Nonlinear control using interval constraints propagation

Luc Jaulin,
Laboratoire E^3I^2
(Extraction et Exploitation de l'Information en Environnements Incertains)
ENSIETA, 2 rue François Verny, 29806 Brest Cédex 09

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Objective of our team: Promote interval methods and constraint propagation within the control community (build solvers, solve applications, . . .)

1 What is control theory ?

Many systems that can be represented a state space equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

where $\mathbf{x}$ is the state vector and $\mathbf{u}$ is the control vector.

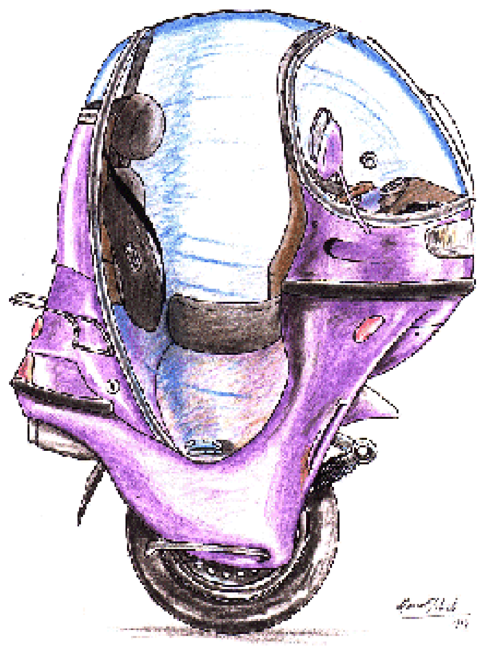
Control problem: Find a controller

$$\mathbf{u} = \mathbf{r}(\mathbf{x}, \mathbf{w}),$$

where $\mathbf{w}$ is the new input vector, such that the closed loop system behaves as desired.

More can be found on the book

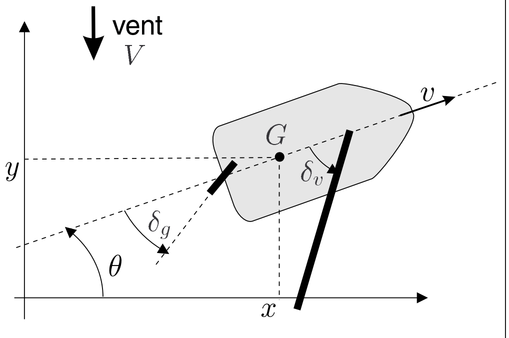
Jaulin L. (2005) « Représentation d'état pour la modélisation et la commande des systèmes » (Coll. Automatique de base), Hermes , 198p



2 Control of a sailboat

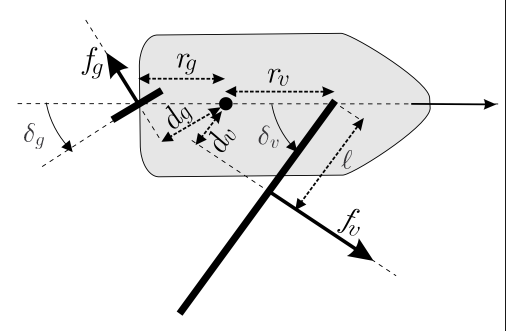
(Collaboration with M. Dao, M. Lhommeau, P. Herrero,
J. Vehi and M. Sainz).

Sailboat :



vent
V
v
G
y
δ_v
δ_g
θ
x

(a)



f_g
r_g
r_v
δ_g
d_g
d_v
δ_v
ℓ
f_v

(b)

$$
\begin{cases}
\dot{x} & = & v \cos\theta, \\
\dot{y} & = & v \sin\theta - \beta V, \\
\dot{\theta} & = & \omega, \\
\dot{\delta}_s & = & u_1, \\
\dot{\delta}_r & = & u_2, \\
\dot{v} & = & \frac{f_s \sin\delta_s - f_r \sin\delta_r - \alpha_f v}{m}, \\
\dot{\omega} & = & \frac{(\ell - r_s \cos\delta_s) f_s - r_r \cos\delta_r f_r - \alpha_\theta \omega}{J}, \\
f_s & = & \alpha_s \left(V \cos(\theta + \delta_s) - v \sin\delta_s \right), \\
f_r & = & \alpha_r v \sin\delta_r.
\end{cases}
$$

The state, input and chosen input vectors are

$$
\mathbf{x} = \begin{pmatrix} x \\ y \\ \theta \\ \delta_s \\ \delta_r \\ v \\ \omega \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \; \mathbf{w} = \begin{pmatrix} \bar{\theta} \\ \bar{v} \end{pmatrix}
$$

Polar speed diagram of a sailboat.

The set of feasible chosen input vectors is

$$\mathbb{W} = \left\{ \quad (\theta, v) \mid \quad \begin{array}{l} \exists (f_s, f_r, \delta_r, \delta_s) \\ 0 = \frac{f_s \sin \delta_s - f_r \sin \delta_r - \alpha_f v}{m} \\ 0 = \frac{(\ell - r_s \cos \delta_s) f_s - r_r \cos \delta_r f_r}{J} \\ f_s = \alpha_s \left(V \cos (\theta + \delta_s) - v \sin \delta_s \right) \\ f_r = \alpha_r v \sin \delta_r \end{array} \right\}.$$

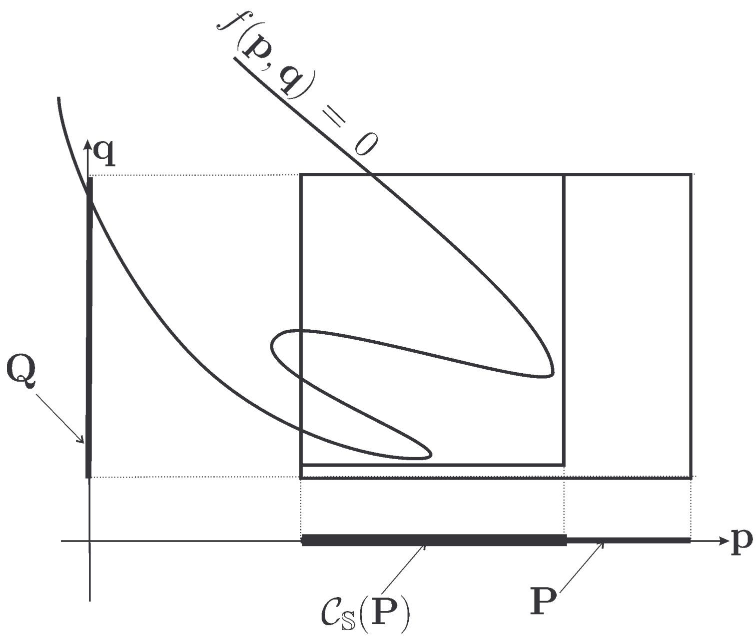
An elimination of f_s, f_r and δ_r yields

$$\begin{aligned}\mathbb{W} = \{ \quad & (\theta, v) \mid \\ & \exists \delta_s \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \\ & \left(\frac{(\alpha_r + 2\alpha_f)v + 2\alpha_s v \sin^2 \delta_s}{V} - 2\alpha_s \cos(\theta + \delta_s) \sin \delta_s \right)^2 \\ & + \left(\frac{2\alpha_s}{r_r} (\ell - r_s \cos \delta_s) \left(\cos(\theta + \delta_s) - \frac{v}{V} \sin \delta_s \right) \right)^2 \\ & - \alpha_r^2 \frac{v^2}{V^2} = 0 \ \}\end{aligned}$$

We shall now provide an algorithm to compute an inner and an outer approximation of the set

$$\mathbb{S} \triangleq \{\mathbf{p} \in \mathbf{P} \mid \exists \mathbf{q} \in \mathbf{Q}, f(\mathbf{p}, \mathbf{q}) = 0\}.$$

A contractor for $\mathbb{S}$ can be obtained using classical approaches.



We also need a contractor for $\neg\mathbb{S}$. Since f is continuous, we have

$$\left\{\begin{array}{lll} \mathbf{p}\in\mathbb{S} & \Leftrightarrow & (\exists\mathbf{q}\in\mathbf{Q}, f(\mathbf{p},\mathbf{q})=0) \\ & \Leftrightarrow & \left(\max\limits_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\geq 0\right)\wedge\left(\min\limits_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\leq 0\right). \end{array}\right.$$

If $\hat{\mathbf{q}}\in\mathbf{Q}$, we have

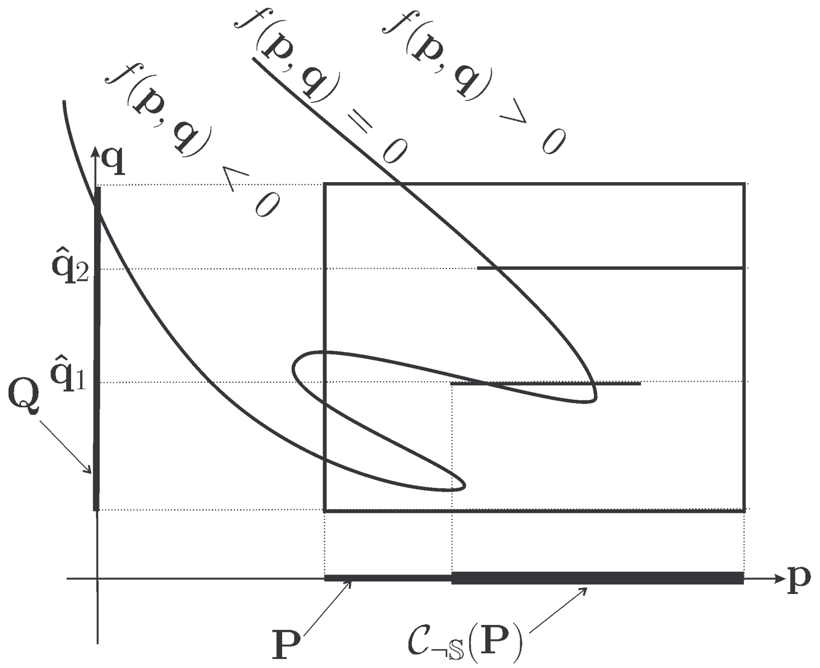
$$\begin{array}{lll} f(\mathbf{p},\hat{\mathbf{q}}) & \geq & 0\Rightarrow \max\limits_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\geq 0, \\ f(\mathbf{p},\hat{\mathbf{q}}) & \leq & 0\Rightarrow \min\limits_{\mathbf{q}\in\mathbf{Q}} f(\mathbf{p},\mathbf{q})\leq 0. \end{array}$$

Thus, if $\hat{\mathbf{q}}_1\in\mathbf{Q}$ and $\hat{\mathbf{q}}_2\in\mathbf{Q}$, we have

$$(f(\mathbf{p},\hat{\mathbf{q}}_1)\geq 0)\wedge(f(\mathbf{p},\hat{\mathbf{q}}_2)\leq 0)\Rightarrow\mathbf{p}\in\mathbb{S}.$$

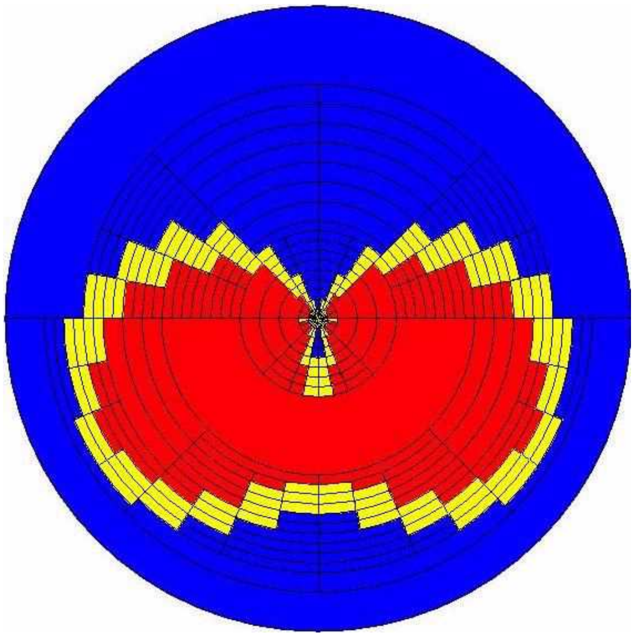
or its contraposite

$$\begin{aligned}\mathbf{p} &\in \neg\mathbb{S} \Rightarrow (f(\mathbf{p},\hat{\mathbf{q}}_1) < 0) \vee (f(\mathbf{p},\hat{\mathbf{q}}_2) > 0)\\ &\Leftrightarrow \min\left(f(\mathbf{p},\hat{\mathbf{q}}_1), -f(\mathbf{p},\hat{\mathbf{q}}_2)\right) < 0\end{aligned}$$



Algorithm Proj1Eq(in: $\mathbf{P}, \mathbf{Q}$)

1 $\mathcal{L} := \{\mathbf{P}\}$;
2 while $\mathcal{L} \neq \emptyset$,
3   pop a box $\mathbf{P}$ out of $\mathcal{L}$
4   Choose $\hat{\mathbf{q}}_1$ to maximize $f(\text{center}(\mathbf{P}),\hat{\mathbf{q}}_1)$
5   Choose $\hat{\mathbf{q}}_2$ to minimize $f(\text{center}(\mathbf{P}),\hat{\mathbf{q}}_2)$
6     $\mathbf{P}_a := \mathcal{C}_{\{\mathbf{p} \mid \min(f(\mathbf{p},\hat{\mathbf{q}}_1),-f(\mathbf{p},\hat{\mathbf{q}}_2))<0\}}(\mathbf{P})$
7   Paint $\mathbf{P}/\mathbf{P}_a$ red; $\mathbf{P} := \mathbf{P}_a$
8     $\mathbf{P}_b \times \mathbf{Q}_b := \mathcal{C}_{\{(\mathbf{p},\mathbf{q}) \mid f(\mathbf{p},\mathbf{q})=0\}}(\mathbf{P} \times \mathbf{Q})$
9   Paint $\mathbf{P}/\mathbf{P}_b$ blue; $\mathbf{P} := \mathbf{P}_b$
10   if $w([\mathbf{p}]) < \varepsilon$, paint $\mathbf{P}$ yellow and goto 2
11   bisect $\mathbf{P}$ and store the two boxes into $\mathcal{L}$;
12 end while.



This picture has been obtained using modal interval techniques with binary contractors.

Control of the sailboat

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

For specific vector $\mathbf{y} = \mathbf{g}(\mathbf{x})$, feedback linearization methods make it possible to find a controller of the form

$$\mathbf{u} = \mathcal{R}_u(\mathbf{x}, \bar{\mathbf{y}}),$$

such that $\mathbf{y}$ converges to $\bar{\mathbf{y}}$. Now, the user wants to choose its own input vector $\mathbf{w} = \mathbf{h}(\mathbf{x})$. The problem of interest is to find a controller

$$\mathbf{u} = \mathcal{R}_w(\mathbf{x}, \bar{\mathbf{w}})$$

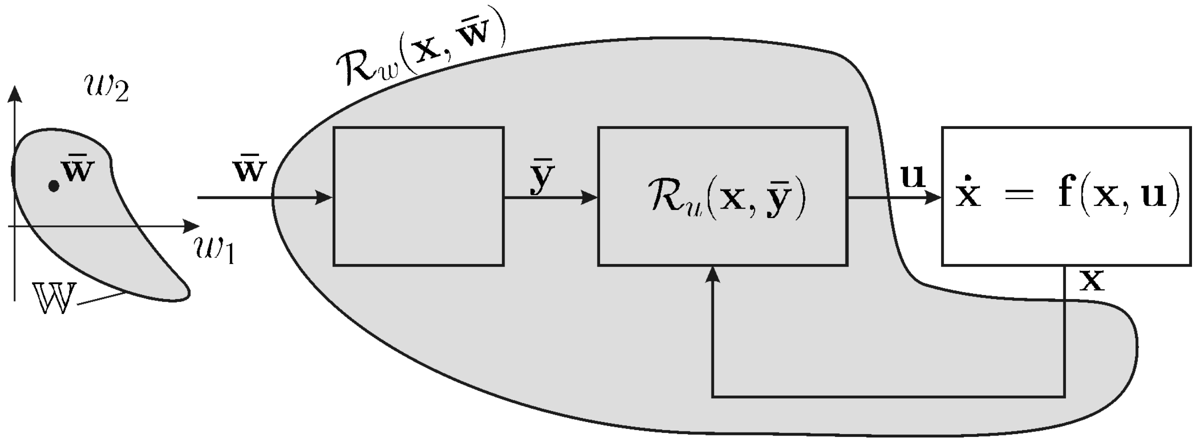
such that $\mathbf{w}$ converges to $\bar{\mathbf{w}}$.

The set of feasible chosen inputs is

$$\mathbb{W} = \{\mathbf{w} \in \mathbb{R}^m | \exists \mathbf{x} \in \mathbb{R}^n, \exists \mathbf{u} \in \mathbb{R}^m, \\ \mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{0}, \mathbf{w} = \mathbf{h}(\mathbf{x}) \}$$

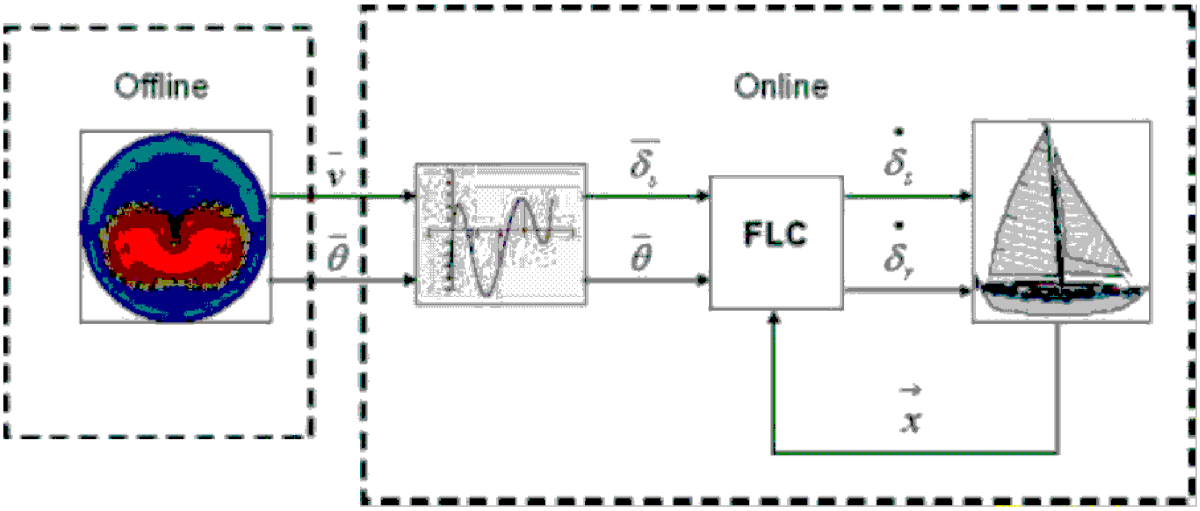
When $\dim \mathbf{u} = \dim \mathbf{w}$, the set $\mathbb{W}$ has a nonempty volume.

The user chooses $\bar{\mathbf{w}}$ inside $\mathbb{W}$. We compute first $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ such that $\mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \mathbf{0}, \bar{\mathbf{w}} = \mathbf{h}(\bar{\mathbf{x}})$. Then we compute $\bar{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}})$. The controller $\mathcal{R}_u(\mathbf{x}, \bar{\mathbf{y}})$ will get $\mathbf{u}$ such that $\mathbf{y}$ converges to $\bar{\mathbf{y}}$. Thus, $\mathbf{x}$ will tend to $\bar{\mathbf{x}}$ and $\mathbf{w}$ to $\bar{\mathbf{w}}$.



For $\mathbf{y} = (\boldsymbol{\delta}_s, \theta)$, the feedback linearization method leads to the controller

$$\mathbf{u} = \mathcal{R}_u(\mathbf{x}, \bar{\mathbf{y}}) = \mathcal{R}_u(\mathbf{x}, \hat{\delta}_s, \hat{\theta}).$$



Offline
Online
$\bar{v}$
$\bar{\theta}$
$\bar{\delta}_s$
$\bar{\theta}$
FLC
$\dot{\delta}_s$
$\dot{\delta}_r$
$\vec{x}$

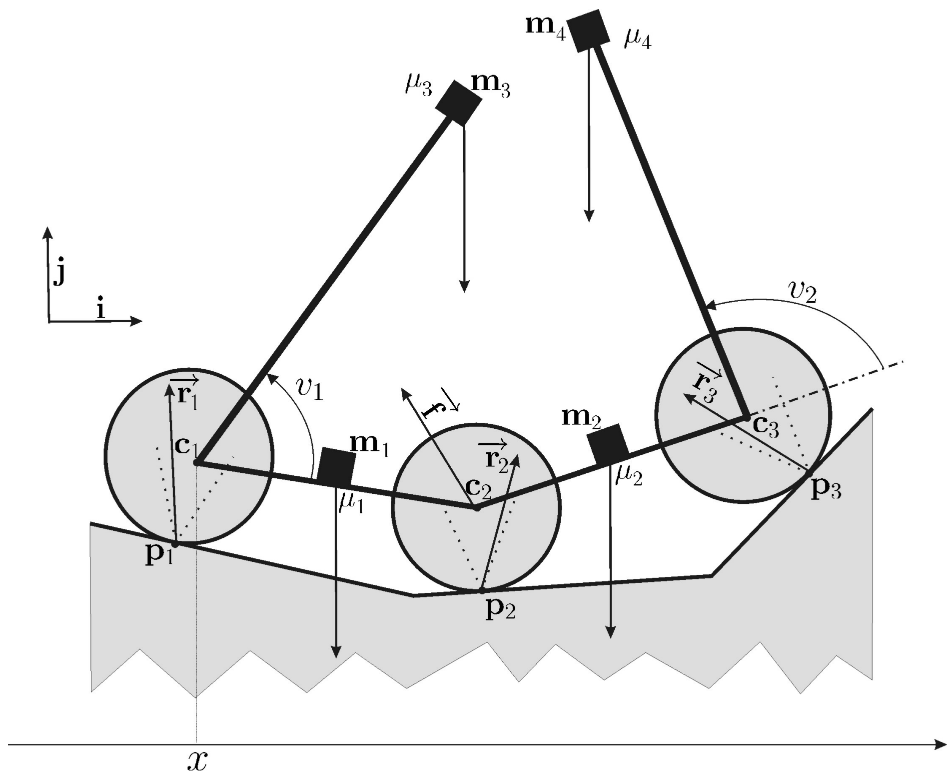
3 Control of a wheeled stair-climbing robot

(Collaboration with students and colleagues from EN-SIETA)

Consider the class of constrained dynamic systems:

$$\begin{array}{ll} \text{(i)} & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \text{(ii)} & (\mathbf{x}(t), \mathbf{v}(t)) \in \mathbb{V}, \end{array}$$

where $\mathbf{v}(t) \in \mathbb{R}^{n_v}$ is the *viable input vector* and $\mathbb{V}$ is the *viable set*.



Assume that the robot has a quasi-static motion.

1) When the robot does not move, we have

$$\left\{ \begin{array}{rcl} -\overrightarrow{\mathbf{p_1 m_1}} \wedge \mu_1 \mathbf{j} + \overrightarrow{\mathbf{p_1 c_2}} \wedge \overrightarrow{\mathbf{f}} - \overrightarrow{\mathbf{p_1 m_3}} \wedge \mu_3 \mathbf{j} & = & 0 \\ -\overrightarrow{\mathbf{p_2 m_2}} \wedge \mu_2 \mathbf{j} - \overrightarrow{\mathbf{p_2 c_2}} \wedge \overrightarrow{\mathbf{f}} + \overrightarrow{\mathbf{p_2 p_3}} \wedge \overrightarrow{\mathbf{r}}_3 & & \\ -\overrightarrow{\mathbf{p_2 m_4}} \wedge \mu_4 \mathbf{j} & = & 0 \\ \overrightarrow{\mathbf{r}}_1 - (\mu_1 + \mu_3)\mathbf{j} + \overrightarrow{\mathbf{f}} & = & 0 \\ \overrightarrow{\mathbf{r}}_2 - \overrightarrow{\mathbf{f}} - (\mu_2 + \mu_4)\mathbf{j} + \overrightarrow{\mathbf{r}}_3 & = & 0, \end{array} \right.$$

This system can be written into a matrix form as

$$\mathbf{A}_1(x).\mathbf{y} = \mathbf{b}_1(x),$$

where

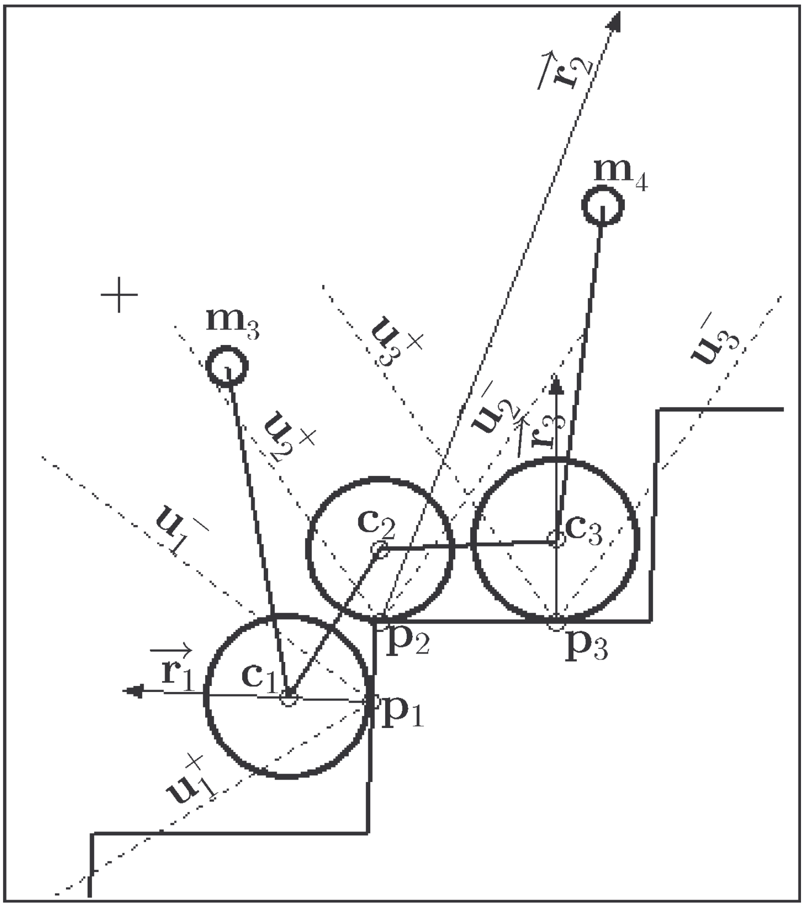
$$\mathbf{y} \;=\; \left(r_{1x}, r_{1y}, r_{2x}, r_{2y}, r_{3x}, r_{3y}, f_x, f_y, m_{3x}, m_{4x}\right)^{\mathsf{T}}.$$

2) None of the wheels will slide if all $\overrightarrow{\mathbf{r}}_i$ belong to their corresponding Coulomb cones:

$$\det(\overrightarrow{\mathbf{r}}_i, \mathbf{u}_i^-) \leq 0 \text{ and } \det(\mathbf{u}_i^+, \overrightarrow{\mathbf{r}}_i) \leq 0,$$

where $\mathbf{u}_i^-$ and $\mathbf{u}_i^+$ denote the two vectors supporting the ith Coulomb cone $\mathcal{C}_i$. These inequalities can be rewritten into

$$\mathbf{A}_2(x).\mathbf{y} \leq \mathbf{0}.$$

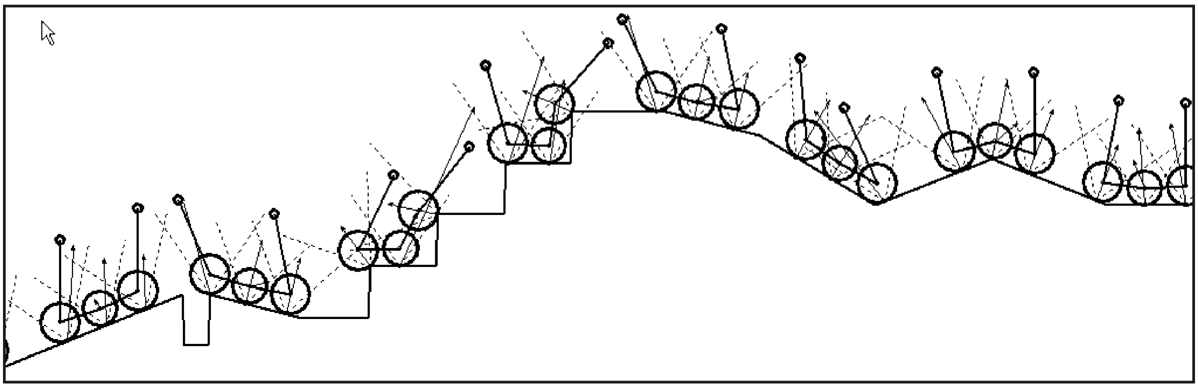


r2
m4
+
m3
u3+
u3−
u2−
r3
u2+
c2
c3
u1−
p2
p3
r1
c1
p1
u1+

3) There is a relation between $\mathbf{y}$ and $\mathbf{v}$ of the form $\mathbf{v} = \mathbf{c}(\mathbf{y})$.

Finally,

$$
\begin{aligned}
\mathbf{A}_1(x).\mathbf{y} &= \mathbf{b}_1(x) \\
\mathbf{A}_2(x).\mathbf{y} &\leq \mathbf{0}. \\
\mathbf{v} &= \mathbf{c}(\mathbf{y})
\end{aligned}
$$



The figure below represents the robot built by the robotics team of the ENSIETA engineering school that has won the 2005 robot cup ETAS. The robot can be seen as a three-dimensional version of the robot treated above. It has been proven to be very competitive on irregular grounds but failed to cross over some compulsory obstacles (such as stairs).

If you want to learn more on 'interval for control', come to the summer school on september 12-16, 2005 in Grenoble.