $\mathbb{X}^+$ and the white boxes are inside $\mathbb{X}^+$ and outside $\mathbb{X}^-$.

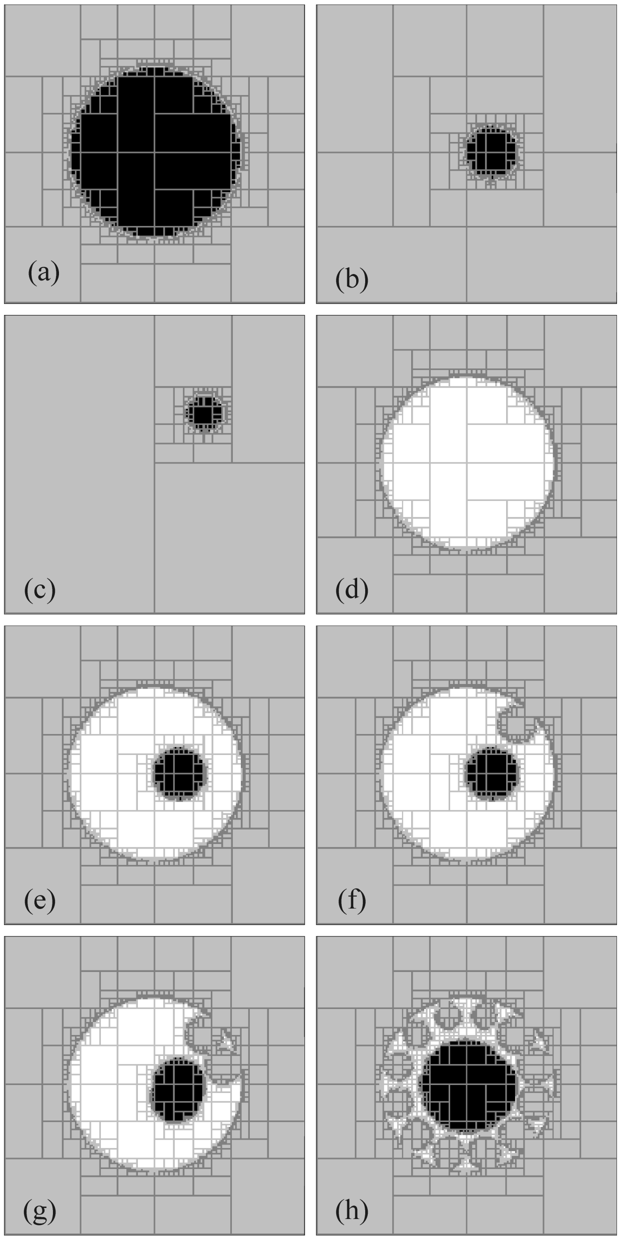


Figure 3: Illustration of the propagation process for set-valued CSP; the frame boxes correspond to $[-3, 3]^2$.

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# 5 Conclusion

Constraint propagation methods are well known methods to solve efficiently nonlinear and non convex problems where the unknown variables belong to discrete sets or when these variables are vectors of $\mathbb{R}^n$. However, to my knowledge, propagation methods have never be used to solve problems where the unknown variables are subsets of $\mathbb{R}^n$. This paper proposes to extend the class of problems that can be solved using constraint propagation to set-valued constraint satisfaction problems (SVCSP). The variables of such CSP are subsets $\mathbb{X}$ of $\mathbb{R}^n$ that can be bracketed by pairs of sets, denoted by $[\mathbb{X}^-, \mathbb{X}^+]$. These pairs, named *i-sets*, form the domains on which the set variables should belong. Operators are provided for i-sets which make possible to build minimal contractors and consequently to allow a resolution based on constraint propagation. An illustrative example has been provided to illustrate the principle of the approach.

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