Computing capture tubes

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Abstract. Many mobile robots such as wheeled robots, boats, or plane are described by nonholonomic differential equations. As a consequence, they have to satisfy some differential constraints such as having a radius of curvature for their trajectory lower than a known value. For this type of robots, it is difficult to prove some properties such as the avoidance of collisions with some moving obstacles. This is even more difficult when the initial condition is not known exactly or when some uncertainties occur. This paper proposes a method to compute an enclosure (a *tube*) for the trajectory of the robot in situations where a guaranteed interval integration cannot provide any acceptable enclosures. All properties that are satisfied by the tube (such as the non-collision) will also be satisfied by the actual trajectory of the robot.

Keywords: capture tube, contractors, interval arithmetic, robotics, stability.

1 Introduction

A dynamic system can generally be described a state equation of the form:

$$\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t). \tag{1}$$

In the situation where the system is uncertain, the state equation becomes a time dependent differential inclusion:

$$\mathcal{S}_{\mathbf{F}} : \dot{\mathbf{x}}(t) \in \mathbf{F}(\mathbf{x}(t), t). \tag{2}$$

Validation of the stability properties of such systems is an important and difficult problem [15]. Most of the time, this problem can be transformed into proving the inconsistency of a *constraint network*. For invariant systems (*i.e.*, $\mathbf{f}$ or $\mathbf{F}$ do not depend on t), it has been shown [10] that the *V-stability* approach combined with interval analysis [16] can solve the problem efficiently. Here, we extend this work to systems where $\mathbf{f}$ depends on time. Moreover, we will show how to compute a *capture tube*, *i.e.*, a set-valued function which associate to each t a subset of $\mathbb{R}^n$ and such that a feasible trajectory cannot escape. For this, we will need to combine guaranteed integration and Lyapunov theory, such as in [19] or [13], in order to compute this capture tube.

The paper is organized as follows. Section 2 defines the notion of capture tube, which is a specific set of trajectories that encloses the unknown trajectory for the robot. Section 3 explains how tubes can be represented inside the computer and how we can calculate a tube for a trajectory which satisfies a differential inclusion. Section 4 provides a new algorithm that is able to calculate an interval of tubes which encloses the smallest capture tube which contains one candidate tube. An illustrative test-case is presented in Section 5 and a conclusion of the paper is given in Section 6.

2 Capture tube

A *tube* $\mathbb{G}$ (see e.g. [1]) is a function which associates to each $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$. Tubes are used for several applications in nonlinear control such as model predictive control [12] or state estimation [2].

Notations. Depending on the context, a tube $\mathbb{G}$ will be seen as a set-valued function $t \mapsto \mathcal{P}(\mathbb{R}^n)$, or also as a subset of $\mathbb{R} \times \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ is the set of subsets of $\mathbb{R}^n$. It will often be written as $\mathbb{G}(\cdot)$ or also $\mathbb{G}(t)$ to recall that it is a function of t. For instance, when we write $\mathbf{x}(t) \in \mathbb{G}(t)$, we mean $\forall t, \mathbf{x}(t) \in \mathbb{G}(t)$ and when we write $(t_a, \mathbf{a}) \in \mathbb{G}(t)$, we mean $\mathbf{a} \in \mathbb{G}(t_a)$. ■

Consider an autonomous system described by a state equation $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ or a differential inclusion $\mathcal{S}_{\mathbf{F}} : \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$. A tube $\mathbb{G}(t)$ is said to be a *capture tube* [5] (or also called *positive invariant tube*) for $\mathcal{S}_{\mathbf{f}}$ or $\mathcal{S}_{\mathbf{F}}$ if we have the following implication:

$$\mathbf{x}(t_a) \in \mathbb{G}(t_a), \tau > 0 \Rightarrow \mathbf{x}(t_a + \tau) \in \mathbb{G}(t_a + \tau). \quad (3)$$

Figure 1 gives some feasible trajectories and a tube $\mathbb{G}(t)$ (in gray). In this figure,

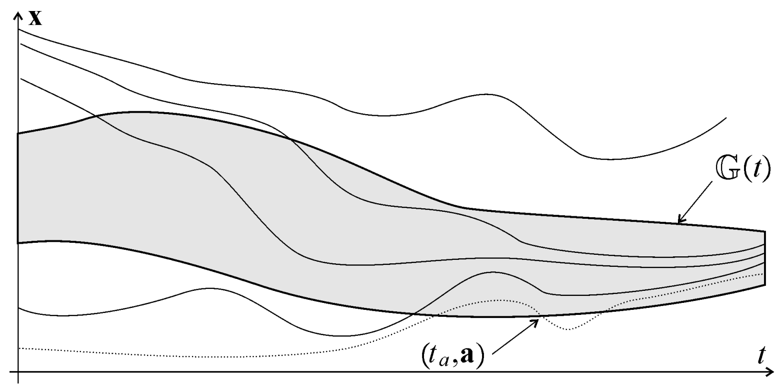

Fig. 1. A tube (painted gray) and possible trajectories for different initial conditions. If a trajectory such as the one represented by the dotted curve exists then the tube is not a capture tube

all the trajectories are consistent with the assumption that $\mathbb{G}(t)$ is a capture tube, except the trajectory represented by the dotted curve at the bottom, which was able to escape from the tube for $t = t_a$. Consider the tube

$$\mathbb{G}(\cdot) : t \mapsto \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}, \tag{4}$$

where $\mathbf{g} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is assumed to be differentiable with respect to both $\mathbf{x}$ and t. The following theorem shows that the problem of proving that $\mathbb{G}(t)$ is a capture tube can be cast into proving that a set of inequalities has no solution.

Theorem 1a. If the system of constraints (called the *cross-out* conditions)

$$\begin{cases} \text{(i)} & \underbrace{\frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t)}_{\dot{g}_i(\mathbf{x}, t)} \geq 0, \\ \text{(ii)} & g_i(\mathbf{x}, t) = 0, \\ \text{(iii)} & \mathbf{g}(\mathbf{x}, t) \leq 0, \end{cases} \tag{5}$$

is inconsistent (*i.e.*, for all $\mathbf{x}$, all $t \geq 0$, and all $i \in \{1, \ldots, m\}$, the inequalities are not satisfied), then $\mathbb{G}(\cdot) : t \mapsto \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$ is a capture tube for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

Sketch of proof (see [21] and [23] for more details). If $\mathbb{G}(t)$ is not a capture tube, it means that there exists one trajectory, which leaves $\mathbb{G}(t)$, *i.e.*, which crosses the ith boundary $g_i(\mathbf{x}, t) = 0$ from inside to outside. This means that there exists a time-space pair $(\mathbf{a}, t_a)$ on the boundary of $\mathbb{G}(t)$ (*i.e.*, such that *(ii)* and *(iii)* are satisfied) and such that $\dot{g}_i(\mathbf{x}, t) \geq 0$ (otherwise the trajectory cannot leave the tube). ■

Example 1. Consider again Figure 1 where we assume that the gray tube corresponds to $\mathbb{G}(\cdot) : t \mapsto \{x \mid g_1(x, t) \leq 0\}$. The dotted trajectory leaves the tube at a time-space point (t_a, a), such that $g_1(a, t_a) = 0$ and $\dot{g}_1(a, t_a) > 0$. If such a trajectory is feasible, then $\mathbb{G}(\cdot)$ cannot be a capture tube.

Example 2. We now illustrate the difficulty to get a capture tube on the simple pendulum described by the state equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 - 0.15 \cdot x_2 \end{cases} \tag{6}$$

where x_1 is the position of the pendulum and x_2 its rotational speed (see Figure 2). To find a positive invariant set (*i.e.*, a capture tube) for such a mechanical system the classical method is to take sublevel sets of the energy of the system. Indeed, since the energy of the system

$$E(\mathbf{x}) = \frac{1}{2}\dot{x}_1^2 - \cos x_1 + 1 = \frac{1}{2}x_2^2 - \cos x_1 + 1 \tag{7}$$

is supposed to decrease with time, we may think that it may be a good candidate for the function g. Let us propose for $g(\mathbf{x}, t)$, which defines our candidate for the capture tube (or positive invariant tube):

$$g(\mathbf{x}, t) = E(\mathbf{x}) - 1 = \frac{1}{2}x_2^2 - \cos x_1, \tag{8}$$

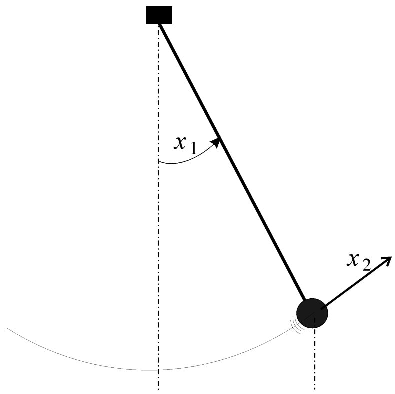

Fig. 2. Simple pendulum

which is here time independent. The cross-out conditions of Theorem 1a are

$$\begin{cases} \text{(i)} & \left(\sin x_1 \ x_2\right) \begin{pmatrix} x_2 \\ -\sin x_1 - 0.15 \cdot x_2 \end{pmatrix} = -0.15 \cdot x_2^2 \ \geq 0, \\ \text{(ii)} & \frac{1}{2}x_2^2 - \cos x_1 = 0. \end{cases} \quad (9)$$

Note that, since $g(\mathbf{x})$ is scalar, we have $i = 1$ and the condition (iii) is a consequence of (ii). This system has two solutions: $\mathbf{x} = \left(\pm\frac{\pi}{2}, 0\right)$. Therefore, Theorem 1a cannot conclude that our tube is positive invariant. Note that, even for this simple two dimensional example which is time-invariant and for which we have a good intuition of a function (the energy) which decreases (almost always), getting a capture tube is difficult. We will see in Section 3 how a capture tube can be computed automatically.

Theorem 1b. If the system of constraints (*cross-out* conditions)

$$\begin{cases} \text{(i1)} & \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \mathbf{a} + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \ \geq 0, \\ \text{(i2)} & \mathbf{a} \in \mathbf{F}(\mathbf{x}, t), \\ \text{(ii)} & g_i(\mathbf{x}, t) = 0, \\ \text{(iii)} & \mathbf{g}(\mathbf{x}, t) \leq 0, \end{cases} \quad (10)$$

is inconsistent for all $\mathbf{x}$, all $\mathbf{a}$, all $t \geq 0$, and all $i \in \{1, \dots, m\}$ then $\mathbb{G}(\cdot) : t \mapsto \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$ is a capture tube for the differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$.

Proof. The proof is a direct consequence of Theorem 1a. See also [23].

Consequence. From Theorems 1a and 1b, we conclude that checking that "a tube defined by inequalities is a capture tube" amounts to checking that a set of constraints (here (5) or (10)) is inconsistent. This type of results was already known since several decades [23] [9]. Now, proving such an inconsistency can easily be performed [21] using contractor-based methods [7].

We now have a procedure to prove that a tube is a capture tube. In practice, such a capture tube is difficult to obtain, especially for nonholonomic robots.

Even if we have a good intuition of the system and if we are very confident on a potential tube, a contractor-based algorithm often finds a counterexample. In the following section, we will give a new method able to compute automatically capture tubes.

3 Computing with tubes

3.1 Representation of tubes

Recall that a tube is a function which associates to any $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$. In the case where these subsets are intervals or boxes, a tube can be represented in the computer by stepwise functions (see [4], [2]) as illustrated in Figure 3. Another possible representation of a tube (see [16]) is an interval expression,

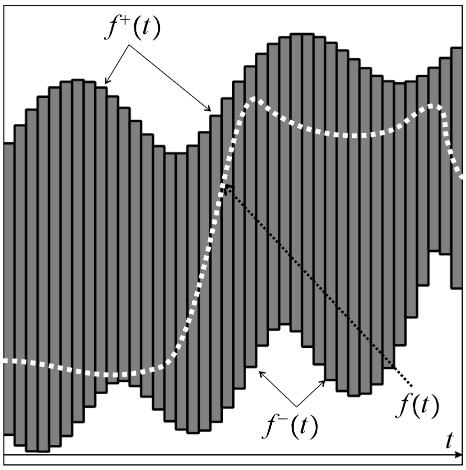

Fig. 3. In numerical computations, a tube $[f](t)$ can be approximated by a lower and an upper stepwise functions $f^-(t)$ and $f^+(t)$. The tube $[f](t)$ encloses an uncertain trajectory $f(t)$

which depends on t. For instance,

$$[f](t) = [1,2] \cdot t + \sin([1,3] \cdot t) \tag{11}$$

corresponds to such a tube. Interval polynomials [16] also enter within this class. An example of a third degree polynomial tube is given by

$$[f](t) = [a_0] + [a_1]t + [a_2]t^2 + [a_3]t^3, \tag{12}$$

where the $[a_i]$ are known intervals. The advantage of interval polynomial is that all operations on scalar polynomials (such as integral, composition, *etc.*) can easily be extended to this class. For instance

$$\int_0^t [f](\tau)\, d\tau = [a_0]\, t + [a_1]\, \frac{t^2}{2} + [a_2]\, \frac{t^3}{3} + [a_3]\, \frac{t^4}{4}. \tag{13}$$

It has been proved [16] for the integration, for the composition, and other operations (such as $+, -, /, \cdot$) that the fundamental inclusion property is satisfied. More precisely, for the integration, this inclusion property is

$$f(\cdot) \in [f](\cdot) \;\Rightarrow\; \forall t, \int_0^t f(\tau)\, d\tau \in \int_0^t [f](\tau)\, d\tau. \tag{14}$$

Remark. For the derivative, this extension cannot be done. For a counterexample, consider the relation

$$\sin(\omega t) \cdot t \in [-1, 1] \cdot t. \tag{15}$$

It is clear that we cannot conclude that

$$\omega \cos(\omega t) \cdot t + \sin(\omega t) \in [-1, 1]. \tag{16}$$

Thus, the fundamental inclusion property, which is required by all set-membership approaches, is not satisfied for the derivative.

3.2 Guaranteed integration

For the problem we consider in this paper, *i.e.*, computing capture tubes, the guaranteed integration will be needed. Guaranteed integration is a set of techniques, which make it possible to compute a tube that encloses the solution of a state equation or to enclose all solutions of a differential inclusion. We here recall the principle of these techniques. For more details on the guaranteed integration of state equations, see [14], [17] or [3], [18]. To our knowledge in the literature, the extension of these techniques to differential inclusion is rarely done. This is why we present here the basic concepts of the guaranteed integration in order to show how they can be extended to the uncertain case, *i.e.*, to differential inclusions. More details and more efficient algorithms for the interval integration of differential inclusions can be found in [11] and [22]

Brouwer theorem. Any continuous function f mapping a compact convex set $\mathbb{X}$ into itself has a fixed point, *i.e.*,

$$\exists x \in \mathbb{X} \mid \; f(x) = x. \tag{17}$$

Note that a direct corollary of this theorem is that these fixed points also belong to the set $f(\mathbb{X})$.

Example 3. Take $f(x) = \sin(x) \cdot \cos(x)$ and $\mathbb{X} = [-2, 2]$. Since

$$f([-2, 2]) \subset \sin([-2, 2]) \cdot \cos([-2, 2]) = [-1, 1] \cdot [-1, 1] = [-1, 1] \subset \mathbb{X}. \tag{18}$$

From the Brouwer theorem, we have

$$\exists x \in [-2, 2] \mid \sin(x) \cdot \cos(x) = x. \tag{19}$$

The Brouwer theorem is the corner stone that will make it possible to compute a tube containing the solution of a state equation. For its extension to differential inclusions, the uncertain case will be treated using a parametric version of the Brouwer theorem.

Parametric Brouwer theorem. If $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, where $\mathbb{X}$ is a convex compact set and f is continuous with respect to $x \in \mathbb{X}$, then

$$\forall u \in \mathbb{U},\ \exists x \in \mathbb{X} \mid f(x, u) = x. \tag{20}$$

Example 4. Take $f(x) = \sin(x + u) \cdot \cos(2x - u)$ and $\mathbb{X} = [-2, 2]$ and $u \in \mathbb{R}$. Since

$$f([-2, 2], \mathbb{R}) \subset [-1, 1] \subset \mathbb{X}, \tag{21}$$

we have

$$\forall u \in \mathbb{R}, \exists x \in [-2, 2] \mid \sin(x + u) \cdot \cos(2x - u) = x. \tag{22}$$

Guaranteed integration of state equations. Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is Lipschitz continuous. The initial condition $\mathbf{x}_0^*$ is known. We want to have an interval enclosure for the trajectory $\mathbf{x}^*(\cdot)$. [1]Define the Picard-Lindelöf operator as

$$\mathcal{T} : \mathbf{x}(\cdot) \to \left(t \mapsto \mathbf{x}_0^* + \int_0^t \mathbf{f}(\mathbf{x}(\tau))\, d\tau \right). \tag{23}$$

Since $\mathbf{f}$ is Lipschitz continuous, $\mathcal{T}$ has a unique fixed point which corresponds to the solution $\mathbf{x}^*(\cdot)$ of the state equation. Take an interval tube $[\mathbf{x}](\cdot)$. By interval tube, we mean that for all t, $[\mathbf{x}](t)$ is a box of $\mathbb{R}^n$ and not any subset of $\mathbb{R}^n$, as it is allowed for general tubes of $\mathbb{R}^n$. From the Brouwer theorem and since $\mathcal{T}$ has a unique fixed point, we have

$$\mathcal{T}([\mathbf{x}](\cdot)) \subset [\mathbf{x}](\cdot) \Rightarrow \mathbf{x}^*(\cdot) \in [\mathbf{x}](\cdot). \tag{24}$$

Figure 4 provides a representation of the tubes $[\mathbf{x}](\cdot)$ and $\mathcal{T}([\mathbf{x}](\cdot))$. Note that, due to the specific form of $\mathcal{T}$, around the initial instant $t = 0$, the tube $\mathcal{T}([\mathbf{x}](\cdot))$ is thin. Note also that we do not have $\mathcal{T}([\mathbf{x}](\cdot)) \subset [\mathbf{x}](\cdot)$ (*i.e.*, $\mathcal{T}([\mathbf{x}](t))$ is included in $[\mathbf{x}](t)$ only for $t \le t_1$) and the trajectory may leave the tubes. If we restrict application of $\mathcal{T}$ over the interval $[0, t_1]$, we get the inclusion. Therefore,

$$\forall t \in [0, t_1], \mathbf{x}^*(t) \in \mathcal{T}([\mathbf{x}](t)), \tag{25}$$

where

$$t_1 = \max\left\{ t \in \mathbb{R}^+ \mid \forall \tau \in [0, t], \mathcal{T}([\mathbf{x}](\tau)) \subset [\mathbf{x}](\tau) \right\}. \tag{26}$$

Of course, the operator can be called several times, *i.e.*,

$$\forall i \ge 0, \forall t \in [0, t_1], \mathbf{x}^*(t) \in \mathcal{T}^i([\mathbf{x}](t)). \tag{27}$$

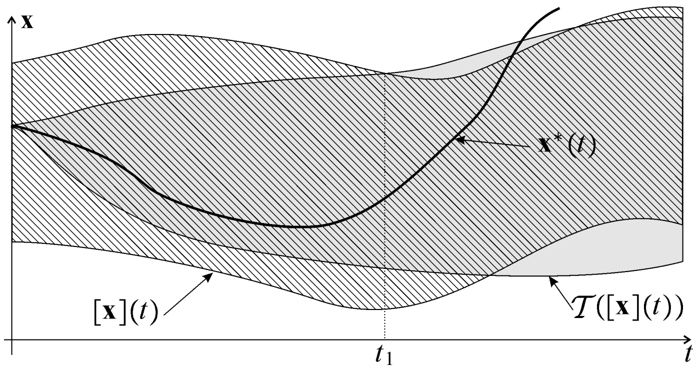

Fig. 4. Illustration of the Picard-Lindelöf operator to the tube $[\mathbf{x}]\,(t)$

Case with uncertainties. Assume now, that $\mathbf{x}_0$ is uncertain and that the system now depends on an uncertain input vector $\mathbf{u}\,(\cdot)$ More precisely, the system is described by

$$\dot{\mathbf{x}} = \mathbf{f}\,(\mathbf{x}, \mathbf{u})\,, \tag{28}$$

where $\mathbf{x}_0 \in [\mathbf{x}_0]$ and $\mathbf{u}\,(\cdot) \in [\mathbf{u}]\,(\cdot)$. By setting $\mathbf{F}\,(\mathbf{x}, t) = \{\mathbf{f}\,(\mathbf{x}, \mathbf{u}) \mid \mathbf{u}\,(t) \in [\mathbf{u}]\,(t)\}$, we obtain that a differential inclusion can be described with this formalism. We assume that $\mathbf{f}$ is Lipschitz continuous with respect to $\mathbf{x}$. The Picard operator

$$\mathcal{T}_{\mathbf{x}_0,\mathbf{u}} : \mathbf{x}\,(\cdot) \to \mathbf{x}_0 + \int_0^t \mathbf{f}\,(\mathbf{x}\,(\tau)\,, \mathbf{u}\,(\tau))\,d\tau, \tag{29}$$

has uncertainty now. For all $\mathbf{x}_0$, and all $\mathbf{u}\,(\cdot)$, the operator $\mathcal{T}_{\mathbf{x}_0,\mathbf{u}}$ has a unique fixed point $\mathbf{x}^*\,(t)$. Consider a tube $\mathbb{X}\,(\cdot)$. If

$$\mathcal{T}_{\mathbf{x}_0,\mathbf{u}}\,(\mathbb{X}\,(\cdot)) \subset \mathbb{X}\,(\cdot) \tag{30}$$

then, from the Brouwer theorem, $\mathbb{X}\,(\cdot)$ contains at least one fixed point, *i.e.*, $\mathbf{x}^*\,(\cdot) \in \mathbb{X}\,(\cdot)$.

Methodology. For a guaranteed integration, we first have to find a potential tube for which we think that it contains the unique solution of the state equation or contain all solutions of the differential inclusion. This candidate could be obtained using an Euler integration method from $[\mathbf{x}_0]$ followed by an inflation. Then we compute a tube $\mathcal{T}^+\,([\mathbf{x}]\,(t))$ which encloses the tube

$$\mathcal{T}\,([\mathbf{x}]\,(t)) = [\mathbf{x}\,(0)] + \int_0^t \mathbf{f}\,([\mathbf{x}]\,(\tau)\,, \tau)\,d\tau, \tag{31}$$

[1] A trajectory $\mathbf{x}$, which is a function from $\mathbb{R}$ to $\mathbb{R}^n$, can be denoted equivalently $\mathbf{x}\,(t)$ or $\mathbf{x}\,(\cdot)$. When no ambiguity may exist, *i.e.*, when t is already used in the same paragraph, we shall often prefer $\mathbf{x}\,(t)$, for simplicity.

or the tube

$$\mathcal{T}([\mathbf{x}](t)) = [\mathbf{x}(0)] + \int_0^t \mathbf{F}([\mathbf{x}](\tau), \tau)\, d\tau, \tag{32}$$

in the case we have to deal with a differential inclusion. As illustrated in Figure 4, we compute

$$t_1 = \max_{t \geq 0} \left\{ t \mid \forall \tau \in [0, t], \mathcal{T}^+([\mathbf{x}](\tau)) \subset [\mathbf{x}](\tau) \right\}. \tag{33}$$

Within the interval $[0, t_1]$, from the Brouwer theorem, we conclude that the tube $\mathcal{T}^+([\mathbf{x}](\cdot))$ encloses the solution.

High order Taylor method. For a more efficient integration [20], we can replace the Picard-Lindelöf fixed point equation:

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \dot{\mathbf{x}}(\tau)\, d\tau \tag{34}$$

by the higher order fixed points Taylor equation with the integral remainder

$$\mathbf{x}(t) = \mathbf{x}_0 + \sum_{i=1}^{k} \frac{1}{i!} \left(\mathbf{x}^{(i)}(0) \right) t^i + \int_0^t \frac{\mathbf{x}^{(k+1)}(\tau)}{k!} (t - \tau)^k\, d\tau. \tag{35}$$

Note that for $k = 0$, we get the Picard-Lindelöf equation. This high order method is particularly suited to situations where $[\mathbf{x}_0]$ is known (or small). Indeed, when $\mathbf{x}_0$, is known, the fixed point Taylor operator becomes

$$\mathcal{T}([\mathbf{x}](t)) = \mathbf{x}_0 + \sum_{i=1}^{k} \frac{1}{i!} \left(\mathbf{x}^{(i)}(0) \right) t^i + \int_0^t \frac{[\mathbf{x}]^{(k+1)}(\tau)}{k!} (t - \tau)^k\, d\tau. \tag{36}$$

All uncertainties, stored inside $[\mathbf{x}]^{(k+1)}$, are divided by $k!$. Now, in practice, the width of $[\mathbf{x}]^{(k+1)}(\tau)$ increases polynomially with k, whereas $k!$ increases exponentially. Thus, the accuracy increases with k. The tube $[\mathbf{x}]^{(k+1)}(t)$ for $\mathbf{x}^{(k+1)}(t)$ is computed from the tube $[\mathbf{x}](t)$ using the expression of the state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$.

Remark. Consider the particular case where $k = 2$ and the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$. We have:

$$\ddot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{f}(\mathbf{x}, \mathbf{u}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \cdot \dot{\mathbf{u}} = \boldsymbol{\psi}^2(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}). \tag{37}$$

For a more general $k \geq 0$, we get:

$$\mathbf{x}^{(k+1)} = \boldsymbol{\psi}^{k+1} \left(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(k)} \right). \tag{38}$$

We have an analytical expression $\boldsymbol{\psi}^{k+1}\left(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(k)}\right)$, but this expression depends on $\dot{\mathbf{u}}, \ldots, \mathbf{u}^{(k)}$. Now, a tube for $\dot{\mathbf{u}}, \ldots, \mathbf{u}^{(k)}$ is not available in the case of differential inclusions. More precisely, $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$ can be cast into the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \mathbf{u} \in [\mathbf{u}]$ but nothing can be deduced on $\dot{\mathbf{u}}, \ddot{\mathbf{u}}$, *etc.* Thus, high order methods will have difficulties to deal with differential inclusions. To deal with uncertain dynamics using a k-order fixed point Taylor method, we need to be able to express the system in the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ with $\mathbf{u} \in [\mathbf{u}], \ldots, \mathbf{u}^{(k)} \in \left[\mathbf{u}^{(k)}\right]$.

4 Computing capture tubes

4.1 Basic idea

If a candidate $\mathbb{G}(t)$ for a capture tube is available, we can prove that $\mathbb{G}(t)$ is a capture tube by checking the inconsistency of a set of nonlinear equations (see the previous sections). This inconsistency can then easily be checked using interval analysis. Now, for many systems such as for nonholonomic systems, we rarely have a candidate for a capture tube and we need to find one. The main contribution of this paper is to provide a method that can help us to find such a capture tube. The idea is to start with a non-capture tube $\mathbb{G}(t)$ (the *candidate*) and to try to characterize the smallest capture tube which encloses $\mathbb{G}(t)$. To do this, we predict for all $(\mathbf{x}, t)$, which satisfy the cross-out conditions, a guaranteed envelope for the trajectory within finite time-horizon window $[t, t+t_2]$ (where $t_2 > 0$ is fixed). If all corresponding $\mathbf{x}(t+t_2)$ belong to $\mathbb{G}(t+t_2)$, then the union of all trajectories and the initial $\mathbb{G}(t)$ (in the (x,t) space) corresponds to the smallest capture tube enclosing $\mathbb{G}(t)$.

4.2 Lattice and capture tubes

First, let us remark that since the set of subsets of $\mathbb{R}^n$ is a lattice with respect to the inclusion $\subset$, the set of tubes $(\mathbb{T}, \subset)$ is also a lattice. When we introduced the basic idea of how we could compute a capture tube, we wrote that we wanted to compute the smallest tube, which encloses the candidate $\mathbb{G}(t)$. This notion of the smallest tube makes sense because of the following theorem.

Theorem 2. Consider a state space system $\mathcal{S}_\mathbf{f} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ or a differential inclusion $\mathcal{S}_\mathbf{F} : \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$. The set of capture tubes $(\mathbb{T}_c, \subset)$ for $\mathcal{S}_\mathbf{f}$ or $\mathcal{S}_\mathbf{F}$ is a sublattice of the set of tubes $(\mathbb{T}, \subset)$.

Proof. Consider two captures tubes $\mathbb{G}_1(t)$ and $\mathbb{G}_2(t)$. If the trajectory $\mathbf{x}(t)$ belongs to both $\mathbb{G}_1(t)$ and $\mathbb{G}_2(t)$, then $\mathbf{x}(t)$ will leave neither $\mathbb{G}_1(t)$ nor $\mathbb{G}_2(t)$. Thus, the intersection $\mathbb{G}_1(t) \cap \mathbb{G}_2(t)$ is a capture tube. The same reasoning can be done for the union of the two tubes. Since $\mathbb{G}_1(t) \cap \mathbb{G}_2(t)$ is the largest tube included in $\mathbb{G}_1(t)$ and $\mathbb{G}_2(t)$ and since $\mathbb{G}_1(t) \cup \mathbb{G}_2(t)$ is the smallest tube which contains $\mathbb{G}_1(t)$ and $\mathbb{G}_2(t)$, we conclude that $(\mathbb{T}_c, \subset)$ is a lattice. Since all capture tubes are also tubes, we get that $(\mathbb{T}_c, \subset)$ is a sublattice of $(\mathbb{T}, \subset)$. ■

Consequences. Since $\mathbb{T}_c$ is a sublattice of $\mathbb{T}$, for any tube $\mathbb{G}(t) \in \mathbb{T}$, we can define the following operator:

$$\operatorname{capt}(\mathbb{G}(t)) = \bigcap \left\{ \overline{\mathbb{G}}(t) \in \mathbb{T}_c \mid \mathbb{G}(t) \subset \overline{\mathbb{G}}(t) \right\}. \tag{39}$$

This set corresponds to the smallest capture tube which encloses $\mathbb{G}(t)$.

Interval of tubes. The set of tubes is a lattice with respect to the inclusion $\subset$. Thus, we can define *intervals of tubes*. This notion is important in this paper, because we need to compute a tube, in a guaranteed way. Now, this tube may probably not be representable in the computer. This new notion of interval of tubes will be needed in order to characterize the tube we want to calculate.

4.3 Computing capture tubes

Since the set of tubes $(\mathbb{T}, \subset)$ is a lattice, we can define intervals of tubes as follows.

Definition. An interval of tubes $[\mathbb{G}]$ is a subset of the set of tubes $\mathbb{T}$ which satisfies

$$[\mathbb{G}] = \{\mathbb{G} \in \mathbb{T} \mid \mathbb{G} \subset \vee [\mathbb{G}] \text{ and } \mathbb{G} \supset \wedge [\mathbb{G}]\}. \tag{40}$$

Here, $\mathbb{G}^+ = \vee [\mathbb{G}]$ denotes the smallest outer bound of $[\mathbb{G}]$ and $\mathbb{G}^- = \wedge [\mathbb{G}]$ denotes the largest inner bound of $[\mathbb{G}]$. The set of intervals of tubes will be denoted by $\mathbb{IT}$. Note that we could also define the notion of interval of capture tubes, but this notion is not interesting in our context since it is very difficult to get (exactly) even one capture tube.

Problem to be solved. Given a tube $\mathbb{G}(\cdot) : t \mapsto \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$ in $\mathbb{T}$, compute an interval $[\mathbb{C}^-(t), \mathbb{C}^+(t)] \in \mathbb{IT}$ such that

$$\text{capt}(\mathbb{G}(t)) \in \left[\mathbb{C}^-(t), \mathbb{C}^+(t)\right]. \tag{41}$$

This is illustrated in Figure 5. Of course, since $\mathbb{G}(t) \subset \text{capt}(\mathbb{G}(t))$, we can take

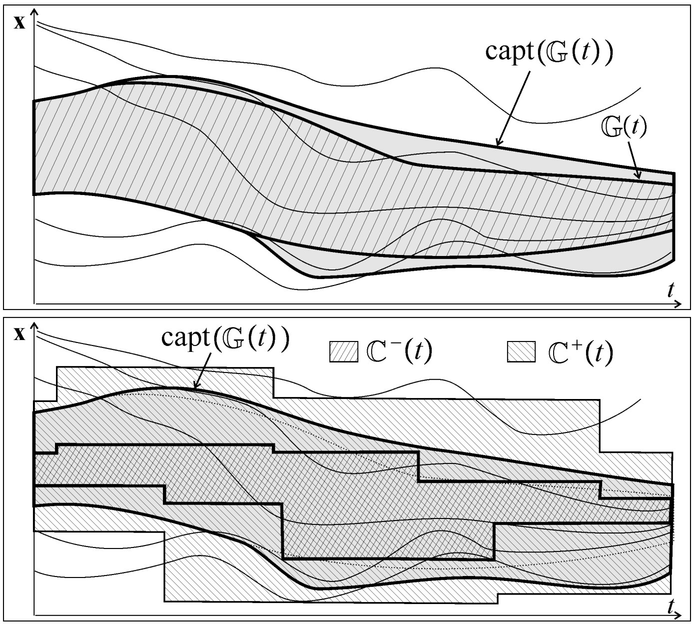

Fig. 5. The capture tube $\text{capt}(\mathbb{G}(t))$, that we want to compute, will be enclosed by an interval of tubes $\left[\mathbb{C}^-(t), \mathbb{C}^+(t)\right]$

$\mathbb{C}^-(t) = \mathbb{G}(t)$. Thus, the main difficulty is to get a tube $\mathbb{C}^+(t)$, which is not too large.

Flow. The flow associated with the system $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is a function $\boldsymbol{\phi}_{t_0,t_1} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \Rightarrow \boldsymbol{\phi}_{t_0,t_1}(\mathbf{x}(t_0)) = \mathbf{x}(t_1). \tag{42}$$

This means that if the trajectory $\mathbf{x}(t)$ is a solution of $\mathcal{S}_{\mathbf{f}}$, we are able to go from the state at instant t_0 to the state at instant t_1 using the flow.

The flow associated with the differential inclusion $\mathcal{S}_{\mathbf{F}} : \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$ is a function $\boldsymbol{\phi}_{t_0,t_1} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$,

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t) \Rightarrow \mathbf{x}(t_1) \in \boldsymbol{\phi}_{t_0,t_1}(\mathbf{x}(t_0)). \tag{43}$$

$\boldsymbol{\phi}_{t_0,t_1}$ should also be the smallest with respect to the inclusion which satisfies this property. Equivalently, $\boldsymbol{\phi}_{t_0,t_1}(\mathbf{x}(t_0))$ corresponds to the set of all states that can be reached at instant $t_1 \geq t_0$ by a trajectory consistent with $\mathcal{S}_{\mathbf{F}}$ and initialized at $\mathbf{x}(t_0)$ for $t = t_0$.

Theorem 3a. Consider the system $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$. The tube

$$\mathbb{C}(\cdot) : t \to \{\mathbf{x} \mid \exists (\mathbf{x}_0, t_0),\ \mathbf{x}_0 \in \mathbb{G}(t_0),\ t \geq t_0,\ \mathbf{x} = \boldsymbol{\phi}_{t_0,t}(\mathbf{x}_0)\}, \tag{44}$$

where $\boldsymbol{\phi}_{t_0,t}$ is the flow function of $\mathcal{S}_{\mathbf{f}}$, corresponds to capt($\mathbb{G}(t)$).

Proof of Theorem 3a. We will show that $\mathbb{C}(t)$, is the smallest capture tube which encloses $\mathbb{G}(t)$. For the proof, we will prove *(i)* that $\mathbb{C}(t)$ contains $\mathbb{G}(t)$, *(ii)* that $\mathbb{C}(t)$ is a capture tube and *(iii)* that $\mathbb{C}(t)$ is the smallest one.

(i) To prove that $\mathbb{G}(t) \subset \mathbb{C}(t)$, it suffices to take $t_0 = t$ and $\mathbf{x}_0 = \mathbf{x}$.

(ii) We now prove that $\mathbb{C}(t)$ is a capture tube. Take a pair $(\mathbf{x}^{t_a}, t_a)$ such that $\mathbf{x}^{t_a} \in \mathbb{C}(t_a)$. From (44), we have

$$\exists (\mathbf{x}_0, t_0),\ \mathbf{x}_0 \in \mathbb{G}(t_0),\ t_a \geq t_0,\ \mathbf{x}^{t_a} = \boldsymbol{\phi}_{t_0,t_a}(\mathbf{x}_0). \tag{45}$$

Take $\tau > 0$ and define the point $\mathbf{x}^{t_a+\tau} = \boldsymbol{\phi}_{t_a,t_a+\tau}(\mathbf{x}^{t_a})$. From (45), we have

$$\exists (\mathbf{x}_0, t_0),\ \mathbf{x}_0 \in \mathbb{G}(t_0),\ t_a \geq t_0,\ \mathbf{x}^{t_a+\tau} = \boldsymbol{\phi}_{t_0,t_a+\tau}(\mathbf{x}_0). \tag{46}$$

Therefore, we have proved that

$$\mathbf{x}^{t_a} \in \mathbb{C}(t_a), \tau \geq 0 \Rightarrow \boldsymbol{\phi}_{t_a,t_a+\tau}\left(\mathbf{x}^{t_a}\right) \in \mathbb{C}(t_a + \tau), \tag{47}$$

i.e., $\mathbb{C}(t)$ is a capture tube.

(iii) We will now prove by contradiction that $\mathbb{C}(t)$ is the smallest capture tube that encloses $\mathbb{G}(t)$. Take a capture tube $\overline{\mathbb{G}}(t)$ such that $\overline{\mathbb{G}}(t) \supset \mathbb{G}(t)$ which is enclosed strictly in $\mathbb{C}(t)$. By strictly, we mean that $\exists (t_1, \mathbf{x}_1)$, $\mathbf{x}_1 \in \mathbb{C}(t_1)$ and $\mathbf{x}_1 \notin \overline{\mathbb{G}}(t_1)$. From (44), $\exists (\mathbf{x}_0, t_0), \mathbf{x}_0 \in \mathbb{G}(t_0),\ \mathbf{x}_1 = \boldsymbol{\phi}_{t_0,t_1}(\mathbf{x}_0)$. The corresponding trajectory crosses the tube $\overline{\mathbb{G}}(t)$ from inside to outside which is inconsistent with the fact that $\overline{\mathbb{G}}(t)$ is a capture tube. ■

Theorem 3b. Consider the system $\mathcal{S}_{\mathbf{F}} : \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$. The tube

$$\mathbb{C}(t) : t \to \{\mathbf{x} \ \mid \exists (\mathbf{x}_0, t_0), \mathbf{x}_0 \in \mathbb{G}(t_0), t \geq t_0,\ \mathbf{x} \in \boldsymbol{\phi}_{t_0,t}(\mathbf{x}_0)\}, \tag{48}$$

where $\boldsymbol{\phi}_{t_0,t}$ is the set membership flow function of $\mathcal{S}_{\mathbf{F}}$, corresponds to capt$(\mathbb{G}(t))$.

Proof. The proof is a direct consequence of Theorem 3a. ■

Theorem 4a. Consider the system $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$. We have

$$\text{capt}(\mathbb{G}(t)) = \mathbb{G}(t) \cup \Delta\mathbb{G}(t), \tag{49}$$

with

$$\Delta\mathbb{G}(t) = t \mapsto \{\mathbf{x} \mid \exists (\mathbf{x}_0, t_0) \text{ satisfying (5)}, \; t \geq t_0, \; \mathbf{x} = \boldsymbol{\phi}_{t_0,t}(\mathbf{x}_0) \text{ and } \mathbf{x} \notin \mathbb{G}(t)\}. \tag{50}$$

Proof. To build capt$(\mathbb{G}(t))$, it suffices to add to the tube $\mathbb{G}(t)$ all pairs $(\mathbf{x}_1, t_1)$ outside $\mathbb{G}(t)$ that can be reached from a pair $(\mathbf{x}_a, t_a)$ in $\mathbb{G}(t)$. The corresponding trajectory will cross the boundary of the tube $\mathbb{G}(t)$ at instant t_0 at the state $\mathbf{x}_0$, *i.e.*, $(\mathbf{x}_0, t_0)$ satisfies (5). This is illustrated in Figure 6. ■

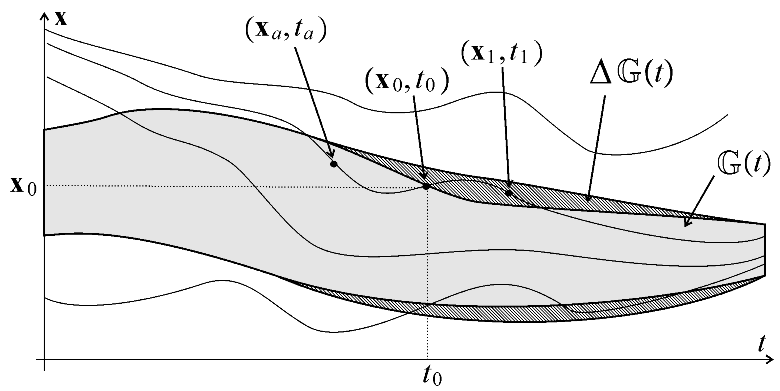

Fig. 6. $\Delta\mathbb{G}(t)$ contains all pairs $(\mathbf{x}_1, t_1)$ outside $\mathbb{G}(t)$ that can be reached from a pair $(\mathbf{x}_0, t_0)$ leaving $\mathbb{G}(t)$

Theorem 4b. Consider the differential inclusion $\mathcal{S}_{\mathbf{F}} : \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$. We have

$$\text{capt}(\mathbb{G}(t)) = \mathbb{G}(t) \cup \Delta\mathbb{G}(t), \tag{51}$$

with

$$\Delta\mathbb{G}(t) = t \mapsto \{\mathbf{x} \mid \exists (\mathbf{x}_0, t_0) \text{ satisfying (10)}, \; t \geq t_0, \; \mathbf{x} \in \boldsymbol{\phi}_{t_0,t}(\mathbf{x}_0) \text{ and } \mathbf{x} \notin \mathbb{G}(t)\}. \tag{52}$$

Consequences. An interval $[\mathbb{C}^-(t), \mathbb{C}^+(t)]$ for capt$(\mathbb{G}(t))$ will be composed by the tube $\mathbb{C}^-(t) = \mathbb{G}(t)$ and by adding to $\mathbb{C}^-(t)$ an enclosure of all trajectories generated from one pair $(\mathbf{x}_0, t_0)$ satisfying (5) or (10).

5 Test case

Consider the pendulum presented in Section 2. Here, we do not consider the sub-level sets of the energy anymore, which only applies on a small class of systems. Instead, we consider, as an candidate tube, the one associated with the function

$$g(\mathbf{x}, t) = x_1^2 + x_2^2 - 1.$$

We have chosen here a time-invariant tube in order to be able to draw pictures. Indeed, both $\mathbb{G}(t)$ and $\Delta\mathbb{G}(t)$ do not depend on t and become subsets of $\mathbb{R}^2$. Our algorithm provides the results shown in Figure 7. Subfigure (a) depicts a subpaving which encloses all points satisfying the cross-out conditions. The guaranteed integration $\Delta\mathbb{G}$ of all these boxes are shown on Subfigure (b). The integration has been performed using the DYNIBEX library[8]. Subfigure (c) represents a subpaving made with boxes shown to be inside $\mathbb{G}$. Since $\mathbb{G} \subset \text{capt}(\mathbb{G})$, this subpaving also corresponds to an inner approximation $\mathbb{C}^-$ of $\text{capt}(\mathbb{G})$. Subfigure (d) shows $\mathbb{C}^+$ which is the union of light gray boxes (back plane) and dark gray boxes (front plane). This union forms an outer approximation of $\text{capt}(\mathbb{G})$.

6 Conclusion

Proving that a controlled nonlinear system always stays inside a time moving bubble (or tube) amounts to proving a set of nonlinear inequalities. Now, in practice, even with a good intuition, finding such a significant capture tube is difficult. This paper proposes a new method for computing an approximation of the smallest tube, which encloses a candidate tube $\mathbb{G}(t)$. Even if $\mathbb{G}(t)$ is generally chosen as rather attractive, it is often possible to cross $\mathbb{G}(t)$ from inside to outside during the initialization of the system. Since this tube may not be representable in the computer, the method calculates an interval of tubes which encloses the capture tube we want to compute. The principle of the approach is to integrate (with a guaranteed interval integration) the state vectors that cross the candidate tube from inside to outside and to add all the corresponding trajectories to the candidate tube. Now, since the less we integrate, the more we are efficient, to deal with large scale systems, it should be necessary to limit the number of integration by giving more importance to the Lyapunov part of the resolution. This could be done, for instance, by computing barrier functions [6].

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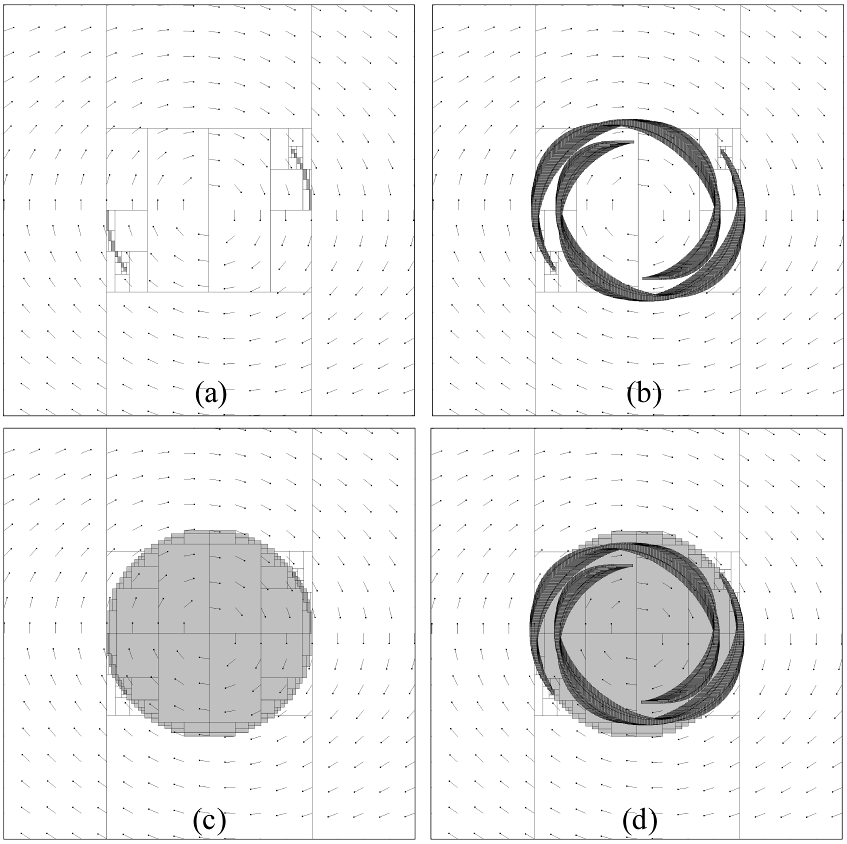

Fig. 7. (a) boxes which enclose the points satisfying the cross-out conditions; (b) guaranteed integration $\Delta\mathbb{G}$ of these boxes; (c) inner approximation $\mathbb{C}^-$ of $\mathrm{Capt}(\mathbb{G})$; (d) outer approximation $\mathbb{C}^+$ of $\mathrm{Capt}(\mathbb{G})$

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