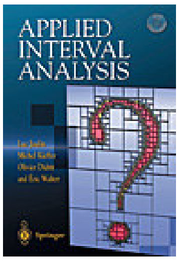
# Calcul par intervalles
# et ses applications



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Séminaire à l'ENSIETA, Brest
Mercredi 15 octobre 2004, 11h-11h45.

# 1 Motivations

## 1.1 Inversion ensembliste

Modèle : $y_{\mathsf{m}}(\mathbf{p},t)=p_1e^{-p_2t}$.

Paramètres : $p_1,p_2$.

Temps de mesure : $t_1,t_2,\ldots,t_m$

Barres de mesure : $[y_1^-,y_1^+],[y_2^-,y_2^+],\ldots,[y_m^-,y_m^+]$

Ensemble de vraisemblance :

$$\begin{aligned}\mathbb{S} &= \left\{\mathbf{p}\in\mathbb{R}^2,\forall i\in\{1,\ldots,m\},y_{\mathsf{m}}(\mathbf{p},t_i)\in[y_i^-,y_i^+]\right\}.\\ &= \mathbf{y}_{\mathsf{m}}^{-1}([\mathbf{y}]).\end{aligned}$$

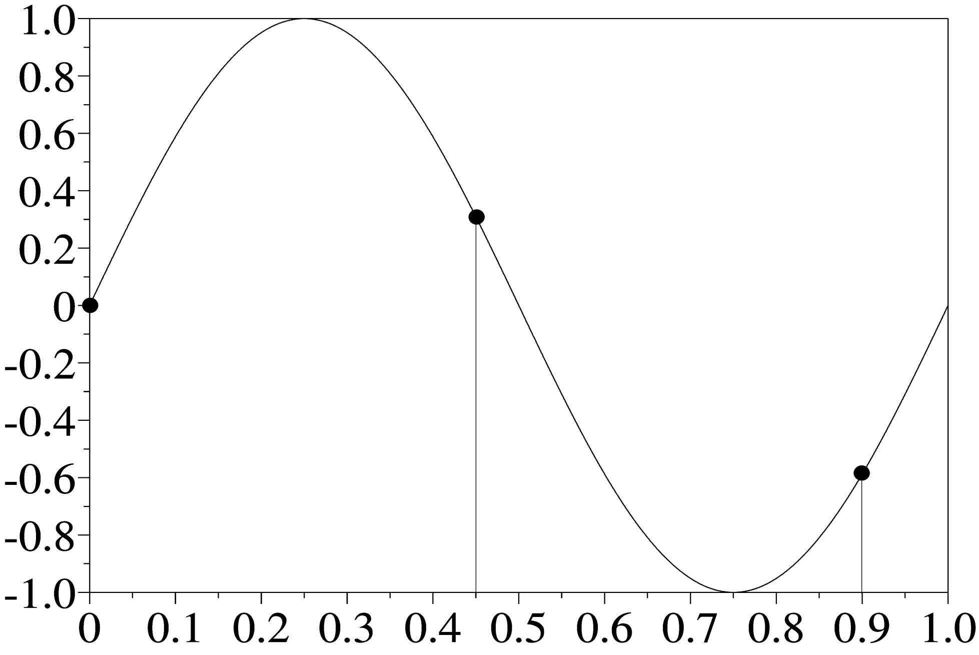
# Logiciel SetDemo

Modèle : $y_{\mathsf{m}}(\mathbf{p}, t) = p_1 \sin(2\pi p_2 t)$.

Paramètres : $p_1, p_2$.

Temps de mesure : $t_1 = 0, t_2 = T_e, t_3 = 2T_e, \ldots$

Barres de mesure : $[y_1^-, y_1^+], [y_2^-, y_2^+], [y_3^-, y_3^+], \ldots$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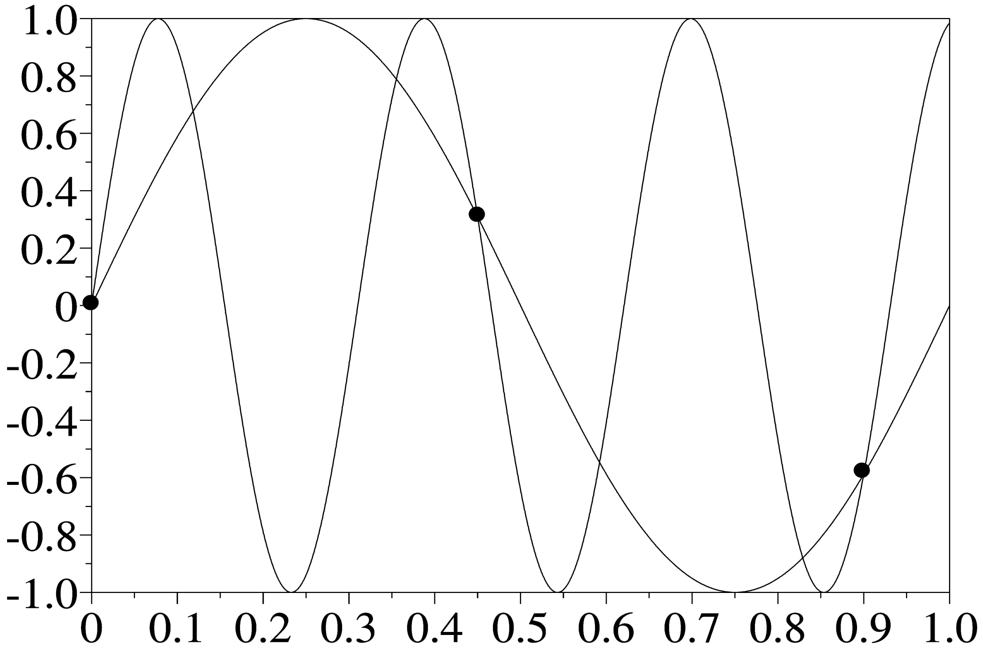
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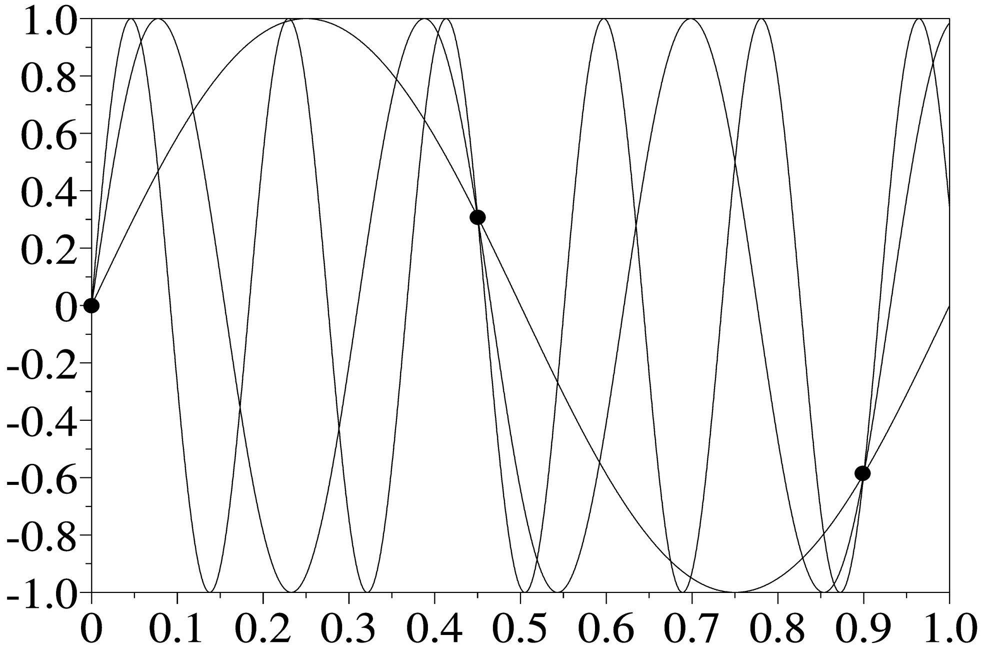
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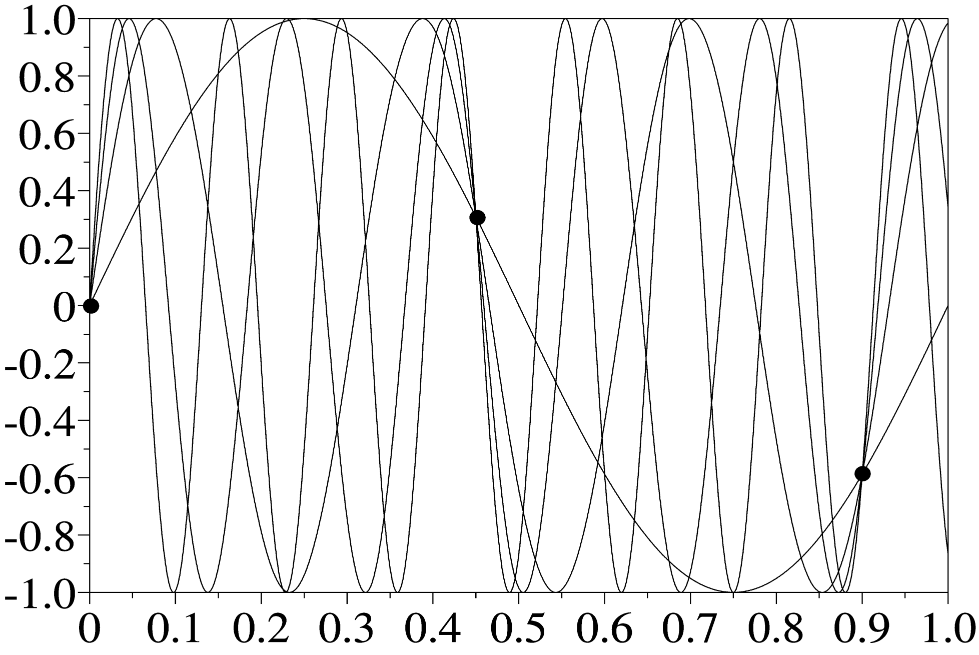
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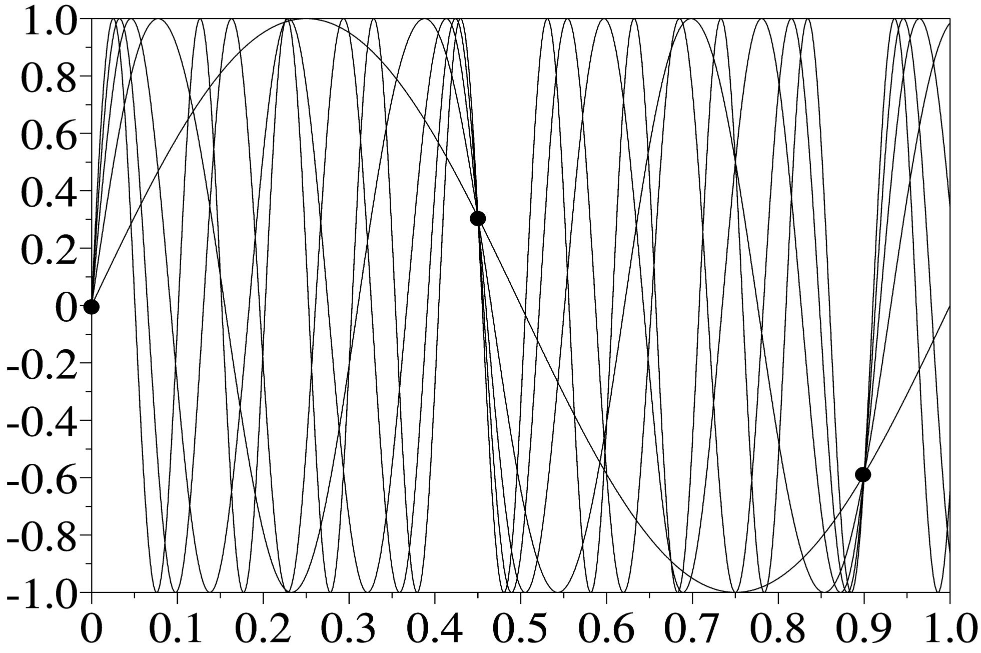
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1.0

(Montrer l’ensemble de vraisemblance avec SetDemo).

# 2 Méthode

## 2.1 Calcul par intervalles

Si $\diamond \in \{+, -, ., /, \max, \min\}$

$$[x] \diamond [y] = [\{x \diamond y \mid x \in [x], y \in [y]\}] \,.$$

Par exemple,

$$\begin{array}{rcl} [-1,3]+[2,5] & = & [?,?], \\ [-1,3].[2,5] & = & [?,?], \\ [-1,3]/[2,5] & = & [?,?], \\ [-1,3] \vee [2,5] & = & [?,?]. \end{array}$$

Si $\diamond \in \{+,-,.,/,\max,\min\}$

$$[x]\diamond[y]=[\{x\diamond y \mid x\in[x],y\in[y]\}]\,.$$

Par exemple,

$$\begin{aligned}
[-1,3]+[2,5] &= [1,8],\\
[-1,3].[2,5] &= [-5,15],\\
[-1,3]/[2,5] &= [-\tfrac{1}{2},\tfrac{3}{2}],\\
[-1,3]\vee[2,5] &= [2,5].
\end{aligned}$$

$$
\begin{aligned}
[x^-, x^+] + [y^-, y^+] &= [\mathsf{?}, \mathsf{?}],\\
[x^-, x^+].[y^-, y^+] &= [\mathsf{?}, \mathsf{?}],\\
[-\mathsf{1}, \mathsf{3}] \vee [\mathsf{2}, \mathsf{5}] &= [\mathsf{?}, \mathsf{?}].
\end{aligned}
$$

$$\begin{aligned}
[x^-,x^+]+[y^-,y^+] = {} & [x^- + y^-, x^+ + y^+],\\
[x^-,x^+].[y^-,y^+] = {} & [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+,\\
& x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+],\\
[x^-,x^+] \vee [y^-,y^+] = {} & [\vee(x^-,y^-), \vee(x^+,y^+)].
\end{aligned}$$

Si $f \in \{\text{cos, sin,sqr, sqrt, log, exp, } \dots\}$

$$f\left([x]\right) = \left[\{f(x) \mid x \in [x]\}\right].$$

Par exemple,

$$\begin{aligned}
\sin\left([0,\pi]\right) &= [?,?], \\
\text{sqr}\left([-1,3]\right) &= [?,?], \\
\text{abs}\left([-7,1]\right) &= [?,?], \\
\text{sqrt}\left([-10,4]\right) &= [?,?], \\
\log\left([-2,-1]\right) &= [?,?].
\end{aligned}$$

Si $f \in \{\text{cos, sin,sqr, sqrt, log, exp, } \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}] \, .$$

Par exemple,

$$\begin{array}{rcl}
\sin([0,\pi]) & = & [0,1], \\
\text{sqr}([-1,3]) & = & [-1,3]^2 = [0,9], \\
\text{abs}([-7,1]) & = & [0,7], \\
\text{sqrt}([-10,4]) & = & \sqrt{[-10,4]} = [0,2], \\
\log([-2,-1]) & = & \emptyset.
\end{array}$$

## 2.2 Projection de contraintes

Soient 3 variables $x, y, z$ telles que

$$\begin{aligned} x &\in [1, 5], \\ y &\in [2, 4], \\ z &\in [6, 10], \\ z &= x + y. \end{aligned}$$

Les valeurs 1 pour $x$ et 10 pour $z$ sont dites inconsistantes.

Projeter une contrainte (ici, $z = x + y$), c'est calculer les plus petits intervalles ne contenant que des valeurs consistantes. Pour notre exemple, cela revient à projeter 3 fois (suivant $x, y$, puis $z$) le sous-ensemble de $\mathbb{R}^3$ donné par

$$\mathbb{S} = \{(x, y, z) \in [1, 5] \times [2, 4] \times [6, 10] \mid z = x + y\} \,.$$

## 2.3 Méthode numérique de projection

Puisque $x \in [1,5], y \in [2,4], z \in [6,10], z = x + y$, on a

$$\begin{aligned} z = x + y \Rightarrow \quad & z \in \quad [?,?] \\ x = z - y \Rightarrow \quad & x \in \quad [?,?] \\ y = z - x \Rightarrow \quad & y \in \quad [?,?] \end{aligned}$$

Puisque $x \in [1,5], y \in [2,4], z \in [6,10], z = x + y$, on a

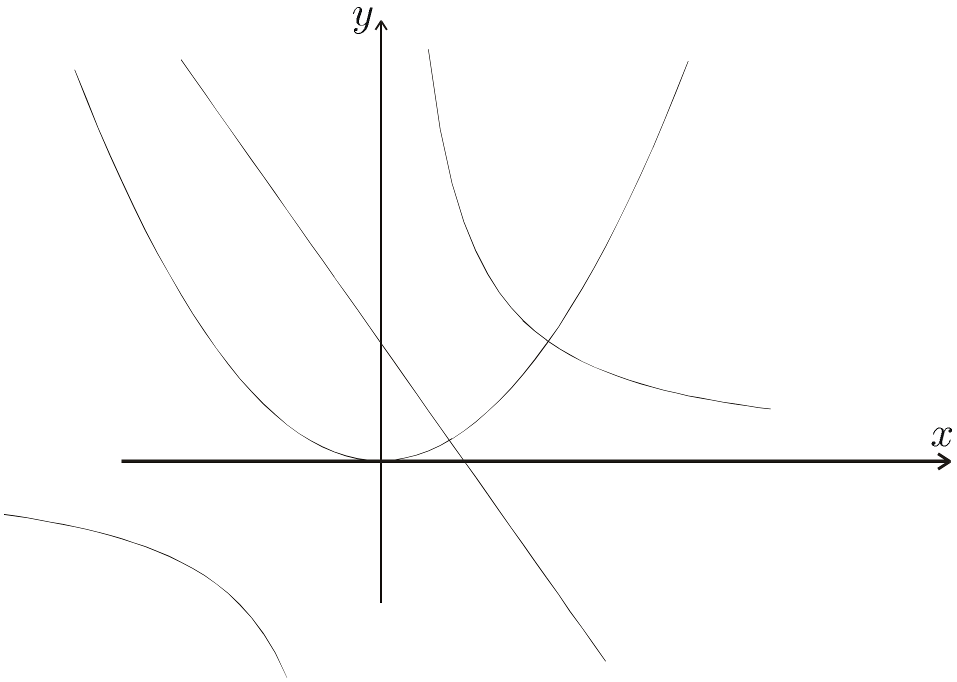
$$\begin{aligned}
z = x + y \Rightarrow \quad z \in \quad & [6,10] \cap ([1,5] + [2,4]) \\
& = [6,10] \cap [3,9] = [6,9]. \\
x = z - y \Rightarrow \quad x \in \quad & [1,5] \cap ([6,10] - [2,4]) \\
& = [1,5] \cap [2,8] = [2,5]. \\
y = z - x \Rightarrow \quad y \in \quad & [2,4] \cap ([6,10] - [1,5]) \\
& = [2,4] \cap [1,9] = [2,4].
\end{aligned}$$

## 2.4 Propagation de contraintes

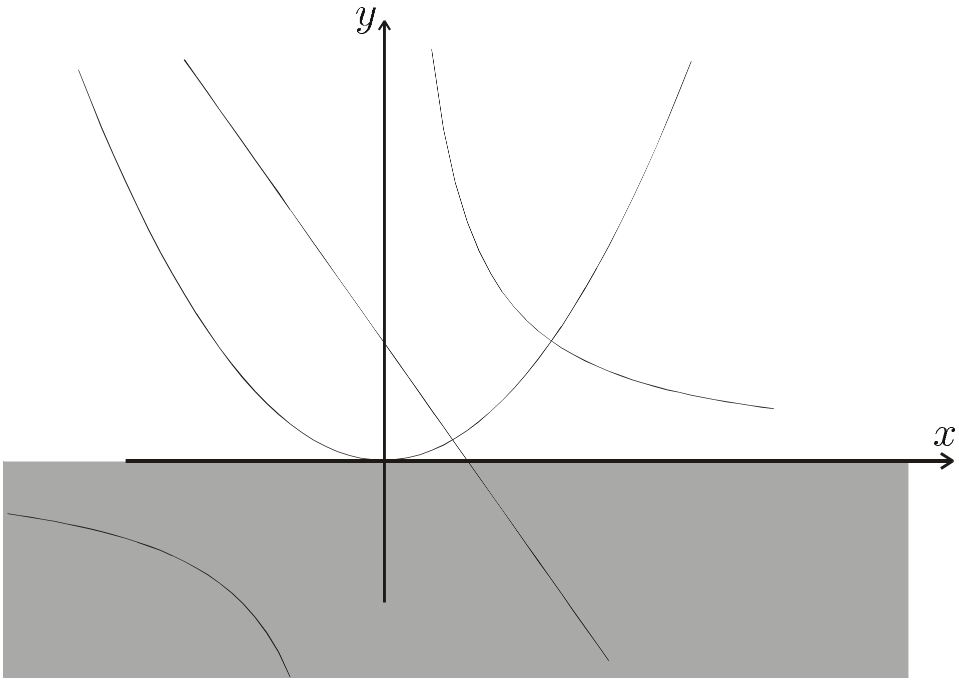
Soient les trois contraintes suivantes

$$\begin{aligned}(C_1) &\ :\ y = x^2,\\ (C_2) &\ :\ xy = 1,\\ (C_3) &\ :\ y = -2x + 1.\end{aligned}$$

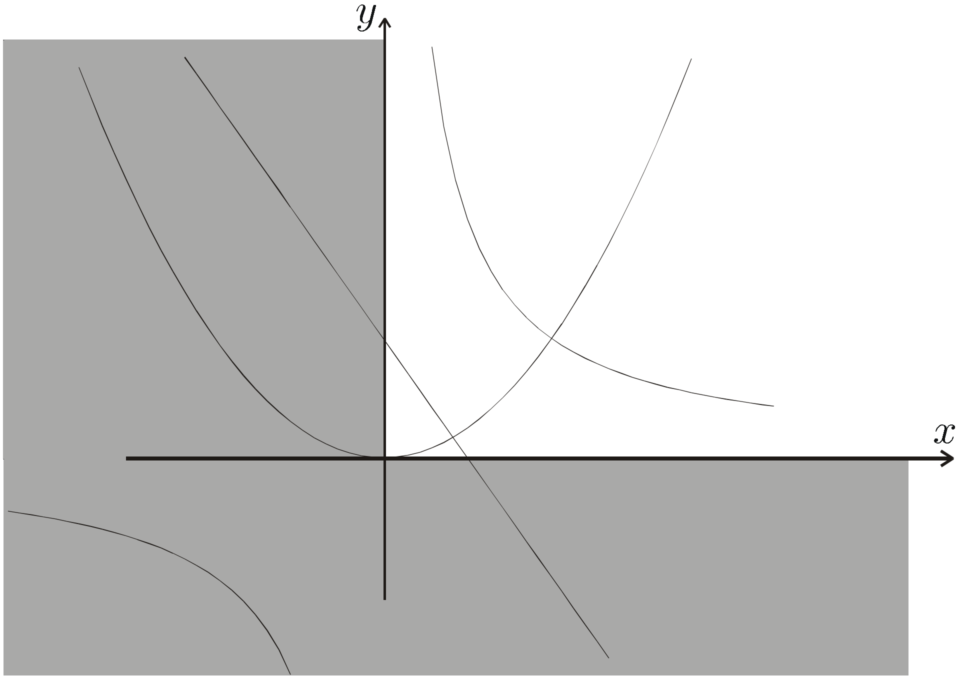
A chaque variable on affecte le domaine $]-\infty, \infty[$. Une *propagation de contrainte* consiste à projeter les contraintes jusqu'à l'équilibre.



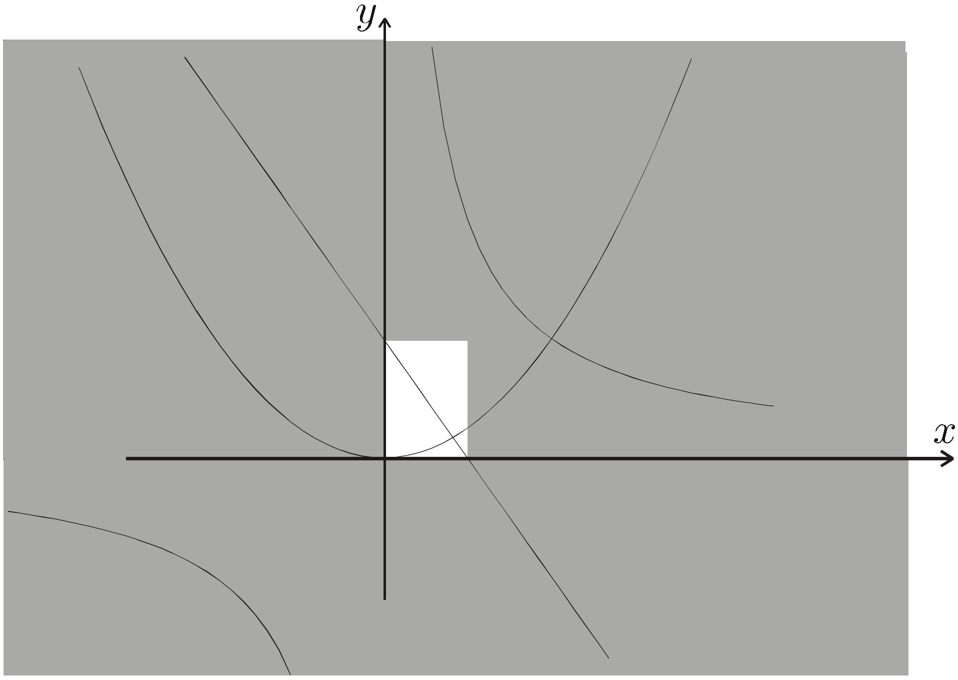
$$\left\{\begin{array}{l} x \in ]-\infty, \infty[ \\ y \in ]-\infty, \infty[ \end{array}\right. \overset{(y=x^2)}{\Longrightarrow} \left\{\begin{array}{c} x \in ]-\infty, \infty[ \\ y \in [0, \infty[ \end{array}\right.$$



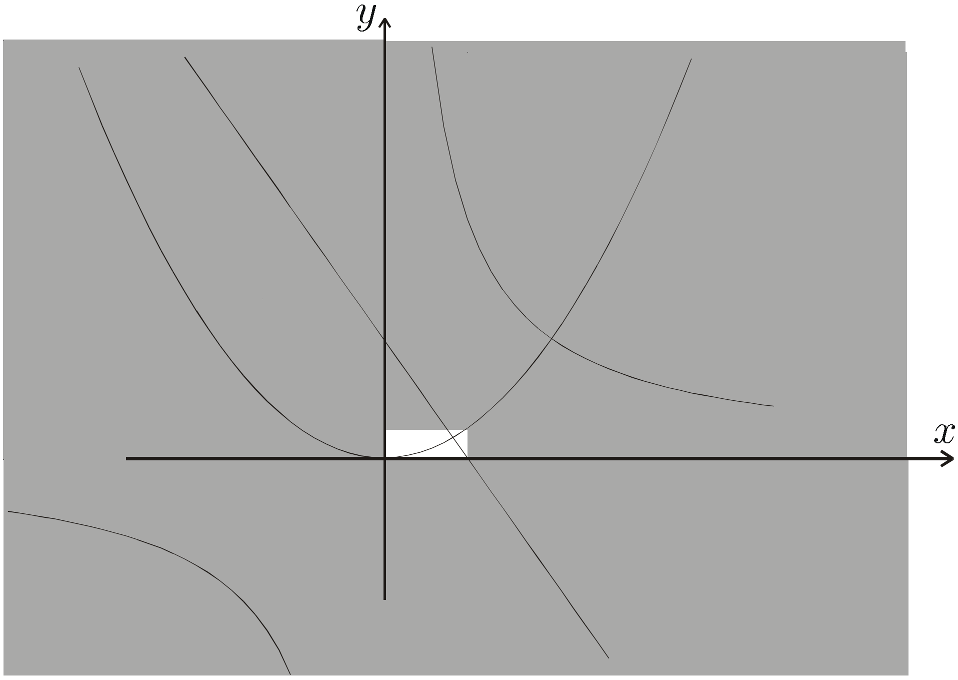
$$\left\{\begin{array}{c} x \in ]-\infty, \infty[ \\ y \in [0, \infty[ \end{array}\right. \overset{(xy=1)}{\Longrightarrow} \left\{\begin{array}{c} x \in [0, \infty[ \\ y \in [0, \infty[ \end{array}\right.$$



$$\begin{cases} x \in [0,\infty[ \\ y \in [0,\infty[ \end{cases} \overset{(y=-2x+1)}{\Longrightarrow} \begin{cases} x \in [0,1/2] \\ y \in [0,1] \end{cases}$$



$$\left\{\begin{array}{c} x \in [0,1/2] \\ y \in [0,1] \end{array}\right. \overset{(y=x^2)}{\Longrightarrow} \left\{\begin{array}{c} x \in [0,1/2] \\ y \in [0,1/4] \end{array}\right.$$



$$\left\{\begin{array}{l} x \in [0, 1/2] \\ y \in [0, 1/4] \end{array}\right. \xRightarrow{(xy=1)} \left\{\begin{array}{l} x \in \emptyset \\ y \in \emptyset \end{array}\right.$$

$$\begin{array}{lll}
(C_1) \Rightarrow & y \in & ]-\infty,\infty[^2 = \ [0,\infty[ \\
(C_2) \Rightarrow & x \in & 1/[0,\infty[ = [0,\infty[ \\
(C_3) \Rightarrow & y \in & [0,\infty[ \cap \ ((-2)\,.[0,\infty[+1) \\
 & & = [0,\infty[ \cap \ (]-\infty,1]) = [0,1] \\
 & x \in & [0,\infty[ \cap \ (-[0,1]/2 + 1/2) \\
 & & = [0,1/2] \\
(C_1) \Rightarrow & y \in & [0,1] \cap [0,1/2]^2 = [0,1/4] \\
(C_2) \Rightarrow & x \in & [0,1/2] \ \cap \ 1/[0,1/4] = \emptyset \\
 & y \in & [0,1/4] \cap 1/\emptyset = \emptyset
\end{array}$$

## 2.5 Décomposition

Pour les contraintes plus complexes, il nous faut effectuer une décomposition

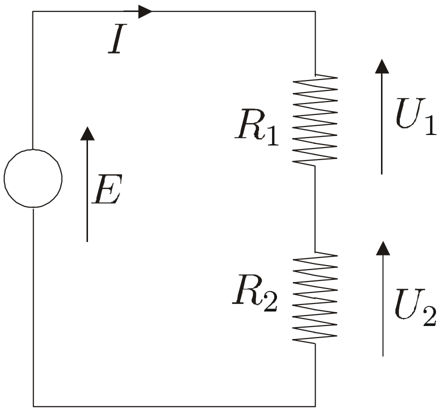
$$\begin{array}{l} x + \mathsf{sin}(y) - xz \leq 0, \\ x \in [-1,1], y \in [-1,1], z \in [-1,1] \end{array}$$

se décompose en

$$\left\{ \begin{array}{l} a = \mathsf{sin}(y) \\ b = x + a \\ c = xz \\ b - c = d \end{array} \right. , \quad \begin{array}{ll} x \in [-1,1] & a \in ]-\infty,\infty[ \\ y \in [-1,1] & b \in ]-\infty,\infty[ \\ z \in [-1,1] & c \in ]-\infty,\infty[ \\ & d \in ]-\infty,0] \end{array}$$

# 3 Applications

## 3.1 Problème d'estimation


$I$
$E$
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$U_1$
$R_2$
$U_2$

Contraintes

$$\begin{aligned} P &= EI;\ E = (R_1 + R_2)\, I; \\ U_1 &= R_1 I;\ U_2 = R_2 I;\ E = U_1 + U_2. \end{aligned}$$

Domaines initiaux

$$\begin{array}{cc} R_1 \in [0, \infty[\Omega, & R_2 \in [0, \infty[\Omega, \\ E \in [23, 26]\mathrm{V}, & I \in [4, 8]\mathrm{A}, \\ U_1 \in [10, 11]\mathrm{V}, & U_2 \in [14, 17]\mathrm{V}, \\ P \in [124, 130]\mathrm{W}, & \end{array}$$

Contraintes

$$\begin{array}{rcl} P & = & EI;\ E = (R_1 + R_2)\, I; \\ U_1 & = & R_1 I;\ U_2 = R_2 I;\ E = U_1 + U_2. \end{array}$$

Domaines contractés

$$\begin{array}{cc} R_1 \in [\mathbf{1.84}, \mathbf{2.31}]\,\Omega, & R_2 \in [\mathbf{2.58}, \mathbf{3.35}]\Omega, \\ E \in [\mathbf{24}, 26]\mathrm{V}, & I \in [\mathbf{4.769}, \mathbf{5.417}]\,\mathrm{A}, \\ U_1 \in [10, 11]\mathrm{V}, & U_2 \in [14, \mathbf{16}]\mathrm{V}, \\ P \in [124, 130]\mathrm{W}, & \end{array}$$

Programme Scilab disponible sur

`http://www.istia.univ-angers.fr/`
`~jaulin/propagR1R2.sce`

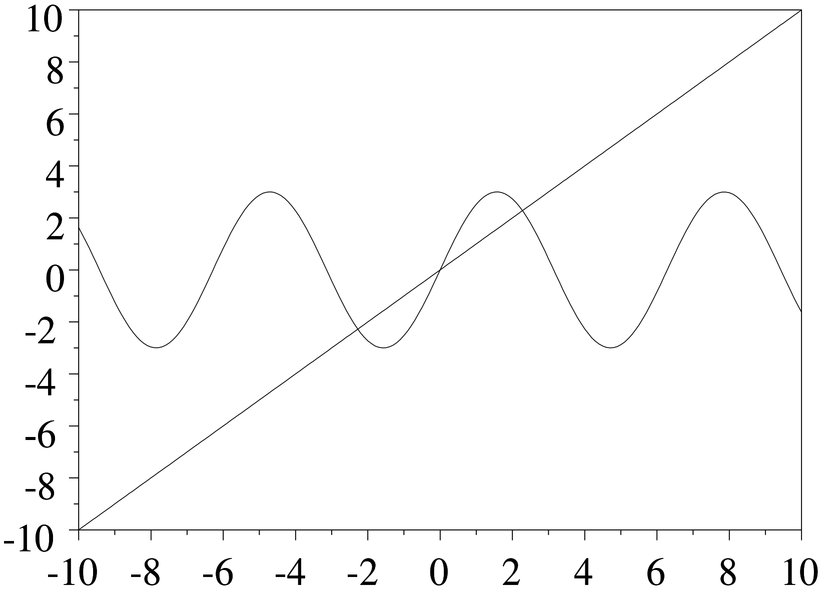
## 3.2 Résolution d'équations non-linéaires

Cherchons par exemple à résoudre

$$3\sin(x) = x, \quad x \in \mathbb{R}.$$

Cette équation se décompose en

$$\left\{ \begin{array}{lcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$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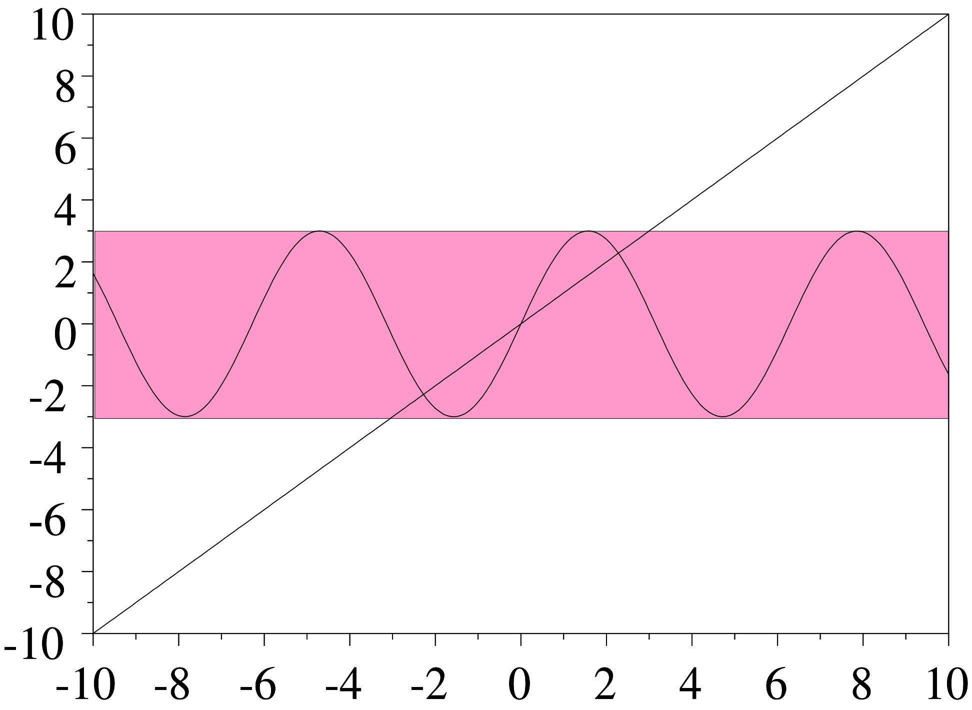
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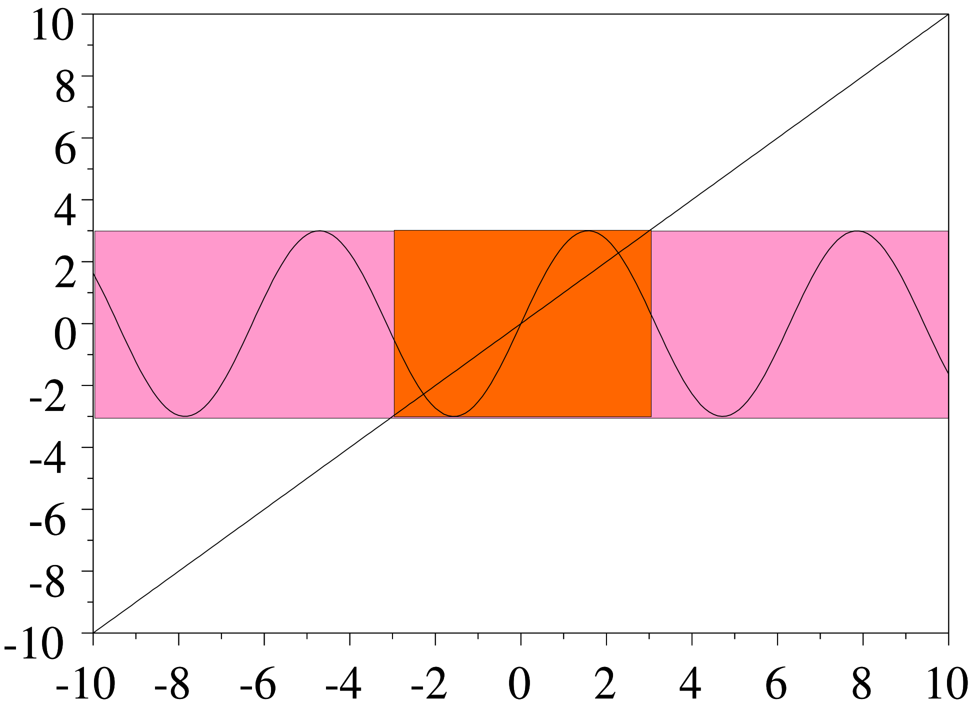
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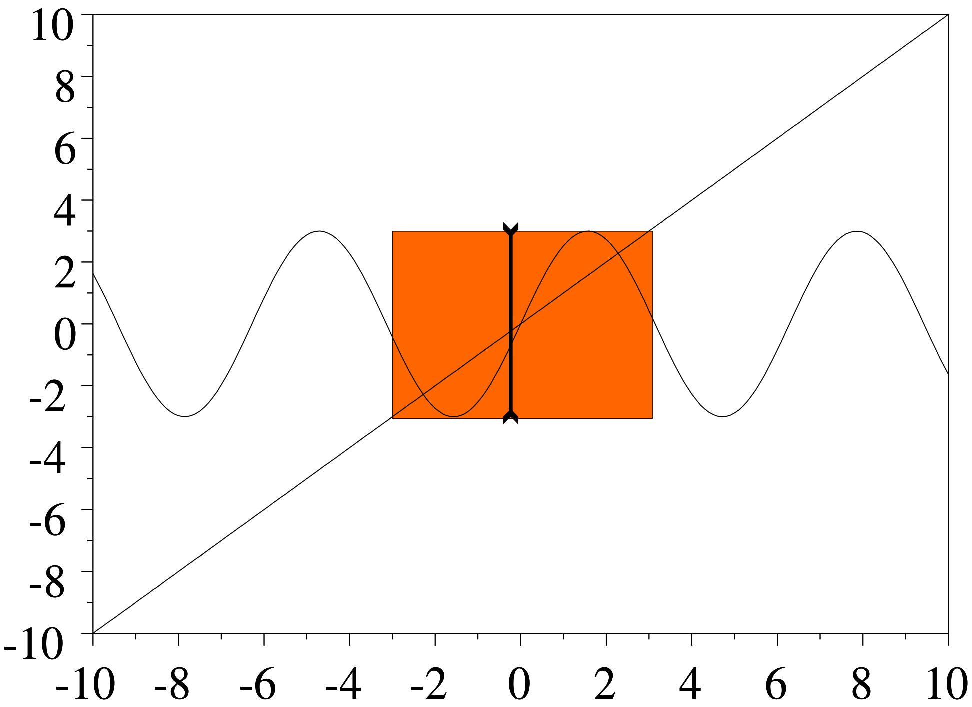
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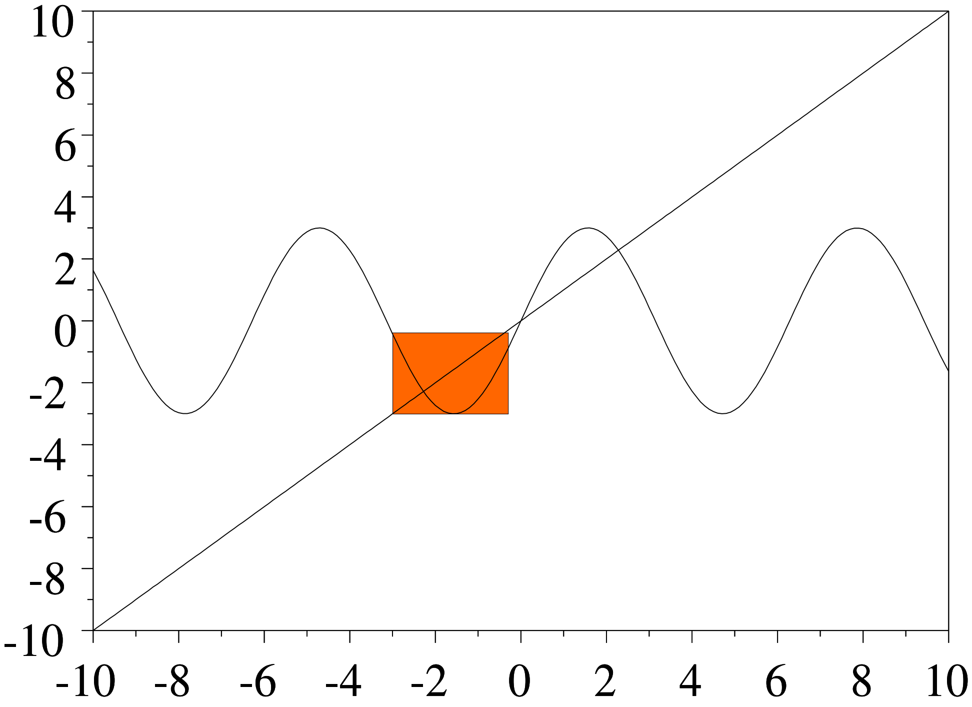
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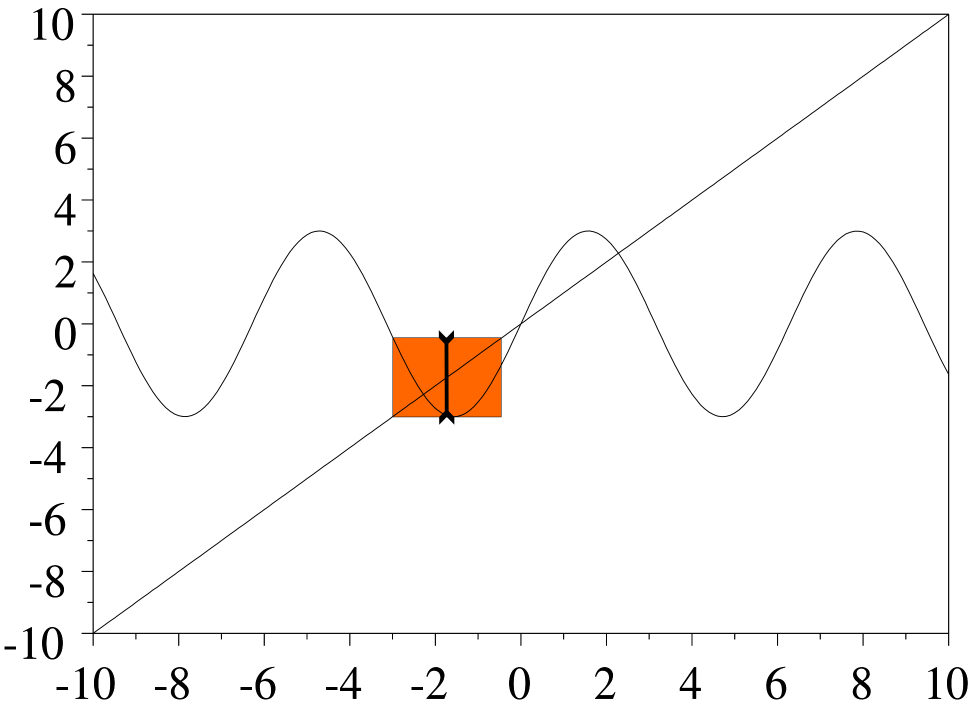
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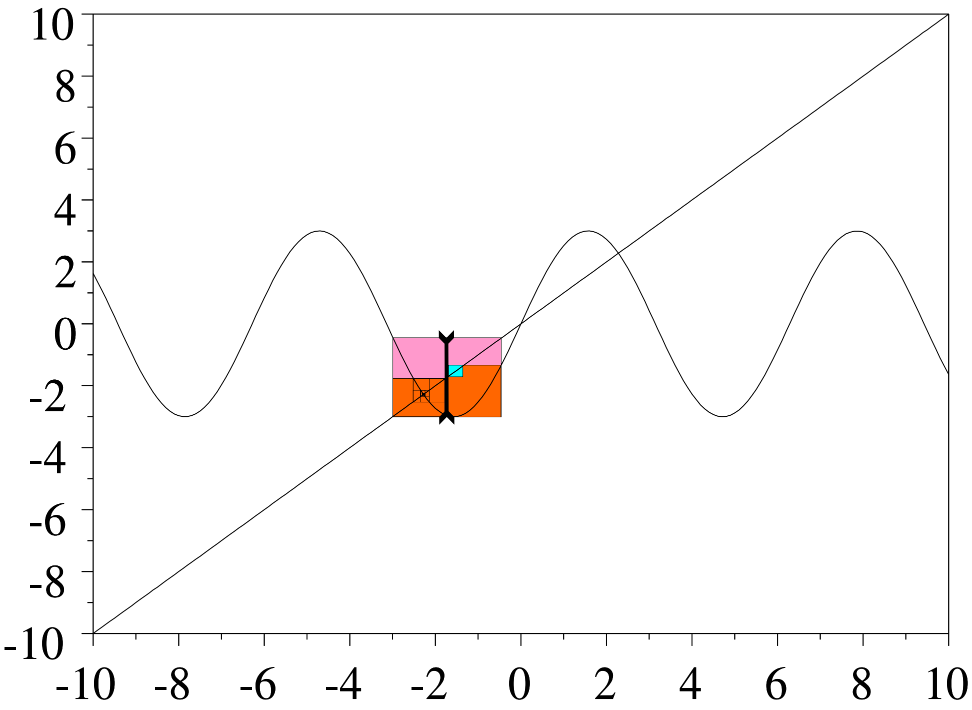
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(Illustrer la résolution à l'aide du solveur Proj2D)

## 3.3 Prouver qu'un ensemble de contraintes est toujours satisfait

Idée : On complémente l'ensemble des contraintes.

Par exemple, montrer que

$$\forall x \in [x], \forall y \in [y],\ f(x,y) \leq 0 \text{ et } g(x,y) \leq 0,$$

revient à montrer que

$$\{(x,y) \in [x] \times [y] \mid f(x,y) > 0 \text{ ou } g(x,y) > 0\} = \emptyset$$

c'est-à-dire que

$$\{(x,y) \in [x] \times [y] \mid \max\left(f(x,y), g(x,y)\right) > 0\} = \emptyset.$$

## 3.4 Optimisation globale

Cherchons à minimiser, avec Proj2d, le critère

$$f(x, y) = \sin x^2 + \sin y.$$

Epigraphe :

$$\mathbb{E} = \{(x, y, a) \mid f(x, y) \leq a\}\,.$$

# 4 Kalman ensembliste

Equation d'état

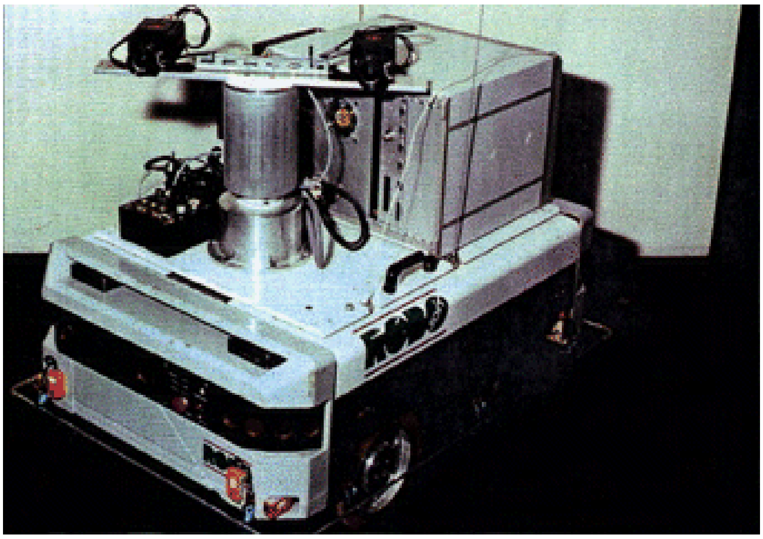
$$\begin{cases} \mathbf{x}(k+1) & = & \mathbf{f}(\mathbf{x}(k),\mathbf{u}(k)) \\ \mathbf{y}(k) & = & \mathbf{g}(\mathbf{x}(k)) \end{cases}$$

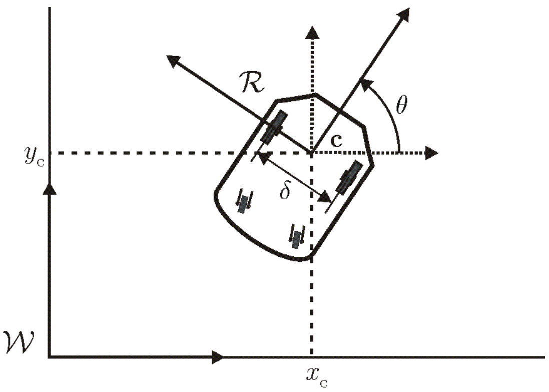
Estimateur ensembliste

$$\begin{array}{ll} \mathbb{X}^{\mathsf{p}}(k) = \mathbf{f}(\mathbb{X}(k-1),\mathbf{u}(k-1)) & // \text{ prédiction} \\ \mathbb{X}_k := \mathbb{X}_k^{\mathsf{p}} \cap \mathbf{g}^{-1}(\mathbb{Y}_k); & // \text{ correction} \end{array}$$

# 5 Localisation

Robot avec 24 télémètres à ultrasons

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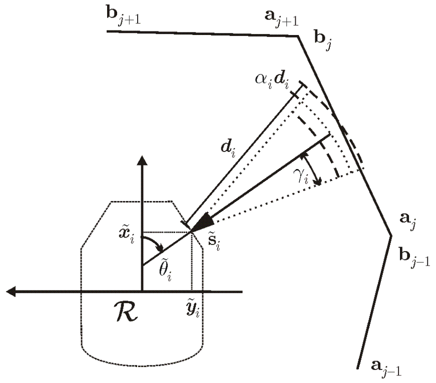
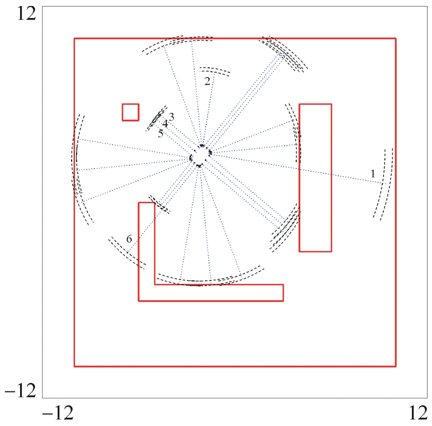
$\mathcal{W}$
$x_{\mathrm{c}}$


$\mathbf{b}_{j+1}$
$\mathbf{a}_{j+1}$
$\mathbf{b}_j$
$\alpha_i \boldsymbol{d}_i$
$\boldsymbol{d}_i$
$\gamma_i$
$\mathbf{a}_j$
$\mathbf{b}_{j-1}$
$\tilde{\boldsymbol{x}}_i$
$\tilde{\mathbf{s}}_i$
$\tilde{\theta}_i$
$\mathcal{R}$
$\tilde{\boldsymbol{y}}_i$
$\mathbf{a}_{j-1}$

# Après inversion ensembliste


12
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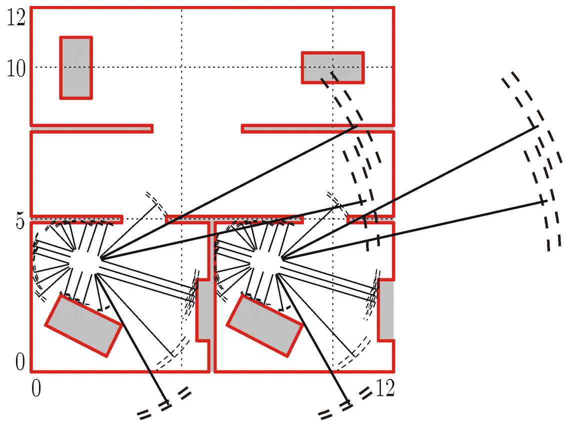
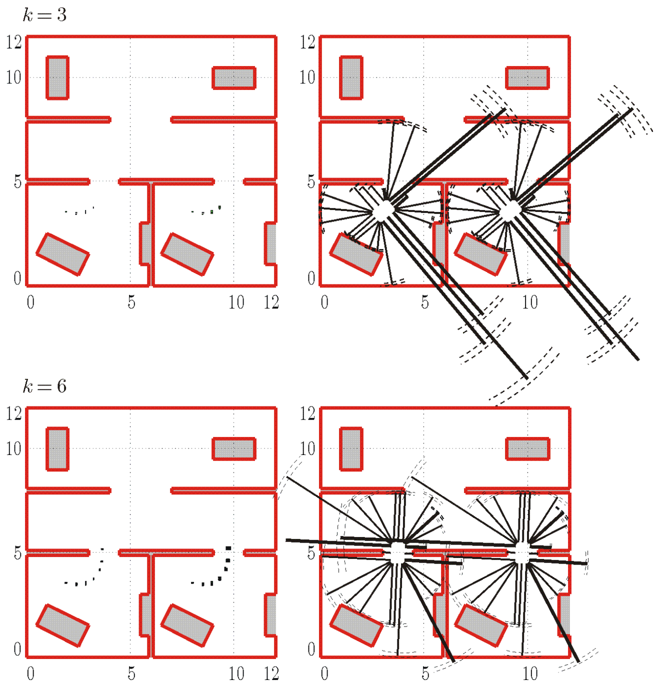
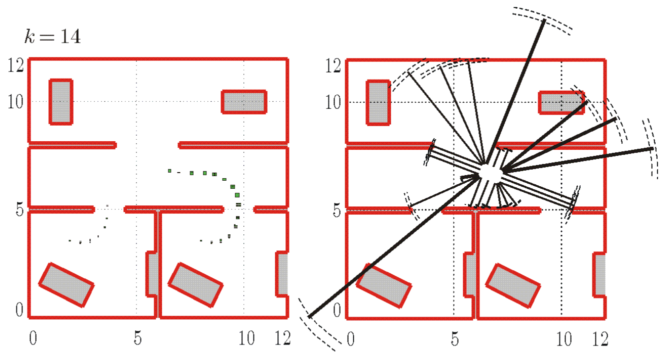
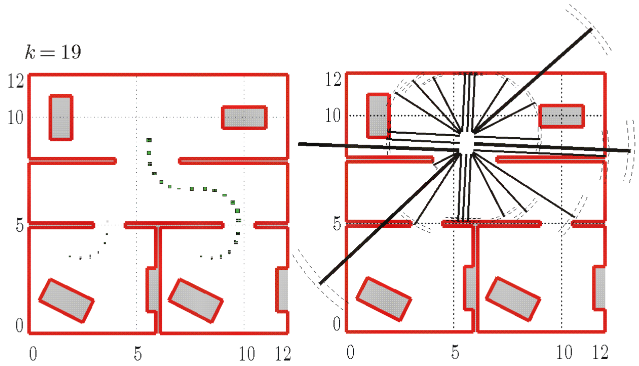


Equations d'état

$$\left\{\begin{array}{rcl}\dot{x} & = & \rho\,\dfrac{\omega_{\mathsf{r}}+\omega_{\mathsf{l}}}{2}\cos\theta,\\ \dot{y} & = & \rho\,\dfrac{\omega_{\mathsf{r}}+\omega_{\mathsf{l}}}{2}\sin\theta\\ \dot{\theta} & = & \rho\,\dfrac{\omega_{\mathsf{r}}-\omega_{\mathsf{l}}}{\delta}\end{array}\right.$$


$k = 3$
$k = 6$


$k = 14$


$k = 19$