# GT "Calcul Ensembliste"

## Beyond the bounded error framework for non linear state estimation

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# Outline

# Introduction

## State space model

The state-space model can be broken down into a state transition and state measurement model :

$$\begin{cases} \mathbf{x}_{k+1} = & f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k) \\ \mathbf{z}_{k+1} = & g(\mathbf{x}_{k+1}, \mathbf{w}_{k+1}) \end{cases}$$

where $\mathbf{x}_k$ is the state at time $k$, $\mathbf{z}_k$ is the measure, $\mathbf{u}_k$ the input, $\mathbf{v}_k$ and $\mathbf{w}_k$ are models noises.

## Different aspects in a state estimation problem

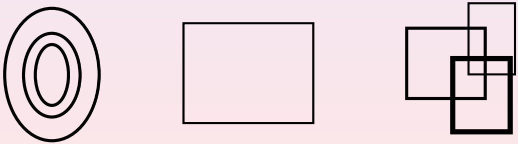
- Data représentation : Depends on the nature of the available data + Nature of the desirable information (guaranteed or gaussian shaped...)
- Data fusion
- Data filtering

**The choice of the estimation method depends on the application to handle**

# Some of data-modelization types

- Probabilistic modelization
- Bounded error modelization
- Modelization using belief functions theory. If $\Omega$ is *the frame of discernment* of the state. **The belief relatively to a given problem** can be quantified via a belief mass function $m_S^{\Omega}$, where

$$m_S^{\Omega} : 2^{\Omega} \longrightarrow [0,1]$$
$$B \longrightarrow m_S^{\Omega}(B)$$



# Some of belief functions combination rules

## Conjonctive rule

- If $m_1$ and $m_2$ are two items of evidence, then their conjonctive combination is given by

$$(m_1 \bigcirc\!\!\!\!\cap\, m_2)(C) = \sum_{\{i,j|A_i \cap B_j = C\}} m_1(A_i)m_2(B_j), \quad \forall C \subseteq \Omega.$$

## Disjonctive rule

- If $m_1$ and $m_2$ are two items of evidence, then their disjonctive combination is given by

$$m_1^\Omega \bigcirc\!\!\!\!\cup\, m_2^\Omega(D) = \sum_{B \cup C = D} m_1^\Omega(B) m_2^\Omega(C).$$

## Probabilistic methods

- **Goal :** Calculate the a posteriori probability $p(\mathbf{x}_{k+1}|\mathbf{z}_{k+1})$ by using :

$$\begin{cases} p(\mathbf{x}_{k+1}|\mathbf{x}_k) & \textbf{Transition model or prediction} \\ p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) & \textbf{Likelihood model} \end{cases}$$

A. Gning, L. Mihaylova, F. Abdallah.
Mixture of Uniform Probability Density Functions for non Linear State Estimation using Interval Analysis.
*Fusion 2010 EICC Edinburgh, UK*, 26-29 July 2010.

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## Bounded errors methods

- **Goal :** Calculate a bounded domain $[\mathbf{x}_{k+1}]$ which contains the state to estimate :

$$[\mathbf{x}_{k+1}] = \mathcal{C}\left([f], [g], [\mathbf{x}_k], [\mathbf{u}_k], [\mathbf{z}_{k+1}]\right) \quad \textbf{where } \mathcal{C} \textbf{ is a set of constraints}$$

F. Abdallah, A. Gning, and P. Bonnifait.
Box particle filtering for nonlinear state estimation using interval analysis.
*Automatica*, 44(3) :807–815,2008.

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## Belief functions theory based methods

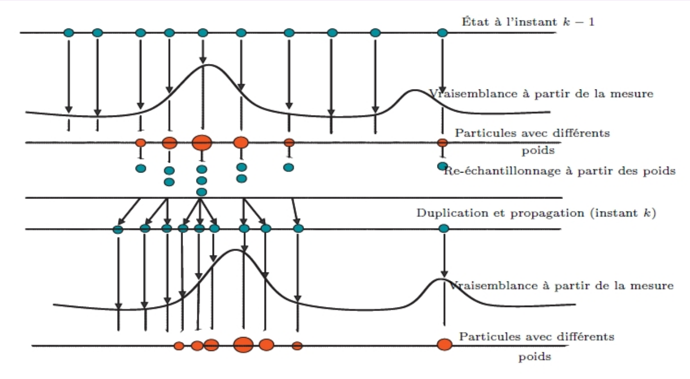
- **Goal :** Calculate a mass function $m^{\mathbf{x}}_{k+1}$ on the state to estimate :

$$m^{\mathbf{x}}_{k+1}([\mathbf{x}]) = \sum_{\{i,j,\ell|[\mathbf{x}]=\mathcal{C}\left(f,g,[\mathbf{x}^i_k],[\mathbf{u}^j_k],[\mathbf{z}^\ell_{k+1}]\right)\}} m^{\mathbf{x}}_k([\mathbf{x}^i_k]) \cdot m^{\mathbf{u}}_k([\mathbf{u}^j_k]) \cdot m^{\mathbf{z}}_{k+1}([\mathbf{z}^\ell_{k+1}])$$

G. Nassreddine, F. Abdallah and T. Denoeux.
State estimation using interval analysis and belief function theory : Application to dynamic vehicle localization.
*IEEE Transactions on Systems, Man and Cybernetics B*, 40(5) :1205–1218,2010.

# Sketch of the particle filter

- The Extended Kalman Filter (EKF)is one of the most used approaches in data fusion problems for non-linear state estimation but...
- A particle filter implements the Bayesian filtering using a set of particles simulating the a posteriori probability of the state to estimate



# Comments on particle filter

- Particle Filters and Sequential Monte Carlo Methods are an evolving and active topic, with good potential to handle hard estimation problems, involving non-linearity and multi-modal distributions.
- In general, these schemes are computationally expensive as the number of particles N needs to be large for precise results.
- Additional work is required on the computational issue, in particular, optimizing the choice of N, and related error bounds.
- Another promising direction is the merging of sampling methods with more disciplined approaches (such as Gaussian filters, and the Rao-Blackwellization scheme...).

# Interval Analysis concepts

## Different aspects of Interval Analysis

- Initially introduced to propagate rounding errors in mathematical computations (earlier in the fifties).
- The goal of interval analysis : to deal with intervals of real numbers instead of real numbers
    - Elementary arithmetic operations, e.g., $+, -, *, \div$, etc., as well as operations between sets of $\mathbb{R}^n$, such as $\subset, \supset, \cap, \cup$, etc., have been naturally extended to interval analysis context
    - An important concept of interval analysis is the so called *inclusion functions*.

# Interval Analysis concepts

## Inclusion Functions

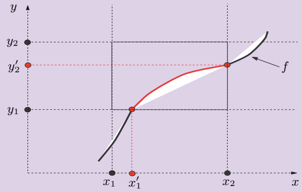
- $[f]$ is an *inclusion function* for $f$ if the image of a box $[\mathbf{x}]$ is a box $[f]([\mathbf{x}]) \supseteq f([\mathbf{x}])$
- The goal is to work only with boxes
- To optimise the interval enclosing the real image set = to decrease the pessimism when boxes are propagated.

# Interval Analysis concepts

## Contraction Operation

- Other interval methods intensively studied to solve equations are contraction operators
- Popular methods : constraint propagation techniques, Krawczyk methods, interval Newton techniques...

**Domain $x \times y$ reduction by using the constraint $f$**



# Interval Analysis concepts

## Example

Let $z = x.\exp(y)$. The initial domain is defined by $[z] = [0,3]$, $[x] = [1,7]$ and $[y] = [0,1]$.

- We decompose the initial constraint into primitives. Thus the constraint $z = x.\exp(y)$ is decomposed into $a = \exp(y)$ and $z = x.a$, where $a$ is an auxiliary variable initialized at $[a] = [0, +\infty[$.
- We use the principle of constraint propagation
- We use inclusion functions, [exp] and $[(\exp)^{-1}] = [\ln]$.

# Interval Analysis concepts

## Example

Let $z = x.\exp(y)$. The initial domain is defined by $[z] = [0, 3]$, $[x] = [1, 7]$ and $[y] = [0, 1]$.

- $[a] = [a] \cap [\exp]([y]) = [1, e]$
- $[z] = [z] \cap [x].[a] = [1, 3]$
- $[x] = [x] \cap ([z]/[a]) = [1/e, 3]$
- $[a] = [a] \cap ([z]/[x]) = [1/3, 3e]$
- $[y] = [y] \cap [\ln]([a]) = [0, 1]$

The variables domain will be reduced to $[z] = [1, 3]$, $[x] = [1/e, 3]$ and $[y] = [0, 1]$.

**If we have several constraints with several commun variables !**
**$\Longrightarrow$ Use forward and backward propagation techniques**

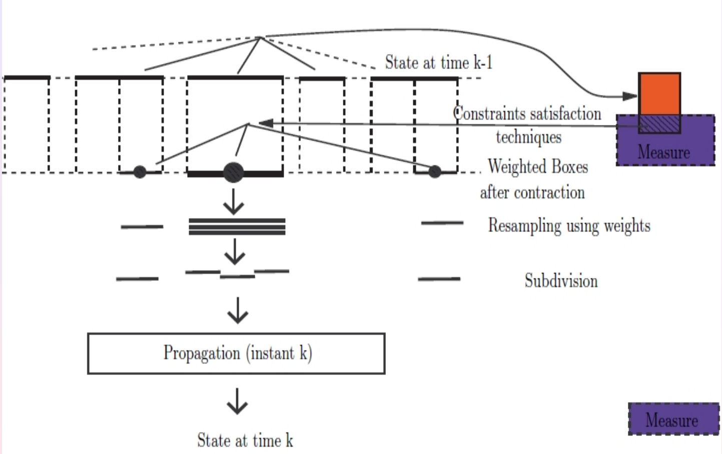
# State estimation under interval representation

## Motivations

1. With the hypothesis that we know only the bounds of the different errors or noises, it seems adequate to represent different informations using intervals
2. An interval represents infinitely many particles continuously distributed throughout the interval ;

3. An interval represents a particle imprecisely located in the interval.

# State estimation under interval representation

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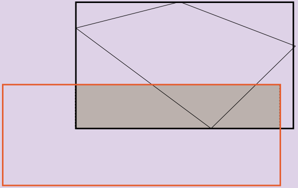
# State estimation under interval representation

## Forward and backward propagation

- Techniques de satisfaction de contraintes

$$[\mathbf{x}_{k+1}] = \mathcal{C}\left([f], [g], [\mathbf{x}_k], [\mathbf{u}_k], [\mathbf{z}_{k+1}]\right) \quad \textbf{where } \mathcal{C} \textbf{ is a set of constraints}$$



# State estimation under interval representation

## Forward and backward propagation

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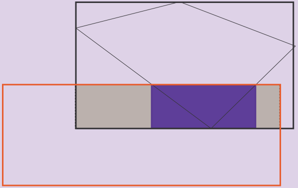


# Application : Localization of a land vehicle

## The model

- **The state $\mathbf{x}_k = x_k \times y_k \times \theta_k$ constituted by the position and the heading angle of the vehicle at time instant** $k$, where :

$$\begin{cases} [x_{k+1}] = & [x_k] + [\delta_{S,k}][\cos]([\theta_k] + \frac{[\delta_{\theta,k}]}{2}) \\ [y_{k+1}] = & [y_k] + [\delta_{S,k}][\sin]([\theta_k] + \frac{[\delta_{\theta,k}]}{2}) \\ [\theta_{k+1}] = & [\theta_k] + [\delta_{\theta,k}] \end{cases}$$

- **The measurement consists in 2D position provided by a Global Position System (GPS)** which is :

$$\begin{cases} [x_{GPS}] = & [x_{GPS} - 3\sigma_{GPS}, x_{GPS} + 3\sigma_{GPS}] \\ [y_{GPS}] = & [y_{GPS} - 3\sigma_{GPS}, y_{GPS} + 3\sigma_{GPS}] \end{cases}$$

# Application : Localization of a land vehicle

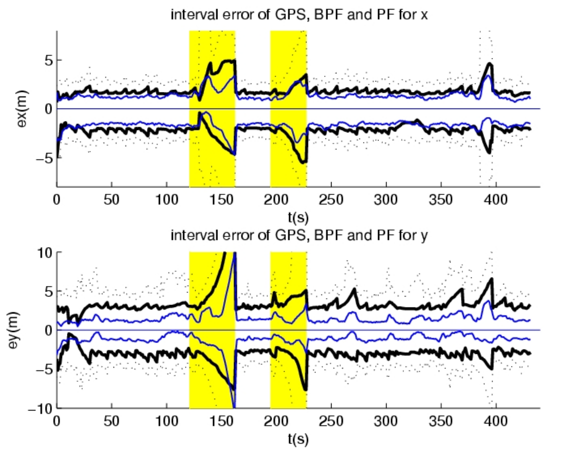
## Comparison of PF and BPF

| | GPS | PF | BPF |
|---|---|---|---|
| mean square error for x(m) | 0.134 | 0.129 | 0.119 |
| mean square error for y(m) | 0.374 | 0.217 | 0.242 |
| particles number | - | 3000 | 10 |
| one step running time (ms) | - | 666 | 149 |
| mean square error for $\theta$(degrees) | - | 0.446 | 0.445 |

# Application : Localization of a land vehicle

## Comparison of PF and BPF

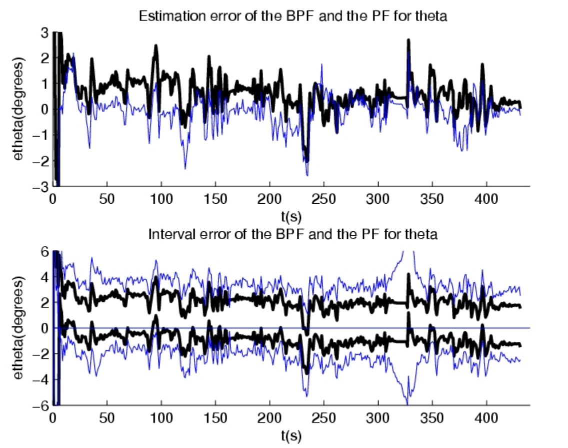
- **The figures show the interval error for x and y estimated for GPS (dashed black), BPF (bold black) and PF (solid blue).**



# Application : Localization of a land vehicle

## Comparison of PF and BPF

- **The figures show the estimated heading error and the interval errors, in degrees, for BPF (bold black) and PF (solid blue).**



# Illustration of the BPF on Real Data

F. Abdallah, A. Gning, and P. Bonnifait.
Box particle filtering for nonlinear state estimation using interval analysis.
*Automatica*, 44(3) :807–815,2008.

# BPF Interpretation as a Bayesian Filter

## Main Objectives

- To derive convergence properties
- To derive more sophisticated procedures (the time update step and the measurement update step)
- To evaluate the number of box particles needed to estimate the posterior for a given problem

# BPF Interpretation as a Bayesian Filter

## Possible interpretation of a box particle

- Infinite number of particles continuously distributed within the box
- One particle imprecisely located within the box

## Interpretation of interest in our case

- Infinite number of particles continuously distributed within the box
- Each box models the support of an unknown pdf
- The uniform pdf is a candidate that fits well with the BPF formulation
  - A common interpretation for a box enclosing a solution set : all possible points inside the box have the same probability of belonging to the solution set.

# State estimation Using a Mixture of Uniform PDFs

## Mixture of uniform PDFs with box supports advantages

- Mixtures of uniform pdfs with box supports belong to piecewise constant functions family
  - Natural simplicity
  - Basis of Riemann integration theory
  - This family is dense in the space of continuous functions.
- With box supports, interval analysis theory offers variety of tools to propagate box supports through linear or non linear functions and even through differential equations.

# Bayesian Inference

## Goal of Bayesian methods for state estimate

- Calculate the a posteriori probability of the state $p(\mathbf{x}_{k+1}|\mathbf{z}_{k+1})$

## Steps

- Measurement update step :

$$p(\mathbf{x}_{k+1}|\mathbf{Z}_{k+1}) = \frac{1}{\alpha_{k+1}} p(\mathbf{x}_{k+1}|\mathbf{Z}_k) p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}),$$

- Time update step :

$$p(\mathbf{x}_{k+1}|\mathbf{Z}_k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k) p(\mathbf{x}_k|\mathbf{Z}_k) d\mathbf{x}_k,$$

where $\mathbf{x}_k$ is the state at time instant $k$, $\mathbf{Z}_k = \{\mathbf{z}_i, i = 1, \cdots, k\}$ the measurements up to time $k$. $\alpha_{k+1} = \int p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) p(\mathbf{x}_{k+1}|\mathbf{Z}_k) d\mathbf{x}_{k+1}$ is a normalisation factor.

# Time Update Step

- Time update step :

$$p(\mathbf{x}_{k+1}|\mathbf{Z}_k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Z}_k)d\mathbf{x}_k,$$

- At time instant $k$, an approximation for the pdf $p(\mathbf{x}_k|\mathbf{Z}_k)$ by a mixture of $l_k$ uniform pdfs with box supports $[\mathbf{x}_\mathbf{k}^{(\mathbf{i})}]$ is available

$$p(\mathbf{x}_k|\mathbf{Z}_k) = \sum_{i=1}^{l_k} w_k^{(i)} U_{[\mathbf{x}_k^{(i)}]}(\mathbf{x}_k)$$

- The prediction step is to be performed : a new approximation using the above mixture of uniform pdfs.

# Time Update Step

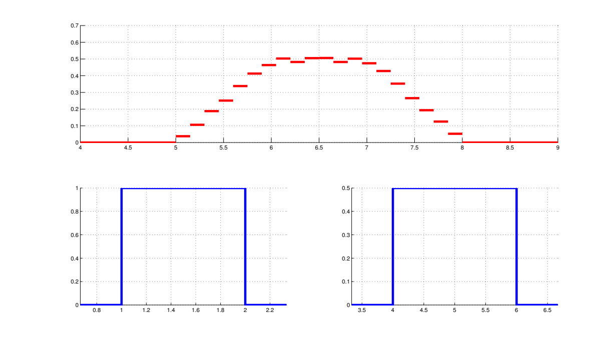
$$p(\mathbf{x}_{k+1}|\mathbf{Z}_k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k) \sum_{i=1}^{l_k} w_k^{(i)} U_{[\mathbf{x}_k^{(i)}]}(\mathbf{x}_k) d\mathbf{x}_k$$
$$= \sum_{i=1}^{l_k} w_k^{(i)} \int p(\mathbf{x}_{k+1}|\mathbf{x}_k) U_{[\mathbf{x}_k^{(i)}]}(\mathbf{x}_k) d\mathbf{x}_k$$

- For $i = 1, \ldots, l_k$, we have :

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) U_{[\mathbf{x}_k^i]}(\mathbf{x}_k) = 0, \quad \forall\, \mathbf{x}_{k+1} \notin [f]([\mathbf{x}_k^i], [\mathbf{v}_{k+1}]).$$

- Using interval analysis techniques, the support for the pdf terms $\int p(\mathbf{x}_{k+1}|\mathbf{x}_k) U_{[\mathbf{x}_k^{(i)}]}(\mathbf{x}_k) d\mathbf{x}_k$ can be approximated by $[f]([\mathbf{x}_\mathbf{k}^{(\mathbf{i})}], [\mathbf{v}_{k+1}])$
- It can be seen that, in the BPF algorithm, each pdf term $\int p(\mathbf{x}_{k+1}|\mathbf{x}_k) U_{[\mathbf{x}_k^{(i)}]}(\mathbf{x}_k) d\mathbf{x}_k$ is modelled using one uniform pdf component having as support the interval $[f]([\mathbf{x}_\mathbf{k}^{(\mathbf{i})}], [\mathbf{v}_{k+1}])$
- This modelization may be note not enough accurate for some cases

# Time Update Step



# Time Update Step

## Solution refinement

- We show that the true prediction pdf can be estimated by uniform pdfs using interval analysis
  - Convergence results are proven using two lemmas
  - The basic idea is to first subdivide the box support of the prediction pdf $p(\mathbf{x}_{k+1}|\mathbf{Z}_k)$, then to estimate and associate probability for each component
  - These probabilities are also obtained by subdividing the prior boxes, $\mathbf{x}_k$, from the previous time step
- A more general version can be given, where we represent the noise $\mathbf{v}_{k+1}$ pdf by a mixture of uniform pdfs : $p(\mathbf{v}_{k+1}) = \sum_{i=1}^{q_k} \lambda_{k+1}^{(i)} U_{[\mathbf{v}_{k+1}^{(i)}]}(\mathbf{v}_{k+1})$ (with only one component if the only information about the noise is that about the bounds).
- Using the representation with uniform pdfs, the BPF time update step can be refined at a computational time expense

# Measurement Update Step

- Measurement update step :

$$p(\mathbf{x}_{k+1}|\mathbf{Z}_{k+1}) = \frac{1}{\alpha_{k+1}} p(\mathbf{x}_{k+1}|\mathbf{Z}_k) p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})$$

- At time instant $k+1$, an approximation of the time update pdf $p(\mathbf{x}_{k+1}|\mathbf{Z}_k)$ by a mixture of $l_{k+1|k}$ uniform pdfs with interval supports $[\mathbf{x}^{(\mathbf{i})}_{\mathbf{k+1|k}}]$ and weights $w_{k+1|k}$ is available

$$p(\mathbf{x}_{k+1}|\mathbf{Z}_k) = \sum_{i=1}^{l_{k+1|k}} w^{(i)}_{k+1|k} U_{[\mathbf{x}^{(i)}_{k+1|k}]}(\mathbf{x}_{k+1})$$

- Assume that the likelihood function is expressed as a uniform pdf mixture, with $s_{k+1}$ components weighted with the normalised coefficient $\beta^{(j)}_{k+1}$

$$p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) = \sum_{j=1}^{s_{k+1}} \beta^{(j)}_{k+1} U_{[\mathbf{z}^{(j)}_{k+1}]}(g(\mathbf{x}_{k+1}))$$

# Measurement Update Step

## Final result

- It is shown that the measurement update has the expression

$$p(\mathbf{x}_{k+1}|\mathbf{Z}_{k+1}) = \frac{1}{\alpha_{k+1}} \sum_{j=1}^{s_{k+1}} \sum_{i=1}^{l_{k+1|k}} \beta_{k+1}^{(j)} w_{k+1|k}^{(i)} U_{[\mathbf{z}_{k+1}^{(j)}]}(g(\mathbf{x}_{k+1})) U_{[\mathbf{x}_{\mathbf{k+1|k}}^{(\mathbf{i})}]}(\mathbf{x}_{k+1})$$

## Interpretation

- The term $U_{[\mathbf{z}_{k+1}^{(j)}]}(g(\mathbf{x}_{k+1})) U_{[\mathbf{x}_{\mathbf{k+1|k}}^{(\mathbf{i})}]}(\mathbf{x}_{k+1})$ is also a constant function with a support being the set
  $\{\mathbf{x}_{\mathbf{k+1}} \in [\mathbf{x}_{\mathbf{k+1|k}}^{(\mathbf{i})}] \mid \exists\, \mathbf{w}_{k+1} \in [\mathbf{w}_{k+1}] \text{ such that } g(\mathbf{x}_{k+1}, \mathbf{w}_{k+1}) \in [\mathbf{z}_{k+1}^{(j)}]\}$.
- This means that using contraction operators, the predicted supports $[\mathbf{x}_{\mathbf{k+1|k}}^{(\mathbf{i})}]$ from the time update pdf $p(\mathbf{x}_{k+1}|\mathbf{Z}_k)$ approximation have to be contracted with respect to the new measurement

# Measurement Update Step

## Interesting properties

- These contraction steps give the new support for the posterior pdf $p(\mathbf{x}_{k+1}|\mathbf{Z}_{k+1})$ at time instant $k+1$
- The contraction steps which have been heuristically introduced in the BPF are derived theoretically using the uniform pdfs representation !

# Conclusion and perspectives

- Several manner to handle state estimation problem
- A state estimate filter, BPF, based on interval data representation was introduced
- The BPF algorithm is studied as a Bayesian method through an interpretation using mixtures of uniform pdfs with box supports
- Theoretical justifications of the BPF procedures are shown
- In addition, it is proven that mixture of uniform pdfs can be used to approximate the posteriors pdf for a state estimation problem
- How to determine the number of uniforms involved in the approximation ? ...Reversible jump Metropolis-Hastings ? Merging(to combine closed uniforms...
- Also, should us look at a good proposal distribution ?
- If we know that a posteriori is bi-modal or...