Set-membership identifiability: definitions and analysis

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Outline

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   - Motivation
   - Definitions
   - Links with classical definitions
2. **Methods to analyse set-membership identifiability**
   - Linear uncertain system
   - Nonlinear uncertain system
3. **Conclusion**

- Identifiability of linear and non linear systems

- Identifiability of linear and non linear systems
- What about uncertain systems?
  - The system has constant parameters but the knowledge about the parameter values is uncertain:
    - corresponds to the plus/minus tolerance value provided by the builder of physical device parameters.
    - The study of such systems can be brought back to the study of *a family of constant parameter systems*.
  - Case for which parameter uncertainty comes from the fact that parameters may vary across time.
    - This case is typical of devices that operate in different environmental conditions, which may affect parameter values.

In this work: only the first situation is considered.

- Why not use the set-membership methods?
  - Subject of a growing interest in various communities and applied for many tasks (for example: fault detection, diagnosis).
  - A lot of works on set-membership (state, parameters) estimations.
  - To our knowledge: no existing definition and method for the *identifiability problem of error-bounded uncertain models*.

Two definitions of global set-membership identifiability are provided:

- a conceptual definition,
- a definition relying on a measure μ (can be put in correspondence with operational set-membership estimation methods).

We consider the uncertain system:

$$\Gamma_1^P = \begin{cases} \dot{x}(t,p) = f(x(t,p),p) + u(t)g(x(t,p),p), \\ y(t,p) = h(x(t,p),p), \\ x(t_0,p) = x_0 \in X_0, \\ p \in P \subset \mathcal{U}_{\mathcal{P}}, \\ t_0 \leq t \leq T, \end{cases} \tag{1}$$

where:

- $x(t,p) \in \mathbb{R}^n$: state variables at time t,
- $y(t,p) \in \mathbb{R}^m$: outputs at time t,
- $u(t) \in \mathbb{R}^r$: input vector at time t,
- $x_0 \in X_0$, X_0: a bounded set,
- f, g, h: real functions, analytic on M (an open set of $\mathbb{R}^n$),
- $p \in P \subset \mathcal{U}_{\mathcal{P}}$: vector of parameters, $\mathcal{U}_{\mathcal{P}} \subset \mathbb{R}^p$: an a priori known set of admissible parameters.

Conceptual definitions

<u>Notation:</u> $Y(P,u)$ (respectively $Y(P)$): the set of outputs, solution of Γ_1^P with the input u (resp. when $u=0$)

- Global set-membership identifiability

Definition: Case of controlled systems

The model Γ_1^P given by (1) is globally set-membership identifiable for $P^* \neq \emptyset$, $P^* \subset \mathcal{U}_{\mathcal{P}}$ if there exists an input u such that $Y(P^*,u) \neq \emptyset$ and
$Y(P^*,u) \cap Y(\bar{P},u) \neq \emptyset,\ \bar{P} \subset \mathcal{U}_{\mathcal{P}} \Longrightarrow P^* \cap \bar{P} \neq \emptyset.$

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- Local set-membership identifiability

Definition

The model Γ_1^P given by (1) is locally set-membership identifiable for P^*, if there exists an open neighbourhood W of P^* in which Γ_1^P is globally set-membership identifiable for P^* with $\mathcal{U}_{\mathcal{P}}$ restricted to W.

μ-set-membership identifiability

Let us now consider a bounded set Π of $\mathbb{R}^p$.
$\mu(\Pi) =$*diameter* of Π.
$\mu(\Pi) =$ the least upper bound of $\{d(\pi_1,\ \pi_2),\ \pi_1, \pi_2 \in \Pi\}$, with d a classical metric on $\mathbb{R}^p$. If Π is not bounded, $\mu(\Pi) = +\infty$.

Definition

The model Γ_1^P given by (1) is globally μ-set-membership identifiable for $P^* \neq \emptyset$, $\mu(P^*)$ as small as possible, if there exists an input u such that $Y(P^*, u) \neq \emptyset$ and
$Y(P^*, u) \cap Y(\bar{P}, u) \neq \emptyset,\ \bar{P} \subset \mathcal{U}_{\mathcal{P}} \Longrightarrow P^* \cap \bar{P} \neq \emptyset$.

If $\mu(P^*) \geq \varepsilon$, then we refer to *$\varepsilon$-set-membership identifiability*.
$\Rightarrow$ Practical importance of ε-set-membership identifiability.

- Extension to the *structural μ-set-membership identifiability* (Γ_1^P is μ-set-membership identifiable for all $P \in \mathcal{U}_\mathcal{P}$ except at a subset of points of zero measure in $\mathcal{U}_\mathcal{P}$)
- Extension to local μ-set-membership identifiability for P^*

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- Extension to local μ-set-membership identifiability for P^*

Proposition

(Structural) global μ-set-membership identifiability for P^* implies global set-membership identifiability for P^* but the inverse is not true.

If the model (1) is neither global (μ-)set-membership identifiable nor local (μ-)set-membership identifiable, it is said *non (μ-)set-membership identifiable*.

Structural global (local) identifiability

$$\Gamma_1^P = \begin{cases} \dot{x}(t,p) = f(x(t,p),p) + u(t)g(x(t,p),p), \\ y(t,p) = h(x(t,p),p), \\ x(t_0,p) = x_0 \in X_0, \\ p \in P \subset \mathcal{U}_{\mathcal{P}}, \\ t_0 \leq t \leq T, \end{cases}$$

Notation: Γ_2^{p,x_0}: a specific model of the family of models represented by Γ_1^P, where $p \in P$.

Definition (Ljung - Glad)

The model Γ_2^{p,x_0} is globally identifiable at p^* with respect to $\mathcal{D}_m \subseteq \mathcal{U}_{\mathcal{P}}$ if there exists a control u such that, $Y(p^*,u) \neq \emptyset$ and $Y(p^*,u) \cap Y(\bar{p},u) \neq \emptyset,\ \bar{p} \in \mathcal{D}_m \Longrightarrow p^* = \bar{p}$

- Extension to local/structural identifiablility

Proposition

1) If Γ_1^P is (structurally) globally μ-set-membership identifiable for P^* then Γ_2^{p,x_0} is (structurally) globally identifiable at a p^* in P^*.
2) If Γ_2^{p,x_0} is (structurally) globally identifiable at p^* in $\mathcal{U}_\mathcal{P}$ then there exists a connected set P^* belonging p^* such that Γ_1^P is (structurally) globally μ-set-membership identifiable for P^*.

It is possible to analyse set-membership identifiability:

- by using the links between classical definitions,
- directly for a linear uncertain system by a sufficient condition,
- directly for a nonlinear uncertain system by a sufficient condition.

Method 1: based on that proposed by *Pohjanpalo*

- y is supposed analytical $\Rightarrow$ y is entirely caracterized by the value of its derivatives at 0
- identifiability studied owing to the power series expansion of the solution y.

We consider the system:

$$\Gamma_1^P = \begin{cases} \dot{x}(t,p) = A(p)x(t,p) + B(p)u(t), \\ y(t,p) = C(p)x(t,p) + D(p)u(t), \\ x(0,p) = x_0 \in X_0, \\ p \in P \subset \mathcal{U}_{\mathcal{P}}, \\ 0 \leq t \leq T, \end{cases}$$

where $A(p)$, $B(p)$, $C(p)$ and $D(p)$: matrices depending on p.

Theorem

If Γ_1^P is globally μ-set-membership identifiable for $P^* \neq \emptyset$ then there exists u such that the system:

$$\left\{ \begin{array}{l} C(p)x_0 + D(p)u(0) = y(0,p), \\ C(p)(A^k(p)x_0 + \sum_{i=1}^{k} A^k(p)B(p)u^{i-1}(0)) + D(p)u^{(k)}(0) \\ \qquad\qquad = y^{(k)}(0,p),\ k = 1, \ldots, +\infty, \end{array} \right. \tag{2}$$

admits for solution the connected set P^*.

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Theorem

If there exists u such that the system (2) admits for only solution the connected set $P^* \neq \emptyset$ then Γ_1^P is globally set-membership identifiable for P^*.

$$\begin{cases} \dot{x}_1 = -(k_{21} + k_{31})x_1 + u, \; x_1(0) = 1, \\ \dot{x}_2 = k_{21}x_1 - x_2, \; x_2(0) = x_{20}, \\ \dot{x}_3 = k_{31}x_1 - c_{13}x_3, \; x_3(0) = 1, \\ y = x_2 + c_{13}x_3, \end{cases} \tag{3}$$

- $k_{21},\; k_{31},\; c_{13}$ are parameters to be identified,
- $x_{20} \in [\underline{x_{20}},\; \overline{x_{20}}]$,
- $y^{(k)}(0) \in [\underline{y^{(k)}}(0), \overline{y^{(k)}}(0)]$ for $k = 0, \ldots, 2$.

To analyse set-membership identifiability, the solutions of the following system (4) deduced from (2) is studied:

$$\begin{cases} x_{20} + c_{13} = y(0), \\ k_{21} - x_{20} + c_{13}(k_{31} - c_{13}) = \dot{y}(0), \\ k_{21}(-k_{21} - k_{31}) + x_{20} + c_{13}(k_{31}(-k_{21} - k_{31}) - c_{13}k_{31} \\ +c_{13}{}^2 + k_{31}) = \ddot{y}(0). \end{cases} \tag{4}$$

According to the previous theorem, it is sufficient to find solutions of (4).
In substituting k_{21} obtained with the second equation in the third equation, one gets the following system:

$$\left\{\begin{array}{l} x_{20} + c_{13} = y(0), \\ \\ \underbrace{\begin{array}{l}(x_{20} + c_{13}^2 + \dot{y}(0))(c_{13} - 1) - c_{13}(-x_{20} - c_{13}^2 - \dot{y}(0)) \\ +c_{13}(-x_{20} - c_{13}^2 - \dot{y}(0) - c_{13} + 1))\end{array}}_{=:\,\alpha(c_{13})} k_{31} = \\ \underbrace{-(x_{20} + c_{13}^2 + \dot{y}(0))(-x_{20} - c_{13}^2 - \dot{y}(0)) - x_{20} - c_{13}^3 + \ddot{y}(0)}_{=:\,\beta(c_{13})} \\ \\ k_{21} = \underbrace{\dot{y}(0) + x_{20} - c_{13}(k_{31} - c_{13})}_{\gamma(c_{13},k_{13})}. \end{array}\right.$$

$$\begin{cases} x_{20} + c_{13} = y(0), \\ \alpha(c_{13})k_{31} = \beta(c_{13}) \\ k_{21} = \gamma(c_{13}, k_{13}). \end{cases} \tag{5}$$

Identifiability conclusions:

- First equation of (5): $c_{13} \in [\underline{y}(0) - \overline{x_{20}}, \overline{y}(0) - \underline{x_{20}}]$.
- If $0 \notin \alpha(c_{13})$, one gets $k_{31} \in [\underline{\beta(c_{13})}, \overline{\beta(c_{13})}]/[\underline{\alpha(c_{13})}, \overline{\alpha(c_{13})}]$ and $k_{21} \in [\underline{\gamma(c_{13}, k_{13})}, \overline{\gamma(c_{13}, k_{13})}]$.
- The system (4) admits for solution the only connected set $P^* = [\underline{\gamma(c_{13}, k_{13})}, \overline{\gamma(c_{13}, k_{13})}] \times [\underline{\beta(c_{13})}, \overline{\beta(c_{13})}]/[\underline{\alpha(c_{13})}, \overline{\alpha(c_{13})}] \times [\underline{y}(0) - \overline{x_{20}}, \overline{y}(0) - \underline{x_{20}}]$.
- The system (3) is globally set-membership for P^*.

Method 2: based on that proposed by D. Vidal et G. Joly-Blanchard:

- elimination order $\{p\} < \{y, u\} < \{x\}$ ($\Rightarrow$ eliminate unobservable state variables),
- differential algebra approach (Kolchin and al., 1973)

$\Rightarrow$ relations between outputs and parameters:

$$R_i(y, u, p) = \theta_0(y, u) + \sum_{k=1}^{n_i} \theta_k^i(p) m_k(y, u),\ i = 1, \ldots, m$$

$\longrightarrow$ $(\theta_k^i)_{1 \leq k \leq l}$ are rational in p, $\theta_u^i \neq \theta_v^i$ $(u \neq v)$,

$\longrightarrow$ $(m_k)_{1 \leq k \leq l}$ are differential polynomials with respect to y and u and $\theta_0 \neq 0$.

Size of the system = number of observations.
Afterwards: $i = 1$ ($\Rightarrow$ one observation).

Injectivity of a function (Lagrange and al., 2007)

Consider a function $f : \mathcal{A} \to \mathcal{B}$ and any set $\mathcal{A}_1 \subseteq \mathcal{A}$. The function f is said to be a partial injection of $\mathcal{A}_1$ over $\mathcal{A}$, noted $(\mathcal{A}_1, \mathcal{A})$-injective, if $\forall a_1 \in \mathcal{A}_1$, $\forall a \in \mathcal{A}$,

$$a_1 \neq a \Rightarrow f(a_1) \neq f(a).$$

f is said to be $\mathcal{A}$-injective if it is $(\mathcal{A}, \mathcal{A})$-injective.

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Consider

$$R(y, u, p) = \theta_0(y, u) + \sum_{k=1}^{n} \theta_k(p) m_k(y, u).$$

Theorem

If $\forall (y, u)$, $\triangle(R)(y, u) = det(m_k(y, u), k = 1, \dots, n) \neq 0$, then Γ_1^P is μ-set-membership identifiable for P^* if the function $\Phi : p \in P^* \to (\theta_1(p), \dots, \theta_n(p)) \in (\mathbb{R})^n$ is P^*-injective.

Consider the model:

$$\begin{cases} \dot{x}_1 = p_2 x_1 + p_1 x_2, \\ \dot{x}_2 = p_1 p_2 x_1 x_2 + u, \\ y = x_2. \end{cases} \tag{6}$$

The package diffalg of Maple gives the following input-output polynomial:

$$R(y,u) = y\ddot{y} - \dot{y}^2 + \dot{y}u - y\dot{u} - p_1^2 p_2 y^3 + p_2(yu - y\dot{y}).$$

Consider the functional determinant:

$$\triangle R(y,u) = \begin{vmatrix} y^3 & yu - y\dot{y} \\ 3y^2\dot{y} & \dot{y}(-\dot{y}+u) + y(-\ddot{y}+\dot{u}) \end{vmatrix} \tag{7}$$

$\triangle R(y,u) = 2y^3\dot{y}^2 - 2y^3\dot{y}u - y^4\ddot{y} + y^4\dot{u} \not\equiv 0.$
Thus the first hypothesis is checked.
The function $\Phi : (p_1, p_2) \in P^* \to (p_1^2 p_2, p_2)$ is P^*-injective.
The system (6) is μ-set-membership identifiable for $P^* \subset [0, +\infty[\times[0, +\infty[$.

- Definitions of set-membership identifiability / μ-set-membership identifiability
  - provide a way to study identifiability for uncertain bounded-error systems
  - have a role to play in many pratical problems (diagnosis/prognosis in uncertain environments)
  - provide the guaranty that two situations corresponding to different parametrized setting are distinguishable

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  - provide a way to study identifiability for uncertain bounded-error systems
  - have a role to play in many pratical problems (diagnosis/prognosis in uncertain environments)
  - provide the guaranty that two situations corresponding to different parametrized setting are distinguishable
- Links between these definitions and a classical one
- Methods to analyse (μ-)set-membership identifiability
- Applications on two examples

- L. Ljung and T. Glad. On global identifiability for arbitrary model parametrizations. Automatica, volume 30, pp 265–276, 1994.
- G. Joly-Blanchard and L. Denis-Vidal. Some remarks about identifiability of controlled and uncontrolled nonlinear systems. Automatica, volume 34, pp 1151–1152, 1998
- E.R. Kolchin. Differential algebra and algebraic groups. Academic Press, 1973.
- S. Lagrange and N. Delanoue and L. Jaulin; On sufficient conditions of injectivity, development of a numerical test via interval analysis. Journal Reliable Computing, volume 13, pp 409–421, 2007.
- H. Pohjanpalo. System identifiability based on the power series expansion of the solution. Math. Biosciences, volume 41, pp 21–33, 1978.