An interval approach for solving equations involving complex numbers

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Abstract. The idea of interval arithmetic, proposed by Moore, is to enclose the exact value of a real number inside an interval. Then, computing with intervals will allow us to enclose the true value for a variable we want to compute. This paper emphases the importance of having a lattice structure for the set of intervals and shows that several interval algorithms could be adapted to other types of domains as soon as these domains have a lattice structure with respect to the inclusion and that we could bisect them. As an illustration, we introduce a new type of domains, the *boxpies*, which correspond to the intersection between one box and one pie. We show that *boxpies* can be used efficiently to characterize the solution set of constraints involving complex numbers.

1 Introduction

This paper proposes to use interval analysis and contractor programming with the objective to solve equations involving complex numbers. Our approach is to use concepts of interval analysis developed by Moore [19], but to adapt and extend these concepts in order to be more efficient. The main idea is to take advantage of the dual representation of complex numbers (Cartesian or polar form) and use to two different type of domains to enclose the solutions: Cartesian intervals (or boxes) and polar intervals (or pies) [3].

The paper is organized as follows. Section 2 defines what is an interval of a set which is a metric lattice (such as $\mathbb{R}$ or $\mathbb{R}^n$) and Section 3 shows how the concept of *interval* can generalized to deal with the case where the variable to be enclosed does not belong to a lattice. As an illustration, Section 4 considers the set of angles for which no order relation exists. It shows that what is important is not that the variables take values inside a lattice, but that the domains used to enclose them belong to a lattice with respect to the set inclusion. Section 5 introduces the notion of *pie* which is an illustration of how vectors of variables with no order relation (such as angles) can be enclosed. Section 6 shows how different types of domains can be merged into a single type. This is illustrated by introducing the new notion of *boxpie*, which is the intersection between one box and one pie. Boxpies are particularly suited to deal with polynomial constraint involving complex numbers. Section 7 recalls the definition of a contractor in the general framework that as been presented in the previous sections. Section 8 provides an illustrative example related to robot localization which is formalized with polynomial equations involving complex variables. In this example, the solution set is approximated by an inner and an outer subpavings made with boxpies. A conclusion is given in Section 9.

2 Intervals

Interval methods, introduced by Moore [18] during his Ph.D. thesis, can applied as soon as the set of domains for the variables has a lattice structure [8] as shown in [21]. A *lattice* $(\mathcal{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds [8]. The least upper bound of x and y is called the *join* and is denoted by $x \vee y$. The greatest lower bound is called the *meet* and is written as $x \wedge y$.

Example 1. The set $(\mathbb{R}^n, \leq)$ is a lattice with respect to the partial order relation given by $\mathbf{x} \leq \mathbf{y} \Leftrightarrow \forall i \in \{1, \dots, n\}, x_i \leq y_i$. We have $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$ and $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, \dots, x_n \vee y_n)$ where $x_i \wedge y_i = \min(x_i, y_i)$ and $x_i \vee y_i = \max(x_i, y_i)$.

Example 2. The set $(\mathbb{F}, \leq)$ of the functions which map $\mathbb{R}$ to $\mathbb{R}$ is a lattice with respect to the partial order relation given by $f \leq g \Leftrightarrow \forall t \in \mathbb{R}, f(t) \leq g(t)$. We have $f \wedge g : t \mapsto \min\{f(t), g(t)\}$ and $f \vee g : t \mapsto \max\{f(t), g(t)\}$

Example 3. The set $\mathbb{IR}$ of closed intervals, as introduced by Moore [19], is a complete lattice with respect to the inclusion $\subset$. The meet corresponds to the intersection and the join corresponds to the interval hull. For instance

$$[1,4] \wedge [2,\infty] = [2,4] \text{ and } [1,4] \vee [8,9] = [1,9]. \tag{1}$$

A lattice $\mathcal{E}$ is *complete* if for all (finite or infinite) subsets $\mathcal{A}$ of $\mathcal{E}$, the least upper bound $\wedge\mathcal{A}$ and the greatest lower bound $\vee\mathcal{A}$ belong to $\mathcal{E}$. When a lattice $\mathcal{E}$ is not complete, it is often possible to add two elements corresponding to $\wedge\mathcal{A}$ and $\vee\mathcal{A}$ to make it complete. For instance, the set $\mathbb{R}$ is not a complete lattice whereas $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is. As a consequence, we have $\wedge\emptyset = \vee\mathcal{E}$ and $\vee\emptyset = \wedge\mathcal{E}$. The Cartesian product of two lattices

$(\mathcal{E}_1, \leq_1)$ and $(\mathcal{E}_2, \leq_2)$ is the lattice $(\mathcal{E}, \leq)$ defined as the set of all $(a_1, a_2) \in \mathcal{E}_1 \times \mathcal{E}_2$ with the order relation $(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow ((a_1 \leq_1 b_1) \text{ and } (a_2 \leq_2 b_2))$.

Intervals. A *closed interval* (or *interval* for short) $[x]$ of a complete lattice $\mathcal{E}$ is a subset of $\mathcal{E}$ which satisfies $[x] = \{x \in \mathcal{E} \mid \wedge [x] \leq x \leq \vee [x]\}$. Both $\emptyset$ and $\mathcal{E}$ are intervals of $\mathcal{E}$. An interval is a sublattice of $\mathcal{E}$. If we denote by $\mathbb{I}\mathcal{E}$ the set of all intervals of a complete lattice $(\mathcal{E}, \leq)$ then $(\mathbb{I}\mathcal{E}, \subset)$ is also a lattice. For two elements $[x] = [x^-, x^+]$ and $[y] = [y^-, y^+]$ of $\mathbb{I}\mathcal{E}$, we have:

$$\begin{array}{rcl} [x] \wedge [y] & = & [x^- \vee y^-, x^+ \wedge y^+] \\ [x] \vee [y] & = & [x^- \wedge y^-, x^+ \vee y^+]. \end{array} \tag{2}$$

The meet $[x] \wedge [y]$ is called the *intersection* and will denoted by $[x] \cap [y]$. The join $[x] \vee [y]$ is called the *interval union* and will be denoted by $[x] \sqcup [y]$.

Remark. In his book, Moore [19] has considered intervals that are derived from the lattices $(\mathbb{R}^n, \leq)$. When $n > 1$, these intervals are named interval vectors. Moore also considered tubes, *i.e.*, interval in the lattice of functions $(\mathbb{F}, \leq)$.

Width. The width function w associates to an interval $[x]$ a positive number. The width should satisfy the following properties

$$\begin{array}{lll} \text{(i)} & [x] \subset [y] \Rightarrow w([x]) \leq w([y]) & \text{(monotonicity)} \\ \text{(ii)} & [x](k) \rightarrow a \Rightarrow w([x](k)) \rightarrow 0 & \text{(convergence)} \end{array} \tag{3}$$

The second property tells us that is a sequence of intervals $[x](k)$ converge to a point a (*i.e.*, a degenerated interval a which is a singleton) then the corresponding width converges to 0. This property requires that the sequence $[x](k)$ are intervals of a lattice $\mathcal{E}$ which is also a metric space. Moore defined the width of an interval of $\mathbb{R}$ as:

$$w\left(\left[x^-, x^+\right]\right) = x^+ - x^-, \tag{4}$$

which is consistent with this property.

Cartesian product. The Cartesian product $(\mathcal{E}, \leq)$ of two lattices $(\mathcal{E}_1, \leq_1)$ and $(\mathcal{E}_2, \leq_2)$ is also a lattice. The intervals of $\mathcal{E}$ are made with the Cartesian product of the intervals of $\mathcal{E}_1$ and $\mathcal{E}_2$, *i.e.*, an interval $[x]$ of $\mathcal{E}$ can be written as

$$[x] = [x_1] \times [x_2] \text{ where } [x_1] \in \mathbb{I}\mathcal{E}_1 \text{ and } [x_2] \in \mathbb{I}\mathcal{E}_2. \tag{5}$$

Moreover, the width w in $\mathcal{E}$ can be derived from the width w_1 and w_2 in $\mathcal{E}_1$ and $\mathcal{E}_2$ as follows:

$$w([x_1] \times [x_2]) = \max(w_1([x_1]), w_2([x_2])). \tag{6}$$

The definition of the width of intervals of $\mathbb{R}^n$ provided by Moore is consistent with this definition.

Bisections. A *bisector* [5] is an operator that takes an interval $[x]$ as an input and which returns two intervals $[a]$ and $[b]$ such that *(i)* $[a]$ and $[b]$ do not overlap; *(ii)* $[x] = [a] \cup [b]$ and *(iii)* $\max(w([a]), w([b]))$ is minimal. This is the choice made by several optimization algorithms [16] [12] such as the Moore-Skelboe algorithm [24].

3 Bisectable Abstract Domains

All interval methods initiated by Moore as well as contractor-based tools can easily be generalized in the case where the unknown variable do not belong to a lattice. What is important [6] [11] is that the domains that are handled forms a lattice [8] with respect to the inclusion $\subset$. More precisely, consider a Riemannian manifold $\mathbb{M}$ (such a $\mathbb{R}$, $\mathbb{R}^n$ or a sphere). Since $\mathbb{M}$ is Riemannian, we can define the distance $d(a, b)$ between two points a and b as the minimal length than can be reached by any path connecting a to b. For any subset $\mathbb{X} \subset \mathbb{M}$, we can define the *diameter* (or *width*) $w(\mathbb{X})$ of $\mathbb{X}$ as the maximal distance $d(a, b)$ that exists between two points a and $b \in \mathbb{X}$. Denote by $\mathcal{P}(\mathbb{M})$ the powerset of $\mathbb{M}$. We define a family of *bisectable abstract domains* (*bad* for short) $\mathbb{IM}$ as a subset of $\mathcal{P}(\mathbb{M})$ which satisfies the following properties.

- $\mathbb{IM}$ is a Moore family[1]. This means that the intersection (not necessary finite) is closed in $\mathbb{IM}$, *i.e.,*

$$[a](1) \in \mathbb{IM}, [a](2) \in \mathbb{IM}, \ldots \Rightarrow \bigcap_i [a](i) \in \mathbb{IM} \tag{7}$$

  From this property, we can deduce that $(\mathbb{IM}, \subset)$ is a lattice. But this lattice is not necessary a sublattice of $\mathcal{P}(\mathbb{M})$. Indeed, even if the meet operator $\cap$ is preserved, the join operator in $\mathbb{IM}$ (denoted by $\sqcup$) is different from that in $\mathcal{P}(\mathbb{M})$ (denoted by $\cup$). More precisely, instead of an equality, we have the inclusion:

$$\underbrace{[a] \cup [b]}_{\in \mathcal{P}(\mathbb{M})} \subset \underbrace{[a] \sqcup [b]}_{\in \mathbb{IM}}. \tag{8}$$

[1]Note that the Moore who gave the name to the Moore family is not the R. Moore who builded the theory of Interval Analysis, but Eliakim Hastings Moore (1862-1932) who studied closure operators.

- $\mathbb{IM}$ is equipped with a *bisector, i.e.*, a function $\beta : \mathbb{IM} \to \mathbb{IM} \times \mathbb{IM}$, such that $\beta([x]) = \{[a], [b]\}$ with the following properties: *(i)* $[a]$ and $[b]$ do not overlap, *(ii)* $[a]$ and $[b]$ cover $[x]$ and no other bisection consistent with *(i)* and *(ii)* generates a lower value for $\max\{w([a]), w([b])\}$.

Cartesian product. Let $(\mathbb{IM}_1, \beta_1)$ and $(\mathbb{IM}_2, \beta_2)$ be two bads associated with the manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$. A bad associated with the Cartesian product $\mathbb{M} = \mathbb{M}_1 \times \mathbb{M}_2$ is $(\mathbb{IM}, \beta)$ where

$$\begin{array}{lll} \mathbb{IM} & = & \mathbb{IM}_1 \times \mathbb{IM}_2 \\ \beta([m_1] \times [m_2]) & = & \left\{ \begin{array}{ll} \beta_1([m_1]) \times [m_2] & \text{if } w_1([m_1]) \geq w_2([m_2]) \\ [m_1] \times \beta_2([m_2]) & \text{otherwise} \end{array} \right. \end{array} \tag{9}$$

This defines what we call the *Cartesian product between two bads.* It is useful to enclose vectors of variables. Note that as defined by Moore, a box $[\mathbf{x}] = [x_1] \times [x_2]$ of $\mathbb{R}^2$ is a Cartesian product of two intervals of $\mathbb{R}$ which is a bad. A bisection of $[\mathbf{x}]$ can be defined as in (9), from the bisection of its interval components $[x_1]$ and $[x_2]$.

Reduced product [7]. let $(\mathbb{IM}_1, \beta_1)$ and $(\mathbb{IM}_2, \beta_2)$ be two bads associated with the same manifold $\mathbb{M}$. We define the *reduced product* $(\mathbb{IM}, \beta) = (\mathbb{IM}_1, \beta_1) \otimes (\mathbb{IM}_2, \beta_2)$ as follows

$$\begin{array}{l} \mathbb{IM} = \{[m_1] \cap [m_2] \text{ such that } [m_1] \in \mathbb{IM}_1 \text{ and } [m_2] \in \mathbb{IM}_2\} \\ \beta([m_1] \cap [m_2]) = \left\{ \begin{array}{ll} \beta_1([m_1]) \cap [m_2] & \text{if } w([m_1]) \geq w([m_2]) \\ [m_1] \cap \beta_2([m_2]) & \text{otherwise.} \end{array} \right. \end{array} \tag{10}$$

Note that the intersection is closed in $\mathbb{IM}$. Indeed, if $[a_1] \cap [a_2] \in \mathbb{IM}$ and $[b_1] \cap [b_2] \in \mathbb{IM}$, we have

$$[a_1] \cap [a_2] \cap [b_1] \cap [b_2] = \underbrace{[a_1] \cap [b_1]}_{\in \mathbb{IM}_1} \cap \underbrace{[a_2] \cap [b_2]}_{\in \mathbb{IM}_2}. \tag{11}$$

The idea of the reduced product, which is not well known by the interval community, is classically used in the community of abstract interpretation [7] where different types of domains are combined during the resolution. This is the case of octagons [17] which corresponds to the intersection of a box with rotated boxes.

4 Angles and arcs

The notion of *bad* will now be illustrated in the case of angles which has not a lattice structure. Consider the equivalence relation on $\mathbb{R}$

$$\alpha \sim \beta \Leftrightarrow \frac{\beta - \alpha}{2\pi} \in \mathbb{Z}. \tag{12}$$

The set $\mathbb{A}$ of all angles corresponds to the quotient

$$\mathbb{A} = \frac{\mathbb{R}}{\sim} = \frac{\mathbb{R}}{2\pi\mathbb{N}}. \tag{13}$$

For simplicity, we will also write $\mathbb{A} = [-\pi, \pi[$. Note that the set $\mathbb{A}$ is a Riemannian manifold. Moreover, if α and β are angles and if $\rho \in \mathbb{R}$, we can define the operations $\alpha+\beta$, $\alpha-\beta$ and $\rho\cdot\alpha$. Due to its circular structure, the set of angles $\mathbb{A}$ is not a lattice and it is thus not possible to define intervals of angles in order to apply interval techniques [9]. Define an *arc* as a pair $\langle\alpha\rangle = \langle\overline{\alpha}, \widetilde{\alpha}\rangle$ such that $\overline{\alpha} \in \mathbb{A}$ and $\widetilde{\alpha} \in [0, \pi]$, where $\overline{\alpha}$ is called the *center* and $\widetilde{\alpha}$ is the *radius*. The set of all arcs is denoted by $\mathbb{IA}$. Note that the intersection in $\mathbb{IA}$ is not closed and thus $\mathbb{IA}$ is not a Moore family. To apply an interval approach on angles, it is thus necessary to take as a domains of angles: unions of arcs, which corresponds to the smallest Moore family which contains $\mathbb{IA}$. A union of non overlapping arcs is called a *circular paving*. The set of circular pavings is denoted by $\mathbb{UA}$ and $(\mathbb{UA}, \subset)$. Figure 1 illustrates the intersection and the union of circular pavings.

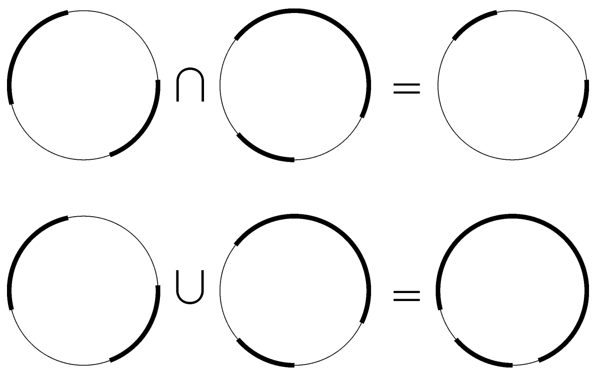


Figure 1: Intersection and union of two circular pavings

5 Pies

In the previous section, we were able to define a family of domains (the circular paving) for angles which is a bad. Since the Cartesian product of bads is a bad, we can thus easily define a bad associated to a finite set of variables. This is what it is done when Moore has defined boxes of $\mathbb{R}^n$ as Cartesian products of intervals. We now illustrate that this by considering an angle variable α and a scalar variable $\rho > 0$. If α belongs to the circular paving $< \alpha >$ and ρ belongs to the scalar interval $[\rho]$ then the pair (α, ρ) belongs

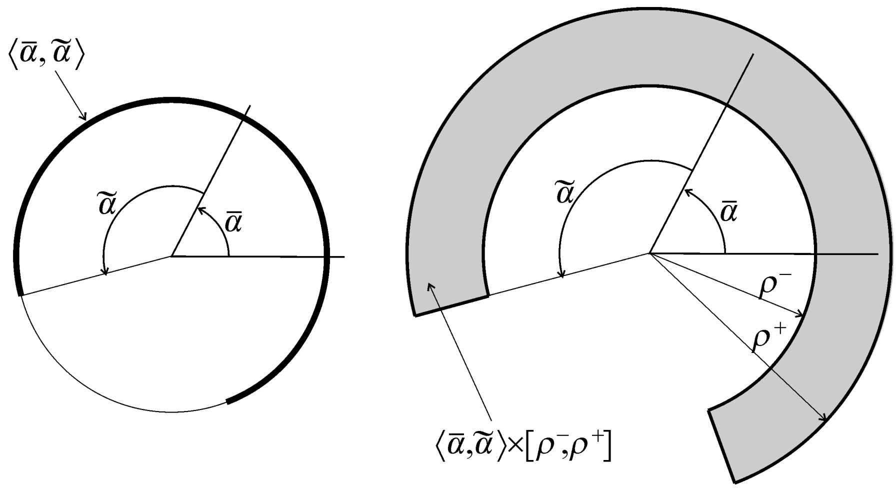


Figure 2: Left: an arc; Right: a pie

to $< \alpha > \times [\rho]$ which is called a *pie*. More formally, a *pie* is an element of $\mathbb{UA} \times \mathbb{IR}$. A pie can also be interpreted as a subset of $\mathbb{R}^2$ as illustrated by Figure 2 right which shows a pie $\langle\overline{\alpha},\widetilde{\alpha}\rangle \times [\rho^-,\rho^+]$ with a single connected component. A pie will often be denoted in its polar form as $[\rho]\, e^{i<\alpha>}$. Note that, due to the fact that the set of pies is bad, the intersection between pies is always a pie. Indeed, if $[\rho_1]\, e^{i<\theta_1>}$ and $[\rho_2]\, e^{i<\theta_2>}$ are two pies, we have

$$[\rho_1]\, e^{i<\theta_1>} \cap [\rho_2]\, e^{i<\theta_2>} = ([\rho_1] \cap [\rho_2])\, e^{i(<\theta_1>\cap<\theta_2>)}. \tag{14}$$

6 Boxpies

Consider the set $\mathbb{C}$ of complex numbers. Two bads could be considered: boxes of $\mathbb{C}$ of the form $[x] + i\,[y]$ and pies of $\mathbb{C}$ of the form $[\rho]\, e^{i<\theta>}$. Both $\mathbb{IC}$ (the boxes) and $\mathbb{UA} \times \mathbb{IR}$ (the pies) are Moore families in $\mathcal{P}(\mathbb{C})$. The union $\mathbb{IC}$ with $\mathbb{UA} \times \mathbb{IR}$ is not a Moore family anymore, but we can define the smallest Moore family $\mathbb{BP}$ of $\mathcal{P}(\mathbb{C})$ which contains both $\mathbb{IC}$ and $\mathbb{UA} \times \mathbb{IR}$. This corresponds to the reduced product $\otimes$ [7] presented in Section 3. Therefore, we can write $\mathbb{BP} = \mathbb{IC} \otimes\ \mathbb{UA} \times \mathbb{IR}$. The family $\mathbb{BP}$ contains boxes and pies. But it also contains all intersections between one box and one pie. An element of $\mathbb{BP}$ is called a *boxpie*. A boxpie can thus be written as

$$[x] + i\,[y] \cap [\rho]\, e^{i<\theta>}. \tag{15}$$

Note that the intersection between two boxpies is also a boxpie. Indeed:

$$\begin{array}{rl} & [x_1] + i\,[y_1] \cap [\rho_1]\, e^{i<\theta_1>} \cap [x_2] + i\,[y_2] \cap [\rho_2]\, e^{i<\theta_2>} \\ = & [x_1] \cap [x_2] + i\,([y_1] \cap [y_2]) \cap ([\rho_1] \cap [\rho_2])\, e^{i(<\theta_1> \cap <\theta_2>)}. \end{array} \tag{16}$$

An arithmetic on boxpies heritates from the good properties of interval arithmetic for the addition, but also of good properties of pie arithmetic [23] for the multiplication.

Selfconsistency. The expression for a boxpie may not be unique. For instance, the boxpie

$$[0,1] + i\,[1,2] \cap [1,2] \cdot e^{i[0,\frac{\pi}{4}]} = [1,1] + i\,[1,1] \cap [\sqrt{2},\sqrt{2}] e^{i\left[\frac{\pi}{4},\frac{\pi}{4}\right]} \tag{17}$$

is a singleton which contains as a single element the complex number $1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$. The representation which is minimal with respect to the inclusion of intervals is said to be *selfconsistent*.

7 Contractors

Many problems of estimation, control or robotics can be represented by *constraint networks* [14]. A constraint networks (see, *e.g.*, [26][27]) is composed of a set of variables $\{x_1, \ldots, x_n\}$, a set of constraints $\{c_1, \ldots, c_m\}$ and a set of domains $\{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$. The domains $\mathbb{X}_i$ should belong to a complete lattice $(\mathcal{L}_i, \subset)$. In the interval literature derived from Moore works, the domains for the variables of a constraint networks are intervals. It is not the case, when dealing with finite domains. The interval nature is not needed as soon as the set of domains has a structure of lattice. In the context of this paper, the sets $\mathcal{L}_i$ will correspond to the set of boxpies $\mathbb{BP}$. Denote by $\mathcal{L}$ the Cartesian product of all $\mathcal{L}_i$'s, *i.e.*, $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$. An element $\mathbb{X}$ of $\mathcal{L}$ is the Cartesian product of n elements of $\mathcal{L}_i$, (*i.e.*, it satisfies $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$). A *contractor* (see e.g. [2]) is an operator

$$\mathcal{C} : \begin{array}{ccc} \mathcal{L} & \rightarrow & \mathcal{L} \\ \mathbb{X} & \mapsto & \mathcal{C}(\mathbb{X}) \end{array} \tag{18}$$

which satisfies

$$\begin{array}{ll} \mathbb{X} \subset \mathbb{Y} \Rightarrow \mathcal{C}(\mathbb{X}) \subset \mathcal{C}(\mathbb{Y}) & \text{(monotonicity)} \\ \mathcal{C}(\mathbb{X}) \subset \mathbb{X} & \text{(contractance)} \end{array} \tag{19}$$

The set of contractors forms also a complete lattice. As a consequence, the meet (or intersection) and join (or union) can also be defined. This lead us to the contractor algebra [5]. When all variables of the constraint networks belong to $\mathbb{R}$, contractor techniques have been shown to be very powerful [25] [1].

Remark. The interval Newton operator developed by Moore [19] also aims at contracting boxes without removing any point from the solution set. Now, do to the fact that this operator is not monotonic, it does not satisfy the definition of a contractor.

Constraint propagation. The principle is to associate to each constraint $c_j \in \{c_1, \dots, c_m\}$ of a constraint network, a contractor $\mathcal{C}_j(\mathbb{X})$ which does not remove any $(x_1, \dots, x_n)$ which is consistent with c_j. Then, we build the contractor $\mathcal{C} = \mathcal{C}_1 \circ \cdots \circ \mathcal{C}_m$. We apply the contractor $\mathcal{C}$ until no more contraction can be observed. From the Tarski theorem, we conclude that the process converges toward the largest subdomain $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$ of the initial domain which cannot be contracted by any $\mathcal{C}_i$.

Contractors. Most of the contractor are build on an arithmetic of domains (which corresponds to the interval arithmetic if these domains are intervals). If $\mathbb{A}$,$\mathbb{B}$ and $\mathbb{C}$ are pies containing the three complex numbers a, b, c, using the arithmetic proposed in [4], it is possible to define the efficient contractors associated with the constraints

$$a + b = c \text{ and } c = a \cdot b. \tag{20}$$

These contractors can thus be used for solving polynomial equations in $\mathbb{C}$. Moreover, due to the non-unicity of the expression of a boxpie, it is important to add a selfconsistent contractor in order to have better contractions.

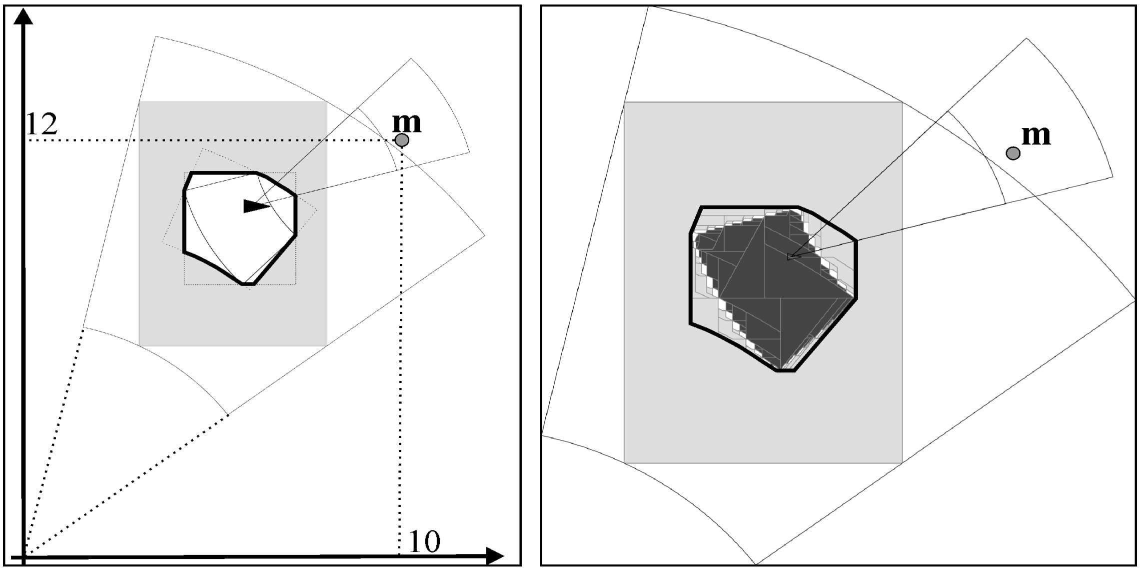
Separators. A separator [13] is a pair of two complementary contractors. Combined with a paver, separators makes it possible to compute an inner and an outer characterization of the solution set. The principle is similar to what has been proposed by Moore [20] and successors (see, *e.g.*, [15] [22] [10]) to characterize an inner and an outer approximations of a set defined by inequalities. The main difference is that Moore used inclusion tests whereas here, we use separators to be more efficient. As shown in [13], from a contractor, it often possible to get the corresponding separator automatically.

8 Application to robot localization

A robot, moving in a plane, is able to see a landmark $\mathbf{m}$ with coordinates $(10, 12)$. More precisely, a sensor in the robot is able to measure the distance d and the azimuth α of $\mathbf{m}$ with a known accuracy. Assume for instance that we collected $\alpha \in [\frac{\pi}{12}, \frac{\pi}{6}]$ and $d \in [4, 6]$. Assume that the position for the robot is known to belong to the box $[3, 8] \times [6, 13]$. Let us represent the position of the robot by a complex number $p \in \mathbb{C}$. We have to solve:

$$10 + 12i - p = de^{i\alpha}, p \in [3, 8] \times [6, 13],\ \ \alpha \in [\frac{\pi}{12}, \frac{\pi}{6}],\ d \in [4, 6]\ .$$

The first contraction yields the boxpie represented in bold in Figure 8, left. On this figure is also represented a black triangle which corresponds to the unknown true position for the robot in $(6, 10)$. A paver is able to give the inner and the outer characterization represented on Figure 8, right.



As a comparison, Figure 3 provides the pavings obtained using boxes and pies as domains, but in a separate way.

9 Conclusion

This paper shows that the interval arithmetic introduced by Moore can be generalized to other types of domains as soon as these domains form a lattice with respect to the inclusion and that we could bisect them. This allowed us to introduce a new type of domains, named *boxpies*. A boxpie corresponds to the intersection between one box and one pie. Most of interval-based algorithms can easily be extended for this type of domains, since we are able to contract boxpies with respect to some constraints and to bisect them. Boxpies is particularly interesting when we deal with equations in $\mathbb{C}$ since they heritate from the accuracy of the Cartesian representation for the addition and the accuracy of the polar representation for the multiplication.

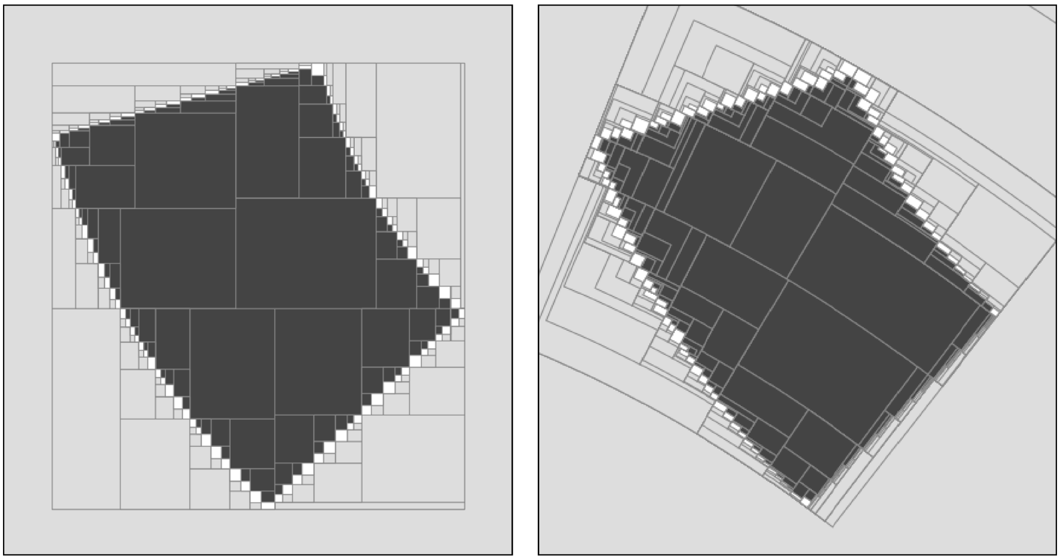


Figure 3: Left: paving obtained using boxes only; Right: paving obtained using pies only

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