A new approach for computing with fuzzy sets using interval analysis

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***Abstract*— We present a new approach for computing with fuzzy sets based on interval analysis techniques. Our proposed method is capable of managing multi-dimensional continuous membership functions of arbitrary form, such as, piecewise affine functions or non-linear expressions without any restrictions regarding convexity. We present a formal representation of fuzzy sets that allows to easily cast fuzzy problems into the set inversion framework. The SIVIA algorithm is presented as a convenient solution to solve this problem via interval analysis. It provides a representation of the resulting fuzzy set in an approximate (with the desired precision) but guaranteed way. Different combination operators were implemented using our method. We show that each operator is implemented following the same procedure and thus that, potentially, any fuzzy problem could be represented as a set inversion problem. Our approach is illustrated by examples at the end of the paper, where a discussion over the obtained results takes place.**

Keywords: fuzzy set theory, interval analysis, possibility theory, combination operators, set inversion.

I. Introduction

The fuzzy set theory was introduced by Zadeh in 1965 [1] in order to mathematically represent the inherent imprecision of some classes of objects. In classical set theory, the elements that fulfil some precise conditions defined by a set (usually called crisp set) are only considered as members of this set; the degree of membership is therefore binary, *i.e.* either the element belongs to the set or it does not. In the real world, it is more likely to find classes or sets where this membership cannot be correctly evaluated in a binary way. Let us consider "the class of young people". Is a 33 years old person young?. A crisp set could define "young" if the age is less than 30. But do we suddenly become "old" the day of our 30$^{\text{th}}$ anniversary?. We are talking of imprecisely defined classes that are not well represented by the classical set theory. To address this problem, Zadeh proposed a new kind of sets with a continuum of grades of membership to model the imprecision of this kind of classes (called fuzzy sets).

The human reasoning process is usually based on imprecise, incomplete or vague knowledge. The fuzzy set theory provides a better representation of the human way of thinking and a convenient way to express its knowledge based on an observed phenomenon. Indeed, fuzzy set theory is well adapted when the *a priori* knowledge of a system is of expert nature [2]. It provides a framework to deal with problems where imprecision arises because of the absence of sharp transitions from class membership to non-membership instead of randomness.

Several authors have devoted themselves to the study of this theory. More recently, the possibility theory [3], [4], derived from that of the fuzzy sets, has become a subject of interest in the research field. This theory inherites the concepts of the fuzzy set theory to model imprecision, and provides the tools to take uncertainty into account. Strong mathematical foundations have been developed for both theories over the last years in both the continuous and the discrete domains.

When working with fuzzy sets, computational analysis is frequently performed in a finite space, *i.e.* a discretization of the space is performed, sampling the real plane or defining discrete supports. These spaces provide a fast and intuitive computational solution to work with fuzzy sets for many applications. However, when dealing with fuzzy quantities, we are interested in performing arithmetic operations over sets defined over the set of real numbers. The propagation of errors could become important and affect the final result if we adopt the sampling approach previously exposed. This problem finds its solution within the concept of fuzzy intervals. However, some constraints apply. Convexity of the membership function is mandatory, which means that all the α-cuts must necessarily be representable by intervals.

There exist two main approaches to work with fuzzy intervals [5]: an exact approach based on parameterized representations and an approximate one which performs classical interval arithmetic operations over the α-cuts. The first approach is frequently used to perform simple arithmetic operations over sets. Although it provides the exact result for some basic, but widely used operators, its applicability is restricted to simple forms, operations are defined between sets of the same type (LR fuzzy intervals) and the result may not be of the same initial type. On the other hand, the approximate approach is more general and well adapted for function evaluations.

The interest of this study is to find a convenient and general way to represent and manipulate fuzzy sets exploring other forms of discrete spaces. We present a new method based on a set approach using interval analysis techniques. The first step of our method is to model a fuzzy problem as one of the set inversion. The SIVIA algorithm (Set Inversion Via Interval Analysis) [6] provides the tools required to solve the set inversion problem via interval analysis techniques. It performs a decomposition of the departing space into 3 partitions (called subpavings) to characterize fuzzy sets given by fuzzy operations. These partitions are composed by unions of non-overlapped intervals and represent in a guaranteed way 3 different groups: the "inside of the fuzzy set" space, the "outside of the fuzzy set" space and the "membership function of the fuzzy set" space.

An efficient way to handle multi-dimensional fuzzy sets is to consider that the variables are independent evoking the Cartesian product property [5]. Regarding this aspect, our method is able to perform a proper representation of the solution through a decomposition of the whole n-dimensional space. As a consequence of this, multi-dimensional fuzzy sets where the variables could be related somehow (fuzzy relations [7] or dependent variables) could be envisaged.

This work finds its applications in the automotive industry. We are working in collaboration with the Autocruise society, rank-1 supplier of radars for driver assistance. Our work involves a study of the data association problem in a multi-radar tracking context, with fusion processes in both the continuous (through the possibility theory) and the discrete (using the belief theory) domains.

In section II, we introduce some basic notions of the fuzzy set theory that will be encountered in the subsequent sections. A short description of the basic concepts of interval analysis applied to this work is given in section III. Our proposed method is described in section IV. At first, a suitable representation of fuzzy sets which allows an easy transition of the problem to the set inversion framework is presented. The SIVIA algorithm is proposed as a convenient solution to address this problem via interval analysis techniques. Some possible extensions of the algorithm are also presented at the end this section. Examples are presented and discussed in section V. Finally, our conclusions and perspectives will be drawn in section VI.

II. Fuzzy set theory

A. Definitions

1) Fuzzy set: let X be a crisp set of objects and x an element of X. In classical set theory a subset A of X is characterized by a membership (characteristic) function $\mu_A(x)$ from X to $\{0,1\}$:

$$\mu_A(x) = \begin{cases} 1 & \text{iff} \quad x \in A \\ 0 & \text{iff} \quad x \notin A \end{cases} \tag{1}$$

A membership function of a fuzzy set A of X, associates each element x with a real number in the interval $[0,1]$. The closer this value is to 1, the more x is an element of A. Fuzzy sets defined over the set of real numbers are also known by the name of fuzzy quantities.

A fuzzy set A of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n \times (0,1]$ such that:

$$\begin{gathered} \forall \alpha_1, \alpha_2 \in (0,1] | \alpha_1 \geq \alpha_2, \quad \forall \mathbf{x} \in \mathbb{R}^n \\ (\mathbf{x}, \alpha_1) \in A \Rightarrow (\mathbf{x}, \alpha_2) \in A \end{gathered} \tag{2}$$

2) Support: the support of a fuzzy set A of X, denoted supp A, is the set of elements of X that belongs, at least with a minimum degree, to A:

$$\text{supp } A = \{x \in X | \mu_A(x) > 0\} \tag{3}$$

3) Core: the core of a fuzzy set A of X, denoted core A, is the set of elements of X that belongs entirely to A:

$$\text{core } A = \{x \in X | \mu_A(x) = 1\} \tag{4}$$

4) Height: the height of a fuzzy set, denoted $h(A)$, is the highest degree of membership of an element of A[1]:

$$h(A) = \sup_{x \in X} \mu_A(x) \tag{5}$$

If $\exists\ x \in X | \mu_A(x) = 1$, the fuzzy set A is said to be normalized.

5) α-Cuts: the α-cut of a fuzzy set $\mu_A(x)$ is defined by:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\} \tag{6}$$

where A_α is a crisp set with characteristic function:

$$\chi_{A_\alpha}(x) = \begin{cases} 1 & \text{if} \quad \mu_A(x) \geq \alpha \\ 0 & \quad \text{otherwise} \end{cases} \tag{7}$$

A fuzzy set can be fully reconstructed by its α-cuts as is shown below:

$$\forall x \in X, \quad \mu_A(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{A_\alpha}(x) \tag{8}$$

A fuzzy quantity is convex if all its α-cuts are convex and can be represented by classic intervals.

6) Fuzzy intervals: a fuzzy interval I of $\mathbb{R}^n$, is a fuzzy quantity of $\mathbb{R}^n$ with membership function $\mu_I(\mathbf{x})$ that obeys the following rule:

$$\forall \alpha \in (0,1], \quad \mu_I^{-1}([\alpha, 1]) \in \mathbb{IR}^n \tag{9}$$

where $\mathbb{IR}^n$ is the set of all the interval vectors of $\mathbb{R}^n$ (more details regarding interval vectors will be presented in section III).

This expression states that a fuzzy interval of $\mathbb{R}^n$ is a convex fuzzy quantity originated by the Cartesian product of n independent fuzzy quantities of $\mathbb{R}$.

From now on, a membership function will be considered as *convex* if all its α-cuts are convex.

B. Basic set operations

1) Intersection and Union: the intersection $A \cap B$ of two fuzzy sets A and B of X, is defined as[2]:

$$\mu_{A \cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right) \tag{10}$$

This implies that an element of X cannot belong at the same time to A and B less than to each one separately.

The union $A \cup B$ of two fuzzy sets A and B of X, is defined as[3]:

$$\mu_{A \cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right) \tag{11}$$

This expression states that an element of X cannot belong at the same time to A or B more than to each one separately.

[1] sup denotes the *supremum*, *i.e.* the upper bound of all the possible values

[2] min denotes the minimization operator

[3] max denotes the maximization operator

2) Cardinality: the cardinality of a fuzzy set A of a finite set X, denoted $|A|$, represents the number of elements of X weighted by their membership degree. It is formally given by:

$$|A| = \sum_{x \in X} \mu_A(x) \tag{12}$$

If X is not a finite set but a measurable one, with a measure $\mathcal{M}$ of X (such as $\int_X d\mathcal{M}(x) = 1$), the cardinality is defined by:

$$|A| = \int_X \mu_A(x) d\mathcal{M}(x) \tag{13}$$

C. Combination operators

There is a vast amount of operators available in the literature to combine fuzzy sets [7]. They are numerous not only in quantity but in variety. The choice depends on the application since the combination is performed differently by each operator. Some of these aggregation functions take into account the conflict between propositions (adaptative operators) to find a convenient way to fuse, while others behave as quite conjunctive or disjunctive operators.

Triangular norms and conorms (t-norms and t-conorms respectively) are generally considered as the most important group of combination tools available in fuzzy logic. t-norms are conjunctive operators; the intersection operator min previously defined in (10), is the strongest of all t-norms. On the contrary, t-conorms are disjunctive; the union operator max (11) is the weakest of all t-conorms. The triangular norms and conorms properties are well adapted to define combination operators that are in close relation with the set operations intersection and union [5]. The probabilistic t-norm (probabilistic product) and t-conorm (probabilistic sum) figure among the most widely used. Their expressions are given as follows [8]:

$$\mu_{A \hat{\cdot} B}(x) = \mu_A(x) \cdot \mu_B(x) \tag{14}$$

$$\mu_{A \hat{+} B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \tag{15}$$

Another group of great interest is the one involving adaptative operators. Their goal is to come up with an adaptative solution (combination operator) based on a measure of conflict between the two propositions [9]. These operators are defined in the frame of the possibility theory. This theory relies on two fundamental measures, a possibility and a necessity measure [3].

Let A be a non-fuzzy (crisp) set of X, and v be a variable on X. If we assume that v takes its values in A, we could define a possibility distribution $\pi(x)$ associating to each element of X the possibility that v lies in x:

$$\Pi(v = x) = \pi(x) = \begin{cases} 1 & \text{iff} \quad x \in A \\ 0 & \text{iff} \quad x \notin A \end{cases} \tag{16}$$

If we release the non-fuzzy restriction of A, the possibility distribution $\pi(x)$ can be represented by the membership function $\mu_A(x)$ conditioning the possible values of v:

$$\Pi(v = x) = \pi(x) = \mu_A(x) \tag{17}$$

Although both concepts have the same mathematical representation, their semantics are different. A fuzzy set A gives a fuzzy value to a variable. This value represents the membership degree of the variable to the class A. From a possibilistic point of view, a fuzzy set A conditions the possible values of a variable v on X. The fuzzy set A represents a "conditional possibility distribution" of the variable v on X.

A possibility distribution satisfies a normalization constraint:

$$\sup_{x \in X} \pi(x) = 1 \tag{18}$$

This constraint is always satisfied under the assumption of a closed world, where at least one element of X is completely possible. The absence of normalisation suggests that the variable v could take a value outside of X (open world) or could not take a value at all (the represented event could not take place) [5].

Getting back to the subject of adaptative operators, previously discussed in the upper paragraphs, the combination of two possibility distributions taking into account the existing conflict between them is proposed by Dubois and Prade [9]:

$$\pi' = \max\left(\frac{\min(\pi_1, \pi_2)}{h}, \min(\max(\pi_1, \pi_2), 1 - h)\right) \tag{19}$$

where h represents a measure of conflict between the sources:

$$h(\pi_1, \pi_2) = 1 - \sup_{x \in X} \min(\pi_1(x), \pi_2(x)) \tag{20}$$

Possibility distributions can be manipulated and combined by all the existing rules in fuzzy set theory. However, as we mentioned earlier, precautions must be taken when interpreting the results since they have different significance [8].

III. Interval Analysis

The following notions will be required for the comprehension of the subsequent sections [6].

A. Interval vectors

A box or vector interval $[\mathbf{x}]$ of $\mathbb{R}^n$ is a vector whose components $[x_i] = [\underline{x_i}, \overline{x_i}]$ for $i = 1, \ldots, n$ are scalar intervals:

$$[\mathbf{x}] = [\underline{x_1}, \overline{x_1}] \times \ldots \times [\underline{x_n}, \overline{x_n}] = [x_1] \times \ldots \times [x_n] = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \tag{21}$$

where $\underline{\mathbf{x}} = (\underline{x_1}, \ldots, \underline{x_n})^T$ and $\overline{\mathbf{x}} = (\overline{x_1}, \ldots, \overline{x_n})^T$. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$.

B. Inclusion function

Consider a function $\mathbf{f}$ from $\mathbb{R}^n \to \mathbb{R}^m$. The interval function $[\mathbf{f}]$ from $\mathbb{IR}^n \to \mathbb{IR}^m$ is an inclusion function for $\mathbf{f}$ if:

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{x}]) \tag{22}$$

This inclusion function is convergent if, for any sequence of boxes $[\mathbf{x}]$ of $\mathbb{IR}^n$:

$$\lim_{k \to \infty} w([\mathbf{x}](k)) = 0 \Rightarrow \lim_{k \to \infty} w([\mathbf{f}][\mathbf{x}](k)) = 0 \tag{23}$$

where $w([\mathbf{x}])$ is the width of $[\mathbf{x}]$, *i.e.* the length of its largest side(s): $w([\mathbf{x}]) = \max(\overline{\mathbf{x}} - \underline{\mathbf{x}})$.

C. Pavings and subpavings

A subpaving $\mathbb{K}$ of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{IR}^n$ with non-zero width. If $\mathbb{A}$ is the subset of $\mathbb{R}^n$ generated by the union of all boxes in the subpaving $\mathbb{K}$, then $\mathbb{K}$ is a paving of $\mathbb{A}$.

IV. A SET APPROACH TO COMPUTE WITH FUZZY SETS

A membership function may take many forms, based on the available *a priori* information or the phenomenon is intending to describe. Some examples, linear and non-linear, are presented as follows and can be visualized in Fig. 1 [7]:

$$\mu_A(x;a,b,c,d) = \begin{cases} 0 & \text{if} \quad x \leq a \\ \frac{x-a}{b-a} & \text{if} \quad a \leq x \leq b \\ 1 & \text{if} \quad b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if} \quad c \leq x \leq d \\ 0 & \text{if} \quad d \leq x \end{cases} \quad \begin{array}{c}\text{Trapezoidal}\\ \text{function}\end{array} \tag{24}$$

$$\mu_A(x;\sigma,m) = \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) \quad \begin{array}{c}\text{Gaussian}\\ \text{function}\end{array} \tag{25}$$

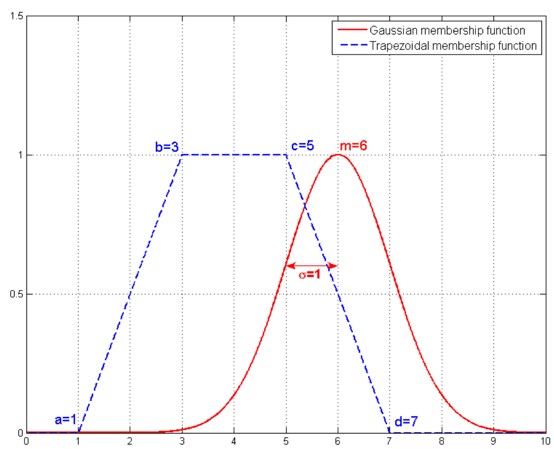


Fig. 1. *Illustration for two kinds of membership functions, each of one is represented by the equations (24) and (25)*

For the linear membership functions, we start by a representation of the fuzzy set in terms of fuzzy intervals. It will be demonstrated that this is not necessary for the non-linear functions and that their representation is straightforward.

All piecewise affine membership functions (convex or not), can be described by the union of fuzzy intervals (derived from the existing linear expressions) that cover *convex* overlapped regions of the original set. This affirmation holds since all fuzzy sets can be reconstructed by their α-cuts and thus by fuzzy intervals for convex fuzzy quantities. Let us take, for example, a trapezoidal fuzzy set A with membership function given by (24). This is a fuzzy interval which can be fully reconstructed with the following expression:

$$A_\alpha = [(b-a)\alpha + a, (c-d)\alpha + d] \tag{26}$$

Convex fuzzy quantities of $\mathbb{R}^n$ resulting from the Cartesian product of independent sets of $\mathbb{R}$, would be represented by interval boxes instead.

A fuzzy interval can be described as a system of inequalities that an element $\mathbf{x}$ of $\mathbb{R}^n$ satisfies if $\mathbf{x} \in A_\alpha$. For (26), such a system is given by one inequality:

$$x \in A_\alpha \quad \Leftrightarrow \mu_A(x;a,b,c,d) \geq \alpha \tag{27}$$

where $\mu_A(x;a,b,c,d)$, previously presented in (24), can be rewritten as follows to obtain a unique analytic expression that characterizes the fuzzy set:

$$\mu_A(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{d-x}{d-c}, 1\right), 0\right) \tag{28}$$

The membership function $\mu_A(\mathbf{x})$ of a fuzzy set A of $\mathbb{R}^n$, resulting from the Cartesian product of *n* fuzzy sets of $\mathbb{R}$, can be evaluated as follows:

$$\forall \mathbf{x} = (x_1, \ldots, x_n), \;\; \mu_A(\mathbf{x}) = \min(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)) \tag{29}$$

A finite covering by *convex* overlapped regions of the original fuzzy set can be performed for piecewise affine membership functions with complex shapes. Let A be a fuzzy set of $\mathbb{R}^n$, resulting from the Cartesian product of *n* fuzzy sets of $\mathbb{R}$ with piecewise affine membership functions. The subset of $\mathbb{R}^n$ which corresponds to a given α-cut of A is given by:

$$A_\alpha = A_\alpha^1 \times \ldots \times A_\alpha^n \tag{30}$$

with:

$$A_\alpha^j = \bigcup_{i=1}^{k_j} A_{\alpha,i}^j \tag{31}$$

where A_α^j is an α-cut of A on the *j*^th^ dimension and $A_{\alpha,i}^j$ is an α-cut of the *i*^th^ fuzzy interval defined over the k_j convex covering regions on that same dimension.

The fuzzy set A can be fully reconstructed by A_α and is characterized by the following expression:

$$\mathbf{x} \in A_\alpha \quad \text{if} \quad \min_{j=1\ldots n}\left(\max_{i=1\ldots k_j}(\mu_{A_i^j}(x_j))\right) \geq \alpha \tag{32}$$

where $\mu_{A_i^j}(x_j)$ corresponds to the membership function of the fuzzy interval whose α-cuts are defined by $A_{\alpha,i}^j$.

As we mentioned before, this procedure is simplified when working with non-linear expressions. The fuzzy interval concept was evoked to overcome the difficulties that arise when trying to represent piecewise affine membership functions with complex shapes. For non-linear membership functions, testing if an element belongs to a given α-cut can be done directly through evaluations over the non-linear functions. Let us take a Gaussian membership function (25) for example:

$$x \in A_\alpha \quad \Leftrightarrow \quad \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) \geq \alpha \tag{33}$$

In order to simplify the expressions we present the following equivalence:

$$(\mathbf{x}, \alpha) \in A \Leftrightarrow \mathbf{x} \in A_\alpha \tag{34}$$

A. The max and min operators via analysis of the inclusion degree

The work presented in this subsection was inspired by the work published in [10].

Let φ be a function evaluated over a set $\mathbb{U} = \{U_1, \dots, U_k\}$ of k fuzzy sets of $\mathbb{R}^n$, that measures the cardinality[4] of the proposition $(\mathbf{x}, \alpha) \in \mathbb{U}$:

$$\varphi(\mathbb{U}; \mathbf{x}, \alpha) = \#\{i \in \{1, \dots, k\} \,|(\mathbf{x}, \alpha) \in U_i\} \tag{35}$$

where:

$$(\mathbf{x}, \alpha) \in U_i \quad \text{if} \quad \mu_{U_i} - \alpha \geq 0 \Leftrightarrow \mu_{U_i} - \alpha \in [0, \infty) \tag{36}$$

The set of elements $\mathbb{S}$ of $\mathbb{R}^{n+1}$ with a minimum given cardinality for the proposition $(\mathbf{x}, \alpha) \in \mathbb{U}$, is given by:

$$\mathbb{S} = \{(\mathbf{x}, \alpha) | \varphi(\mathbb{U}; \mathbf{x}, \alpha) \in \mathbb{Y}\} \tag{37}$$

with:

$$\mathbb{Y} = [k - q, k] \tag{38}$$

where q is a relaxing parameter in the interval $[0, k-1]$.

This approach allows the representation of the solution space of the proposition $(\mathbf{x}, \alpha) \in \mathbb{U}$ with a minimum inclusion degree. It must be noticed that if $q = 0$, $\mathbb{S}$ contains the elements $(\mathbf{x}, \alpha)$ that are common to all the elements of $\mathbb{U}$. This corresponds to the definition of the intersection operator min (10). On the other hand, if we choose $q = k-1$, $\mathbb{S}$ contains the elements $(\mathbf{x}, \alpha)$ that are at least included in one element of $\mathbb{U}$. This time the union operator max (11) is in cause.

The expression of $\mathbb{S}$ presented in (37) can be rewritten as:

$$\mathbb{S} = \varphi^{-1}(\mathbb{Y}) \tag{39}$$

The problem of characterizing $\mathbb{S}$ has been cast into the framework of a set inversion problem.

B. SIVIA algorithm

The algorithm SIVIA can provide an easy way to determine $\mathbb{S}$ via the definition of $\mathbb{Y}$ and the notion of inclusion function [6]. Let $[\varphi]$ be a convergent inclusion function of φ, so that for any box $[\mathbf{p}]$ of $\mathbb{IR}^{n+1}$, $[\varphi([\mathbf{p}])]$ is a box guaranteed to contain all values of $\varphi(\mathbf{p})$ for all $\mathbf{p}$ in $[\mathbf{p}]$, where $\mathbf{p} = (\mathbf{x}, \alpha) \in \mathbb{R}^{n+1}$.

The SIVIA algorithm to be presented is parameterized by ε. For a given value of ε, SIVIA(ε) generates 3 subpavings: $\mathbb{K}_{ok}$ ("inside of the fuzzy set"), $\mathbb{K}_{out}$ ("outside of the fuzzy set") and $\mathbb{K}_{ind}$ ("membership function of the fuzzy set"). The subpaving $\mathbb{K}_{ok}$ contains all the boxes that have been proved to be included in $\mathbb{S}$. $\mathbb{K}_{out}$ contains all the boxes outside $\mathbb{S}$. $\mathbb{K}_{ind}$ contains all indeterminate boxes with width smaller than ε.

We start by defining a possibly very large or infinite prior box $[\mathbf{p_0}]$ which is guaranteed to contain $\mathbb{K}_{ind} \cup \mathbb{K}_{ok}$. A pseudo-code of the algorithm is presented in the algorithm 1.

The subpavings generated by SIVIA verify the following property: $\mathbb{K}_{ok} \subset \mathbb{S} \subset \mathbb{K}_{ind} \cup \mathbb{K}_{ok}$; a guaranteed characterization of the solution space $\mathbb{S}$ is thus obtained via set inversion and interval analysis techniques.

[4]# denotes the cardinality

Algorithm 1: SIVIA(ε) algorithm

Data : ε: required accuracy
Data : $[\varphi]$, $[\mathbf{p_0}]$ and $\mathbb{Y}$
Result: $\mathbb{K}_{ok}$, $\mathbb{K}_{ind}$ and $\mathbb{K}_{out}$
stack:=$\emptyset$, $\mathbb{K}_{ok}$:=$\emptyset$, $\mathbb{K}_{ind}$:=$\emptyset$ and $\mathbb{K}_{out}$:=$\emptyset$
stack $\leftarrow [\mathbf{p_0}]$
while *stack not empty* **do**
  $[\mathbf{p}] \leftarrow$ bottom of the stack
  if $[\varphi([\mathbf{p}])] \cap \mathbb{Y} = \emptyset$ **then** $\mathbb{K}_{out} \leftarrow [\mathbf{p}]$
  else
    if $[\varphi([\mathbf{p}])] \subset \mathbb{Y}$ **then** $\mathbb{K}_{ok} \leftarrow [\mathbf{p}]$
    else
      if $w([\mathbf{p}]) < \varepsilon$ **then** $\mathbb{K}_{ind} \leftarrow [\mathbf{p}]$
      else
        bisect $[\mathbf{p}]$ along the principal plane (longest side)
        stack the resulting boxes

C. Extending the method

All problems in fuzzy set theory can be potentially represented as a set inversion problem and thus solved via interval analysis techniques. The main interest of this approach is its ability to characterize the resulting fuzzy set in an approximate but guaranteed way reconstructing the reciprocal image of a defined space $\mathbb{Y}$.

Let us take the probabilistic t-norm (14) and t-conorm (15) for example. We proceed in the very same way we just did in section IV-A:

$$(\mathbf{x}, \alpha) \in A \hat{\cdot} B \quad \Leftrightarrow \quad \mu_{A \hat{\cdot} B}(\mathbf{x}) \geq \alpha \tag{40}$$

$$(\mathbf{x}, \alpha) \in A \hat{+} B \quad \Leftrightarrow \quad \mu_{A \hat{+} B}(\mathbf{x}) \geq \alpha \tag{41}$$

Let $\Psi_{A \hat{\cdot} B}$ and $\Psi_{A \hat{+} B}$ be two functions evaluated over two fuzzy sets A and B of $\mathbb{R}^n$:

$$\Psi_{A \hat{\cdot} B}(\mathbf{x}, \alpha) = \mu_{A \hat{\cdot} B}(\mathbf{x}) - \alpha \tag{42}$$

$$\Psi_{A \hat{+} B}(\mathbf{x}, \alpha) = \mu_{A \hat{+} B}(\mathbf{x}) - \alpha \tag{43}$$

The sets of elements $\mathbb{S}_{A \hat{\cdot} B}$ (respectively $\mathbb{S}_{A \hat{+} B}$) of $\mathbb{R}^{n+1}$, that obey the rule $(\mathbf{x}, \alpha) \in A \hat{\cdot} B$ (respectively $(\mathbf{x}, \alpha) \in A \hat{+} B$) are given by:

$$\mathbb{S}_{A \hat{\cdot} B} = \{(\mathbf{x}, \alpha) | \Psi_{A \hat{\cdot} B}(\mathbf{x}, \alpha) \in \mathbb{Y}_{A \hat{\cdot} B}\} \tag{44}$$

$$\mathbb{S}_{A \hat{+} B} = \left\{(\mathbf{x}, \alpha) | \Psi_{A \hat{+} B}(\mathbf{x}, \alpha) \in \mathbb{Y}_{A \hat{+} B}\right\} \tag{45}$$

where:

$$\mathbb{Y}_{A \hat{\cdot} B} = \mathbb{Y}_{A \hat{+} B} = [0, \infty) \tag{46}$$

The expressions provided in (44) and (45) can be respectively rewritten as:

$$\mathbb{S}_{A \hat{\cdot} B} = \Psi^{-1}_{A \hat{\cdot} B}(\mathbb{Y}_{A \hat{\cdot} B}) \tag{47}$$

$$\mathbb{S}_{A \hat{+} B} = \Psi^{-1}_{A \hat{+} B}(\mathbb{Y}_{A \hat{+} B}) \tag{48}$$

The problem of characterizing $\mathbb{S}_{A\hat{\cap}B}$ and $\mathbb{S}_{A\hat{\cup}B}$ has been cast again to the framework of a set inversion problem. This problem can be solved with the SIVIA algorithm previously presented in algorithm 1.

Another combination operator of interest is the adaptative operator presented in (19). Extending the method to implement this operator is also an easy task. The only inconvenient is that the parameter h (20) needs to be calculated before the fusion process takes place.

The proposition $(\mathbf{x}, \alpha) \in A$, where A is a fuzzy set given by the intersection of two possibility distributions $\pi_1(\mathbf{x})$ and $\pi_2(\mathbf{x})$, can be directly evaluated as follows:

$$(\mathbf{x}, \alpha) \in A \quad \Leftrightarrow \quad \min(\pi_1(\mathbf{x}), \pi_2(\mathbf{x})) \geq \alpha \tag{49}$$

If A is scaled by h, the following equivalence applies:

$$(\mathbf{x}, \alpha) \in A \quad \Leftrightarrow \quad \frac{\min(\pi_1(\mathbf{x}), \pi_2(\mathbf{x}))}{h} \geq \alpha \tag{50}$$

The proposition $(\mathbf{x}, \alpha) \in B$, where B is a fuzzy set given by the union of two possibility distributions $\pi_1(\mathbf{x})$ and $\pi_2(\mathbf{x})$, can be evaluated through the following expression:

$$(\mathbf{x}, \alpha) \in B \quad \Leftrightarrow \quad \max(\pi_1(\mathbf{x}), \pi_2(\mathbf{x})) \geq \alpha \tag{51}$$

The last part of equation (19) is an additional linear constraint. Let C be a fuzzy set with membership function $\mu_C(\mathbf{x}) = 1 - h$. The proposition $(\mathbf{x}, \alpha) \in C$ can be tested as follows:

$$(\mathbf{x}, \alpha) \in C \quad \Leftrightarrow \quad 1 - h \geq \alpha \tag{52}$$

We finally obtain that the proposition $(\mathbf{x}, \alpha) \in D$, where D is a fuzzy set characterized by the possibility distribution π' (19), can be evaluated as follows:

$$(\mathbf{x}, \alpha) \in D \quad \Leftrightarrow \quad \mathbf{f}_{\pi'} \geq 0 \tag{53}$$

where:

$$\mathbf{f}_{\pi'} = \max\left(\frac{\min(\pi_1, \pi_2)}{h}, \min(\max(\pi_1, \pi_2), 1 - h)\right) - \alpha \tag{54}$$

The set of elements $\mathbb{S}_D$ of $\mathbb{R}^{n+1}$, that obey the rule $(\mathbf{x}, \alpha) \in D$ is given by:

$$\mathbb{S}_D = \{(\mathbf{x}, \alpha) | \mathbf{f}_{\pi'} \in \mathbb{Y}_D\} \tag{55}$$

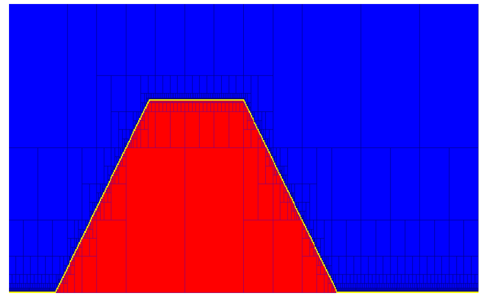
where:

$$\mathbb{Y}_D = [0, \infty) \tag{56}$$

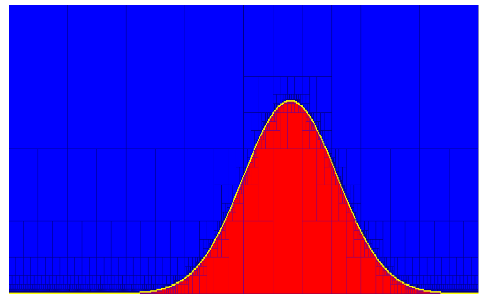
This expression can be rewritten as:

$$\mathbb{S}_D = {\mathbf{f}_{\pi'}}^{-1}(\mathbb{Y}_D) \tag{57}$$

and the set $\mathbb{S}_D$ of $\mathbb{R}^{n+1}$ can be characterized via interval analysis with the SIVIA algorithm presented in algorithm 1.



(a) Trapezoidal fuzzy set

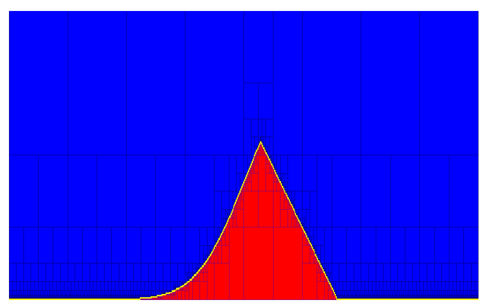


(b) Gaussian fuzzy set

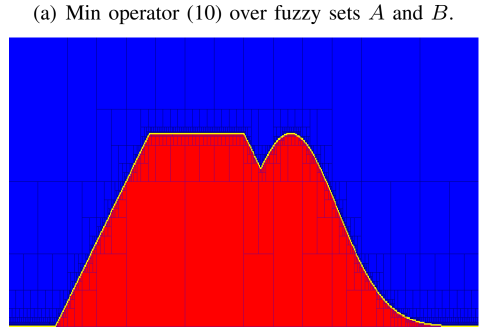
Fig. 2. *Representation of a trapezoidal and a Gaussian membership function via set inversion. The required equations to characterize each set are (27) and (33) respectively. The frame corresponds to the search domain* $[\mathbf{p_0}] = [0, 10] \times [0, 1.5]$ *with* $\varepsilon = 0.01$

V. Simulation results

Let us start by presenting the two fuzzy sets that are going to be the subject of study in almost all the simulations of this



(a) Min operator (10) over fuzzy sets A and B.



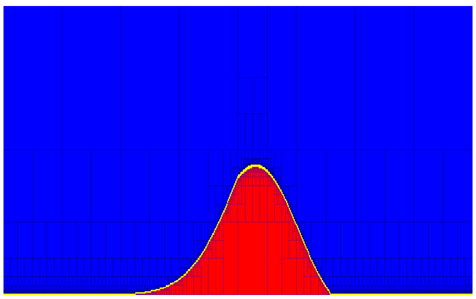
(b) Max operator (11) over fuzzy sets A and B.

Fig. 3. *The results represent the union and intersection of two fuzzy sets. The frame corresponds to the search domain* $[\mathbf{p_0}] = [0, 10] \times [0, 1.5]$ *with* $\varepsilon = 0.01$

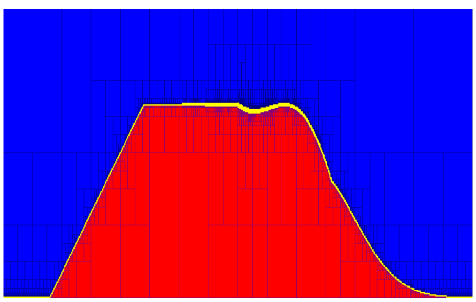
section. These fuzzy sets are represented by the membership functions previously defined in (24) and (25). Let A and B be two fuzzy sets with a trapezoidal membership function $\mu_A(x;1,3,5,7)$ and a symmetric Gaussian membership function $\mu_B(x:1,6)$ respectively. These fuzzy sets are represented via set inversion in Fig. 2.

The min (10) and max (11) operators over the two fuzzy sets A and B are represented via set inversion with (39), where $\mathbb{U}=\{A,B\}$, q is fixed to 0 for the min operator and to 1 for the max. The results are presented in Fig. 3.

The probabilistic t-norm and t-conorm, previously presented in (14) and (15), provide an example of the results when extending the method to handle additional operators. These results are presented in Fig. 4.



(a) Probabilistic t-norm (14) over fuzzy sets A and B



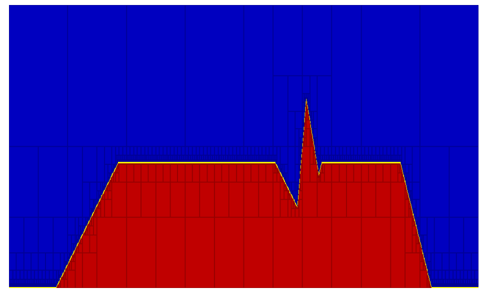
(b) Probabilistic t-conorm (15) over fuzzy sets A and B

Fig. 4. *Results for the probabilistic t-norm and t-conorm. The increased undetermined border of the membership function is a well known phenomenon produced by the propagation of the incertitude by interval arithmetics. The frame corresponds to the search domain* $[\mathbf{p_0}]=[0,10]\times[0,1.5]$ *with* $\varepsilon=0.01$

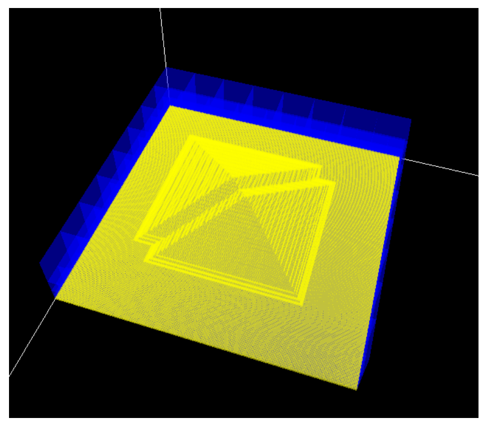
The adaptative method presented in (19) was used to fuse two trapezoidal fuzzy sets with a considerable conflict degree. The combination was performed over the fuzzy set A and a new trapezoidal fuzzy set C with membership function $\mu_C(x;6,7,8,9)$. For this simulation the measure of conflict h (20) is $1/3$. The obtained results are shown in Fig. 5(a), where we can appreciate the compromise between disjunctive and conjunctive behaviour based on the existing conflict.

This adaptative operator was also tested over two pyramidal fuzzy sets of $\mathbb{R}^2$, each one representing the resulting fuzzy set from the Cartesian product of two independent triangular sets of $\mathbb{R}$. The results are presented in Fig. 5(b) and Fig. 5(c), showing the quite conjunctive behaviour of this operator when the sources are in accordance. The measure of conflict h (20), is calculated by the method and guaranteed to belong to the interval $[0.84375, 0.890625]$ with the given accuracy ($\varepsilon=0.05$). Its true value is $h=0.875$.

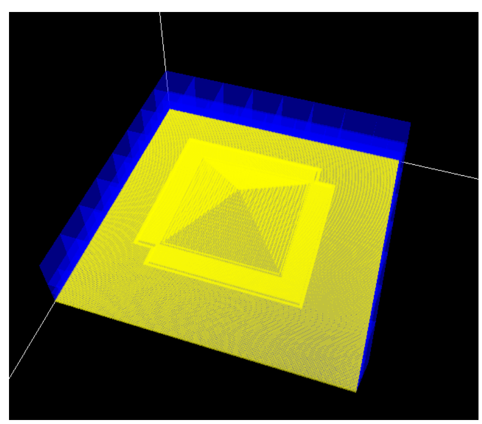
The main difference between the capabilities of our method and the approach based on the Cartesian product to handle multi-dimensional fuzzy sets, becomes evident with the last example; our method can perform an efficient decomposition of the whole n-dimensional space into subpavings of $\mathbb{IR}^n$. This advantage becomes significant when manipulating fuzzy rela-



(a) Results of the adaptative operator over fuzzy sets A and C



(b) Pyramidal fuzzy sets before the fusing process takes place



(c) Resulting fuzzy set from the fusion of two pyramidal fuzzy sets (Fig. 5(b)) using an adaptative operator

Fig. 5. *Results for the adaptative combination operator presented in (19). The frames correspond to the search domain* $[\mathbf{p_0}]=[0,10]\times[0,1.5]$ *for the two-dimensional space with* $\varepsilon=0.01$ *and* $[\mathbf{p_0}]=[0,7]\times[0,7]\times[0,1.5]$ *for the three-dimensional space with* $\varepsilon=0.05$

tions of $\mathbb{R}^n$ or membership functions with related variables, where each variable is a real number.

VI. Conclusions and perspectives

A new general set approach to compute fuzzy sets based on interval analysis techniques has been presented. Its main principle is to cast fuzzy problems into the set inversion framework. This problem is then solved via interval analysis techniques with the SIVIA algorithm. Our method is able to manage multi-dimensional continuous membership functions of arbitrary form, such as, piecewise affine functions or non-linear expressions without any restrictions regarding convexity.

In a first time we presented a general form of representation for fuzzy sets which allows to easily cast a fuzzy problem into the set inversion framework in both the linear and non-linear case. The SIVIA algorithm allows to address the set inversion problem in a convenient way through interval analysis techniques. The min and max operators where implemented via analysis of the inclusion degree, followed by the probabilistic t-norm and t-conorm, and an adaptative combination operator, to put forward the extension capabilities of the method.

The decomposition of the space into subpavings not only provides a guaranteed representation of the solution, in fact, it could be really useful when working with fuzzy sets. For example, the set $\mathbb{K}_{ok}$ in combination with $\mathbb{K}_{ok} \cup \mathbb{K}_{ind}$, could be used to obtain the upper and lower bounds where the cardinality of a fuzzy set given by (13) is guaranteed to belong. On the other hand, the "membership function of the fuzzy set" space characterized by $\mathbb{K}_{ind}$, provides bounds regarding the membership degree.

If more than one operation is performed in a given problem involving fuzzy sets, the characterization of the solution for such a problem needs to be addressed by only one unique set inversion. This way, we both optimize (since performing operations over subpavings at each stage is quite costly) and obtain a set guaranteed to contain the final result with a desired precision.

For the multi-dimensional case, precautions must be taken to ensure performance. The SIVIA algorithm complexity grows exponentially with the dimension of the space, because of the number of bisections involved, while linearly with the number of constraints. A big prior box also increases the number of bisections. There exists another interval analysis technique known as constraint propagation [6]. It reduces the size of the prior box, which is guaranteed to contain the solution, to its minimum size and thus the number of bisections. This technique boosts the actual SIVIA algorithm and increases its performance, especially in the multidimensional case.

Function evaluations and arithmetic operations over fuzzy sets like the ones performed with fuzzy intervals when evoking the extension principle [8] will be the subject of future publications. Some work has already been carried out in our laboratory regarding these aspects.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] M. Rombaut, "Fusion : état de l'art et perspectives," DGA, Tech. Rep. DSP 99.60.078, Oct. 2001.

[3] D. Dubois and H. Prade, *Théorie des possibilités: applications à la représentation des connaissances en informatique*, ser. Méthode + Programmes. Paris, France: Masson, 1987.

[4] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol. 1, pp. 3–28, 1978.

[5] B. Bouchon-Meunier and C. Marsala, *Logique floue, principes, aide à la décision*, ser. Informatique et systèmes d'information. Paris, France: Hermès Science Publications, 2003.

[6] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied interval analysis*. London, Great Britain: Springer-Verlag, 2001.

[7] G. J. Klir and B. Yuan, *Fuzzy sets and fuzzy logics: theory and applications*. New Jersey, USA: Prentice Hall PTR, 1995.

[8] D. Dubois and H. Prade, *Fuzzy sets and systems: theory and applications*, ser. Mathematics in science and engineering. Massachussets, USA: Academic Press, Inc., 1980, vol. 144.

[9] I. Bloch, *Fusion d'informations en traitement du signal et des images*, ser. Traitement du signal et de l'image. Paris, France: Hermès Science Publications, 2003.

[10] L. Jaulin, E. Walter, and O. Didrit, "Guaranteed robust nonlinear parameter bounding," in *CESA'96 IMACS multiconference (Symposium on modelling, analysis and simulation)*, Lille, France, 1996, pp. 1156–1161.

[11] D. Dubois, H. Fargier, and J. Fortin, "A generalized vertex method for computing with fuzzy intervals," in *Proc. of the International Conference on Fuzzy Systems*, Budapest, Hungary, Jul. 2004, pp. 541–546.

[12] M. Mareš, *Computation over fuzzy quantities*. Florida, USA: CRC Press, Inc., 1994.

[13] H. Q. Yang, H. Yao, and J. D. Jones, "Calculating functions of fuzzy numbers," *Fuzzy sets and systems*, vol. 55, no. 3, pp. 273–283, 1993.

[14] R. Baekeland and E. E. Kerre, "Piecewise linear fuzzy quantities: a way to implement fuzzy information into expert systems and fuzzy databases," in *Uncertainty and intelligent systems, 2nd international conference on information processing and management of uncertainty in knowledge-based systems*, Urbino, Italy, Jul. 1988, pp. 119–126.

[15] H. Steyaert, F. V. Parys, R. Baekeland, and E. E. Kerre, "Implementation of piecewise linear fuzzy quantities," *International journal of intelligent systems*, vol. 10, no. 12, pp. 1049–1059, 1995.