UNIVERSITE D'ORLEANS

Reachability computation with hybrid dynamical systems. Application to state estimation.

Interval methods approaches.

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Outline

- **Interval Taylor Methods for IVP ODE**
- **Comparison Theorems for Differential Inequalities**
- *Set-membership Estimation with Nonlinear Continuous Systems*
- *Hybrid and Cyber-Physical Systems*
- **Nonlinear Hybrid Reachability**
- *Set-membership Parameter Estimation with Hybrid Systems*

- **Interval Methods for IVP ODE**

Nonlinear Set Integration

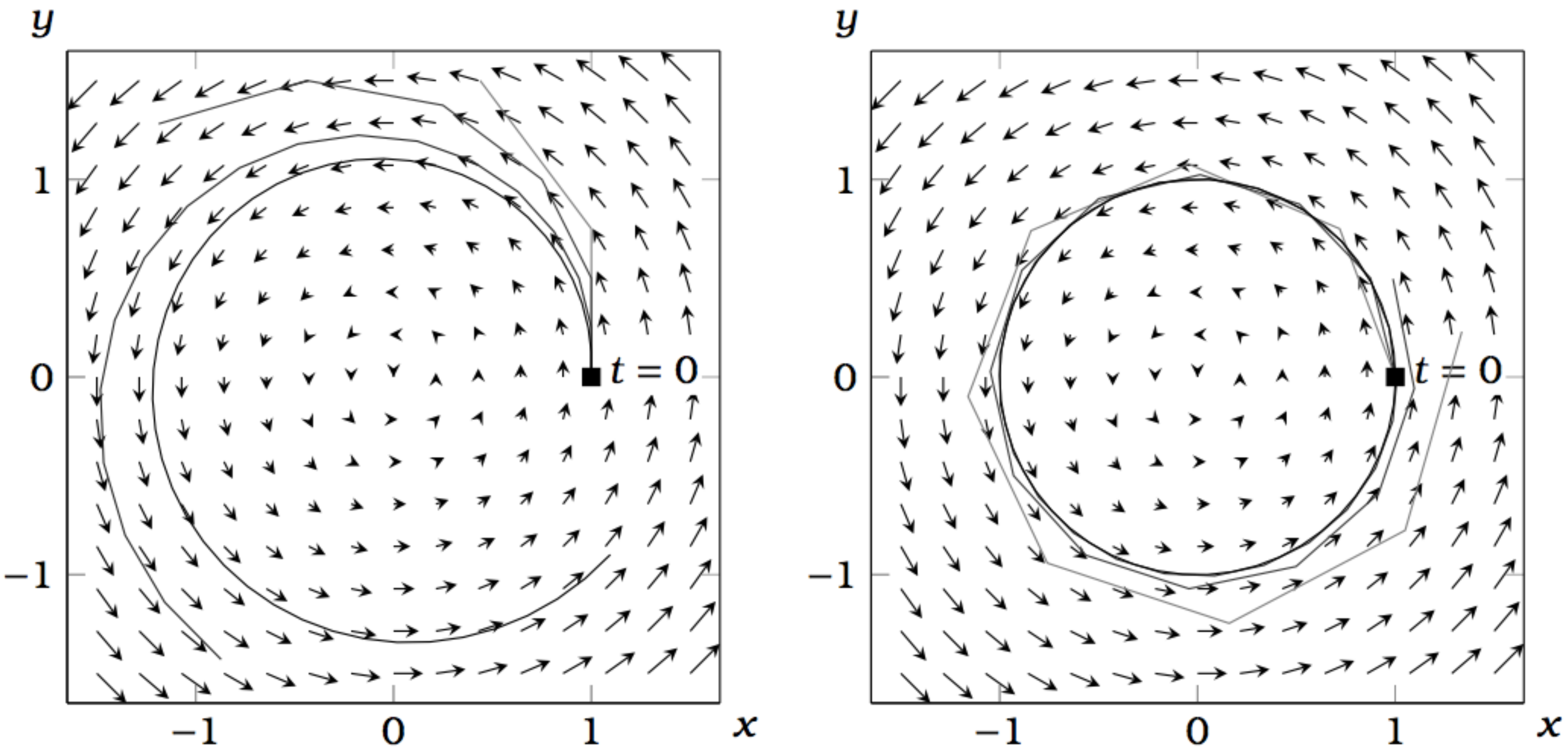
- **Guaranteed set integration with Taylor methods**
  - (Moore,66) (Lohner,88) (Nedialkov,99)
  - IVP ODE

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \ \mathbf{x}(t_0) \in [\mathbf{x}_0], \ \mathbf{p} \in [\mathbf{p}]$$

IVP ODE

Standard Methods

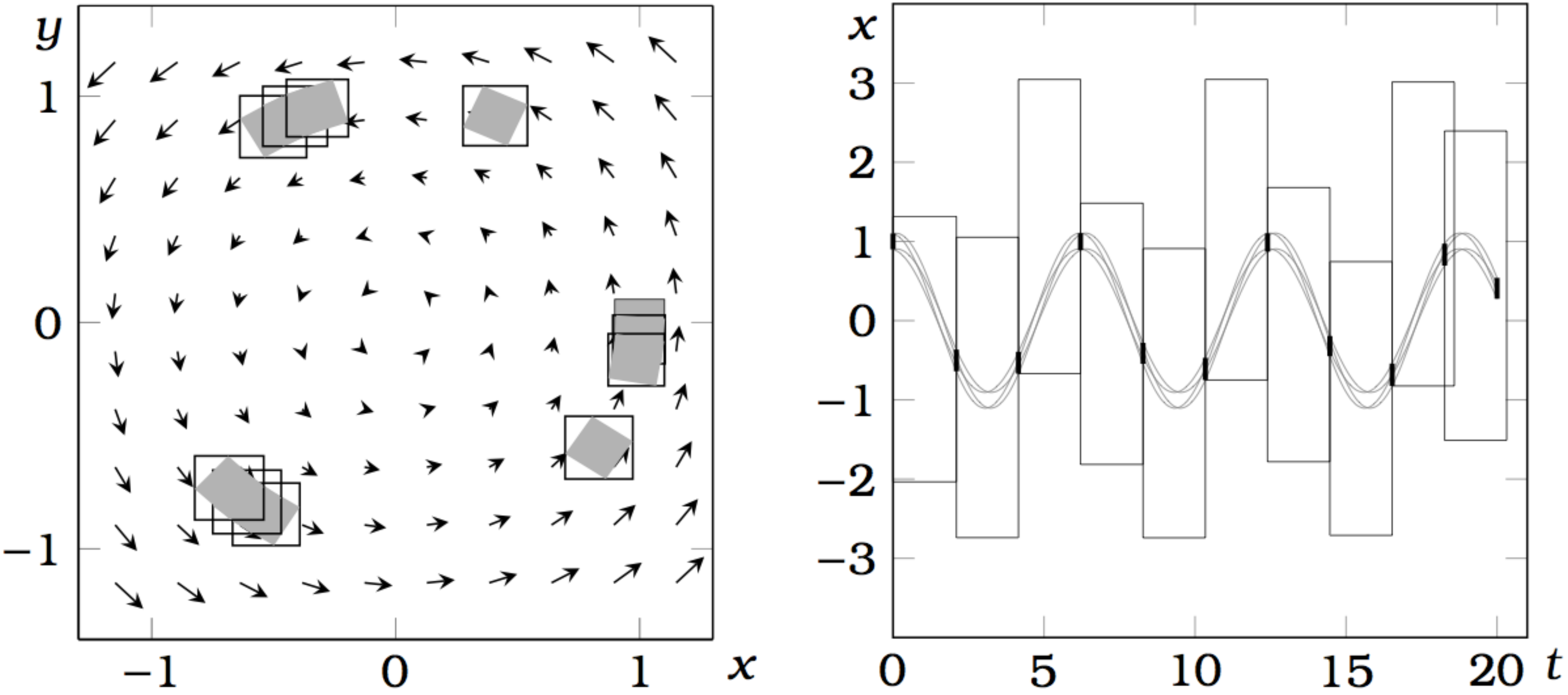
$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \check{\mathbf{p}}, t), \quad t_0 \leq t \leq t_N, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$



Nonlinear Set Integration

- **Guaranteed set integration**

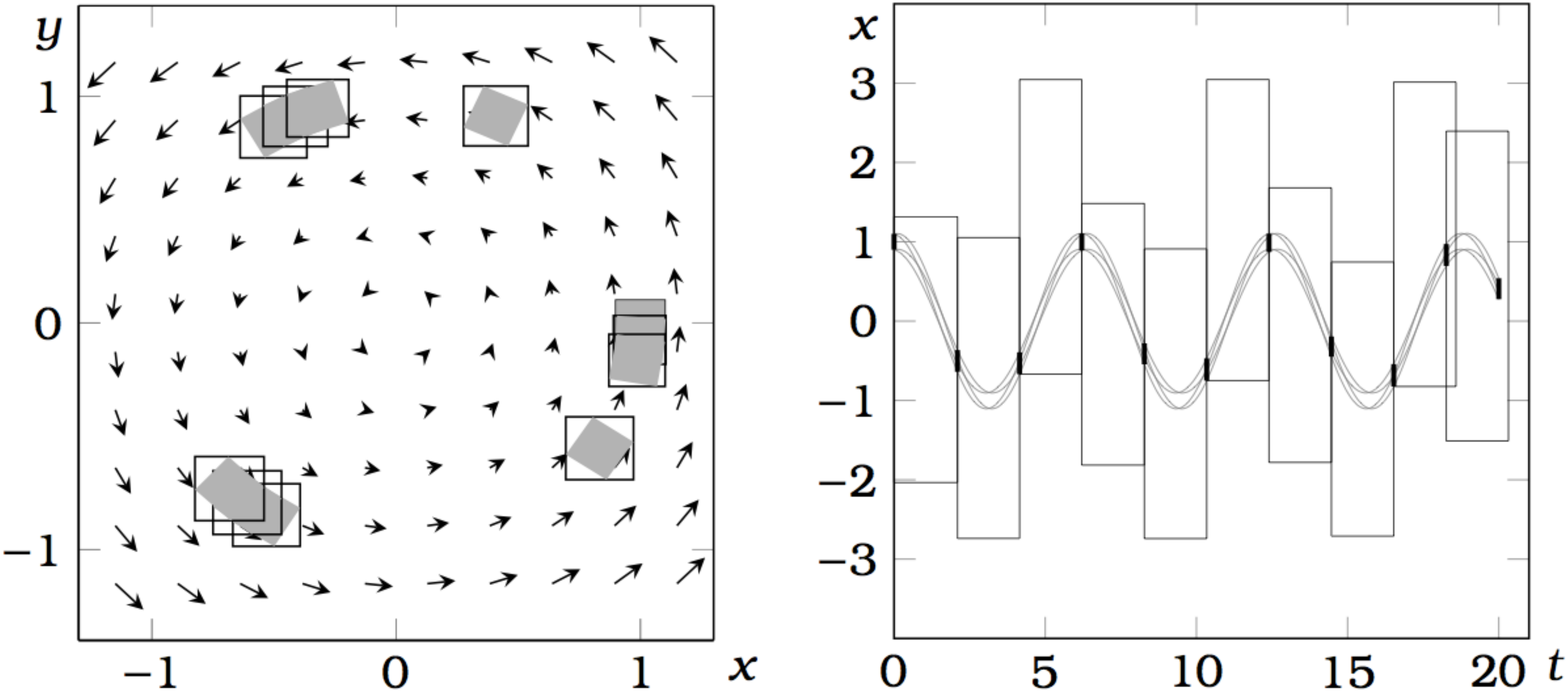
$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \; \mathbf{x}(t_0) \in [\mathbf{x}_0], \; \mathbf{p} \in [\mathbf{p}]$$



Nonlinear Set Integration

- **Guaranteed set integration**

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \boxed{\mathbf{x}(t_0) \in [\mathbf{x}_0], \mathbf{p} \in [\mathbf{p}]}$$



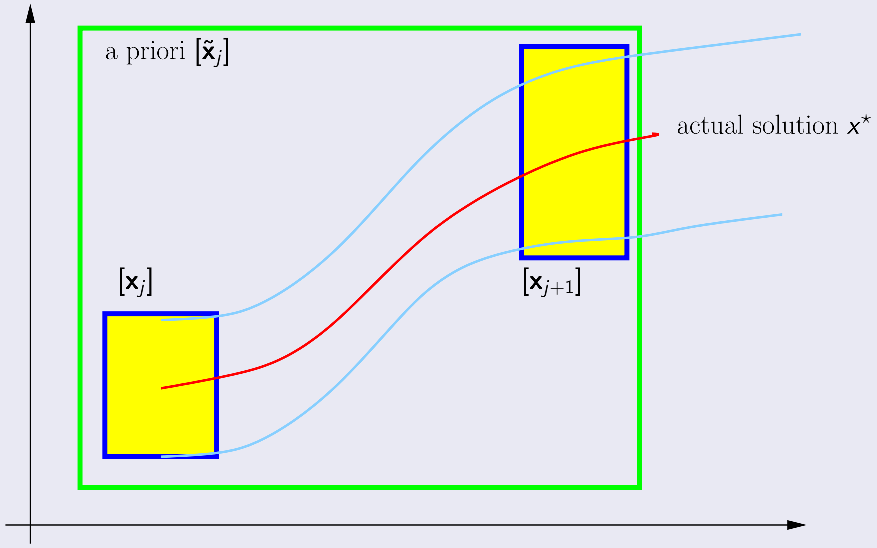
Nonlinear Set Integration

Guaranteed set integration with Taylor methods

- (Moore,66) (Lohner,88) (Nedialkov,99)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0]\,,\ \mathbf{p} \in [\mathbf{p}]$$

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



- Proof of existence
- Yield *a priori* solution $[\tilde{\mathbf{x}}_j]$: $\forall \tau \in [t_j,\ t_{j+1}] \quad x(\tau) \in [\tilde{\mathbf{x}}_j]$

Nonlinear Set Integration

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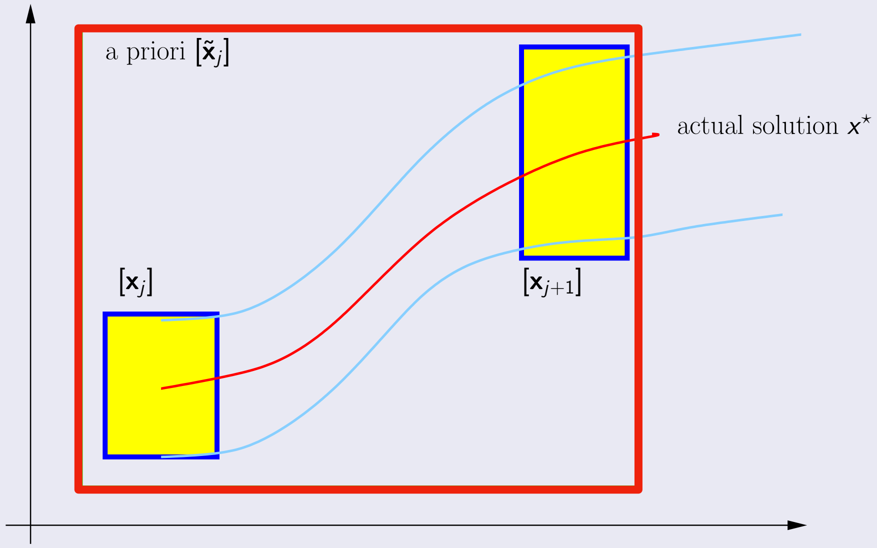

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Nonlinear Set Integration

Banach fixed-point theorem

Let operator $\Phi : \mathbb{X} \to \mathbb{X}$ be defined on a complete metric space $\{\mathbb{X}, d\}$ with a metric $d(.,.)$. Let γ satisfy $0 \leq \gamma < 1$. If Φ satisfies the Lipschitz condition

$$\forall \mathbf{z}_1,\ \mathbf{z}_1 \in \mathbb{X}, \quad d\left(\Phi\left(\mathbf{z}_1\right), \Phi\left(\mathbf{z}_2\right)\right) \leqslant \gamma d\left(\mathbf{z}_1, \mathbf{z}_2\right),$$

then Φ has a unique fixed-point $x^\star \in \mathbb{X}$.

- Metric can be exponential norm of $u(t) \in C^0[t_j,\ t_{j+1}]$

$$\|u\|_\alpha = \max_{t \in [t_j, t_{j+1}]} (e^{-\alpha(t-t_j)} \|u(t)\|), \quad \alpha > 0$$

- If $T\mathbb{X} \subseteq \mathbb{X}$, then T has a unique fixed-point in $\mathbb{X}$

Nonlinear Set Integration

Picard-Lindelöf operator

$\mathbb{U}$: set of continuous functions, $\nu_j = \nu(t_j)$, $\nu(t) \in \mathbb{U}$, $t \in [t_j, t_{j+1}]$

$$\Phi(\nu(t)) = \nu_j + \int_{t_j}^{t} \mathbf{f}(\tau, \nu(\tau)) d\tau$$

Property

$$\Phi(\nu^\star) = \nu^\star \Leftrightarrow \dot{\nu}^\star = \mathbf{f}(\nu^\star)$$

A priori solution $[\tilde{\mathbf{x}}_j]$

If $\Phi([\tilde{\mathbf{x}}_j]) \subseteq [\tilde{\mathbf{x}}_j]$ then $[\tilde{\mathbf{x}}_j] \supseteq \{\mathbf{x}(t) \mid t \in [t_j, t_{j+1}]\}$

Nonlinear Set Integration

Guaranteed set integration with Taylor methods

- (Moore,66) (Lohner,88) (Nedialkov,99), (Nedialkov et al, 01)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \ \mathbf{x}(t_0) \in [\mathbf{x}_0], \ \mathbf{p} \in [\mathbf{p}]$$

$$[\mathbf{x}_j] + [0, h]\mathbf{f}([\tilde{\mathbf{x}}_j]) \subseteq [\tilde{\mathbf{x}}_j]$$

```
A priori Enclosure (input : $[\mathbf{x}_j]$, $h$, $\alpha > 0$ ; output : $[\tilde{\mathbf{x}}_j]$, $h$)
  ① Initialization : $[\tilde{\mathbf{x}}_j] := [\mathbf{x}_j] + [0, h]\,\mathbf{f}\,([\mathbf{x}_j])$ ;
  ② While $([\mathbf{x}_j] + [0, h]\,\mathbf{f}\,([\tilde{\mathbf{x}}_j]) \not\subset [\tilde{\mathbf{x}}_j])$ do

          $[\tilde{\mathbf{x}}_j] := [\tilde{\mathbf{x}}_j] + [-\alpha, \alpha]\,|[\tilde{\mathbf{x}}_j]|$
          $h := h/2$

     end while
```

Nonlinear Set Integration

Guaranteed set integration with Taylor methods

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Nonlinear Set Integration

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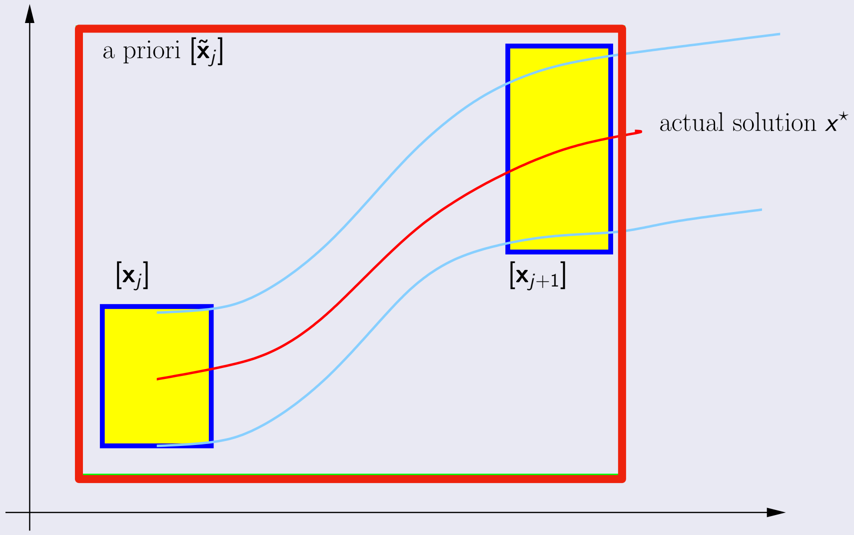
Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



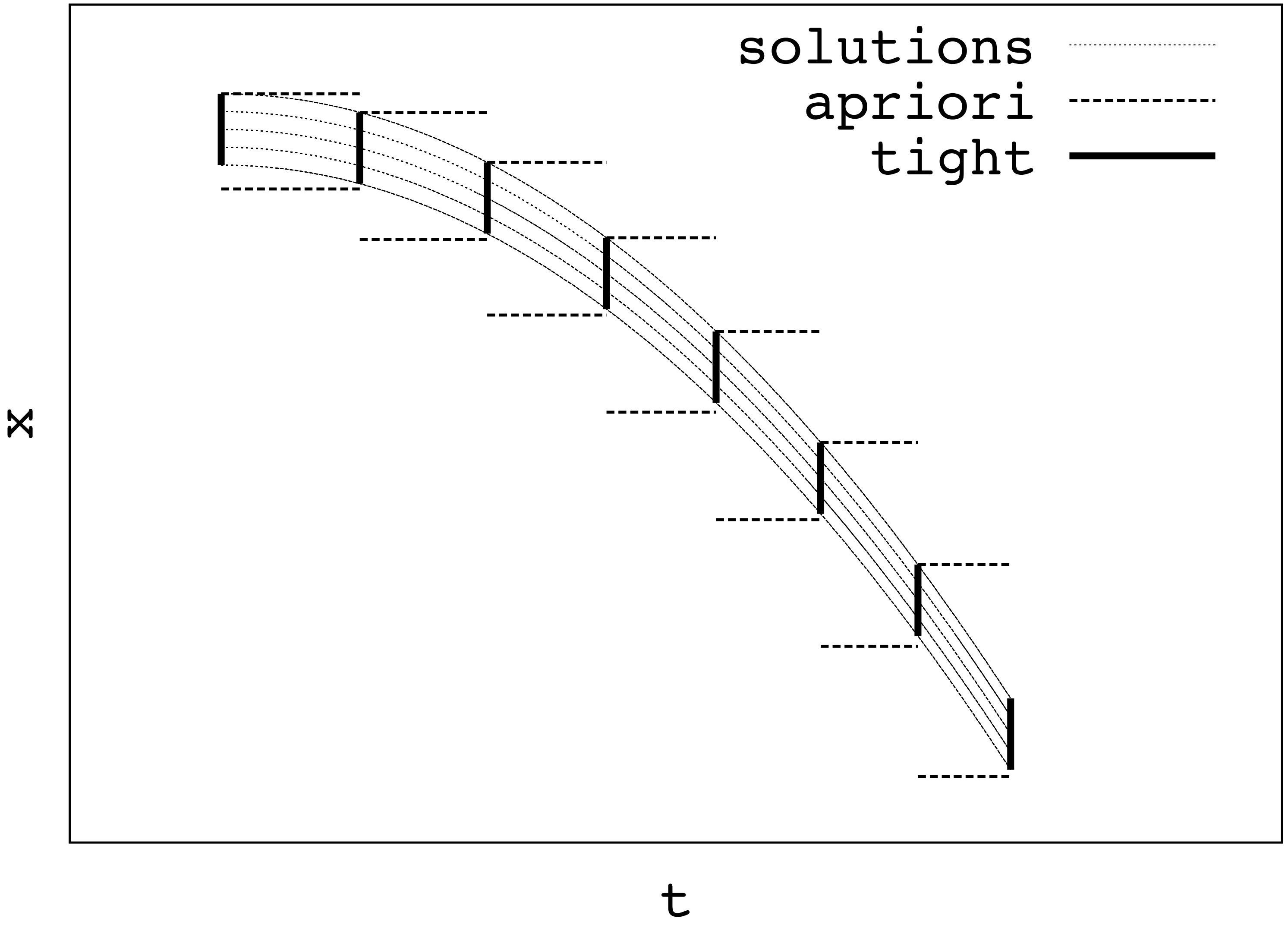
- Compute tight enclosure $[\mathbf{x}_{j+1}] \ni \mathbf{x}(t_{j+1})$

$$[\mathbf{x}_{j+1}] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t_{j+1} - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t_{j+1} - t_j)^k \mathbf{f}^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

Nonlinear Set Integration

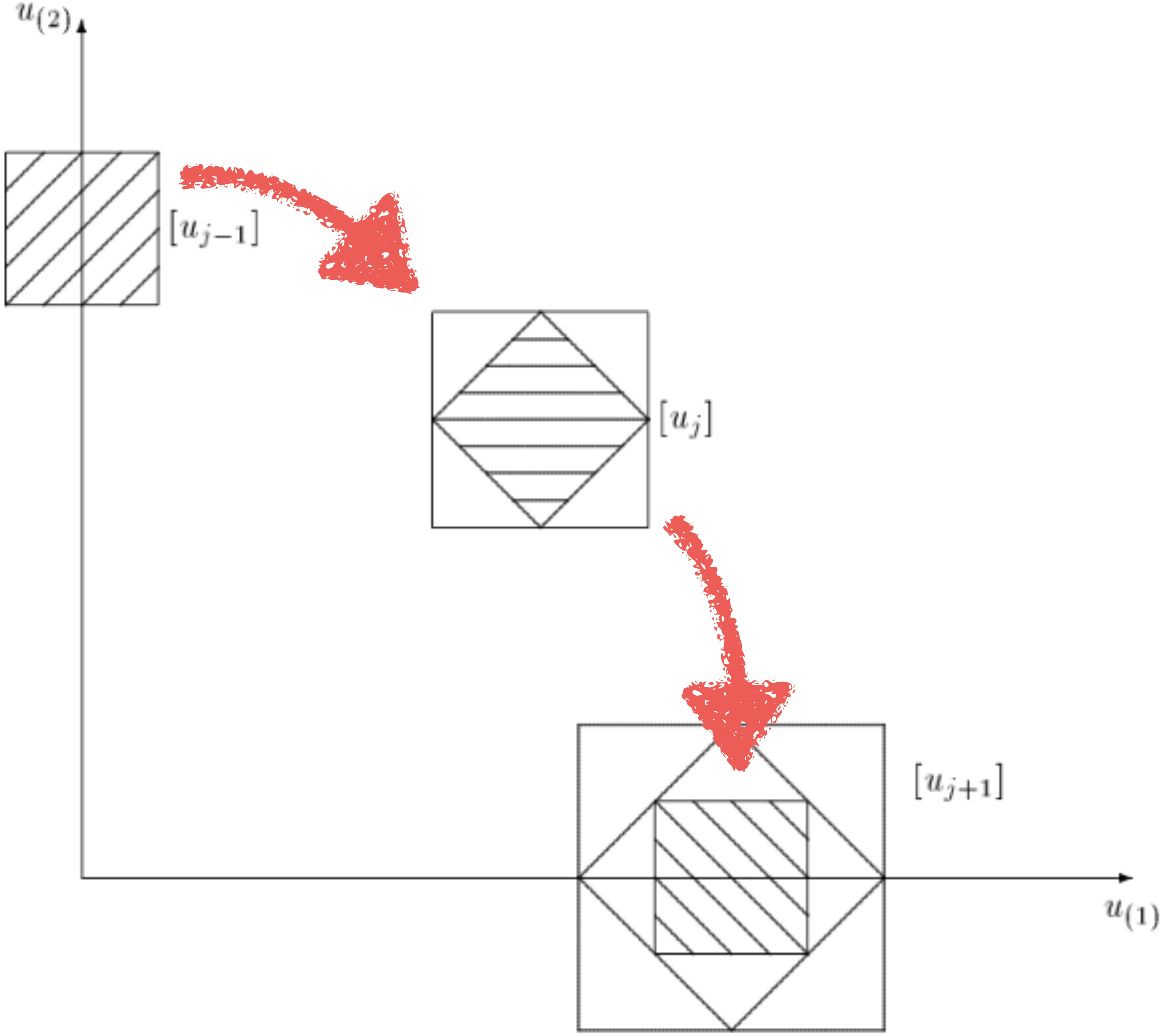
- **Guaranteed set integration with Taylor methods**
  - (Moore,66) (Lohner,88) (Nedialkov,99)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \; \mathbf{x}(t_0) \in [\mathbf{x}_0], \; \mathbf{p} \in [\mathbf{p}]$$



Nonlinear Set Integration

- **Wrapping effect (Moore,66)**



Nonlinear Set Integration

Guaranteed set integration with Taylor methods

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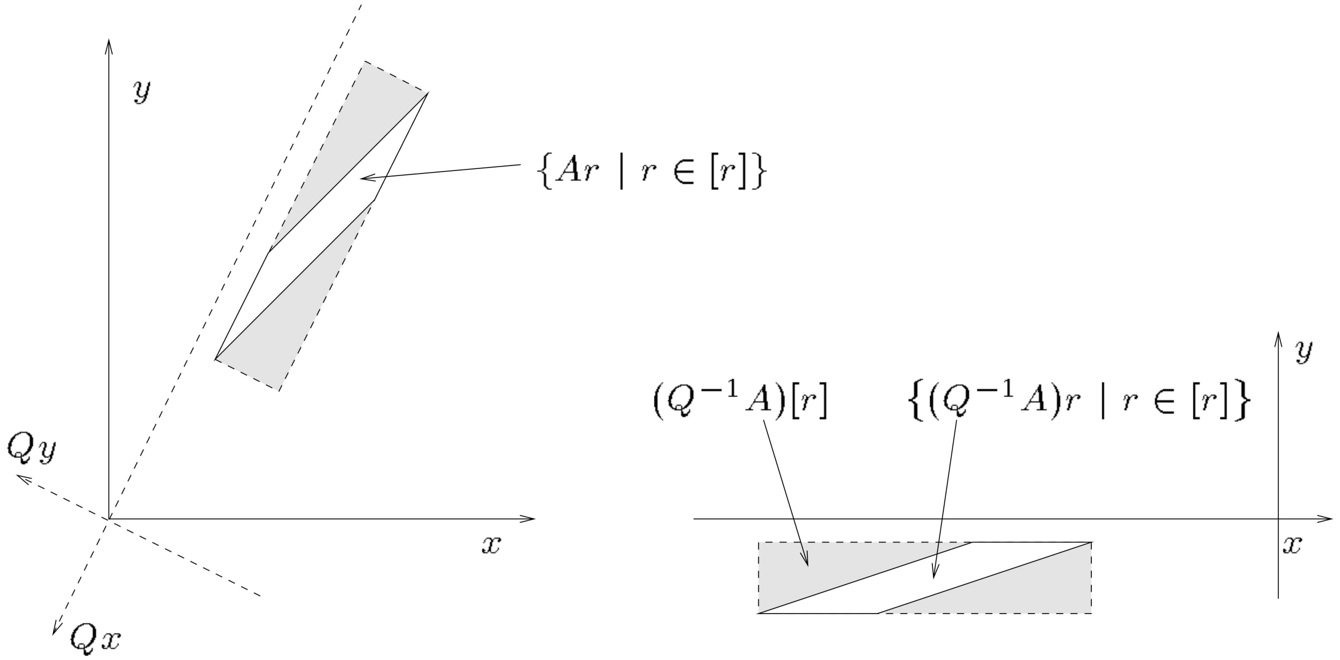
Mean-value approach

$$[\mathbf{x}](t) \in \left\{ \mathbf{v}(t) + \mathbf{A}(t)\mathbf{r}(t) \quad | \quad \mathbf{v}(t) \in [\mathbf{v}](t), \ \mathbf{r}(t) \in [\mathbf{r}](t) \right\}.$$



Nonlinear Set Integration

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$$\begin{aligned}
\mathbf{f}^{[1]} &= \mathbf{x}^{(1)} = \mathbf{f} \\
\mathbf{f}^{[2]} &= \tfrac{1}{2}\mathbf{x}^{(2)} = \frac{1}{2}\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}\mathbf{f} \\
\mathbf{f}^{[i]} &= \tfrac{1}{i!}\mathbf{x}^{(i)} = \frac{1}{i}\frac{\mathrm{d}\mathbf{f}^{[i-1]}}{\mathrm{d}\mathbf{x}}\mathbf{f}, \ i \geqslant 2
\end{aligned}$$

Nonlinear Set Integration

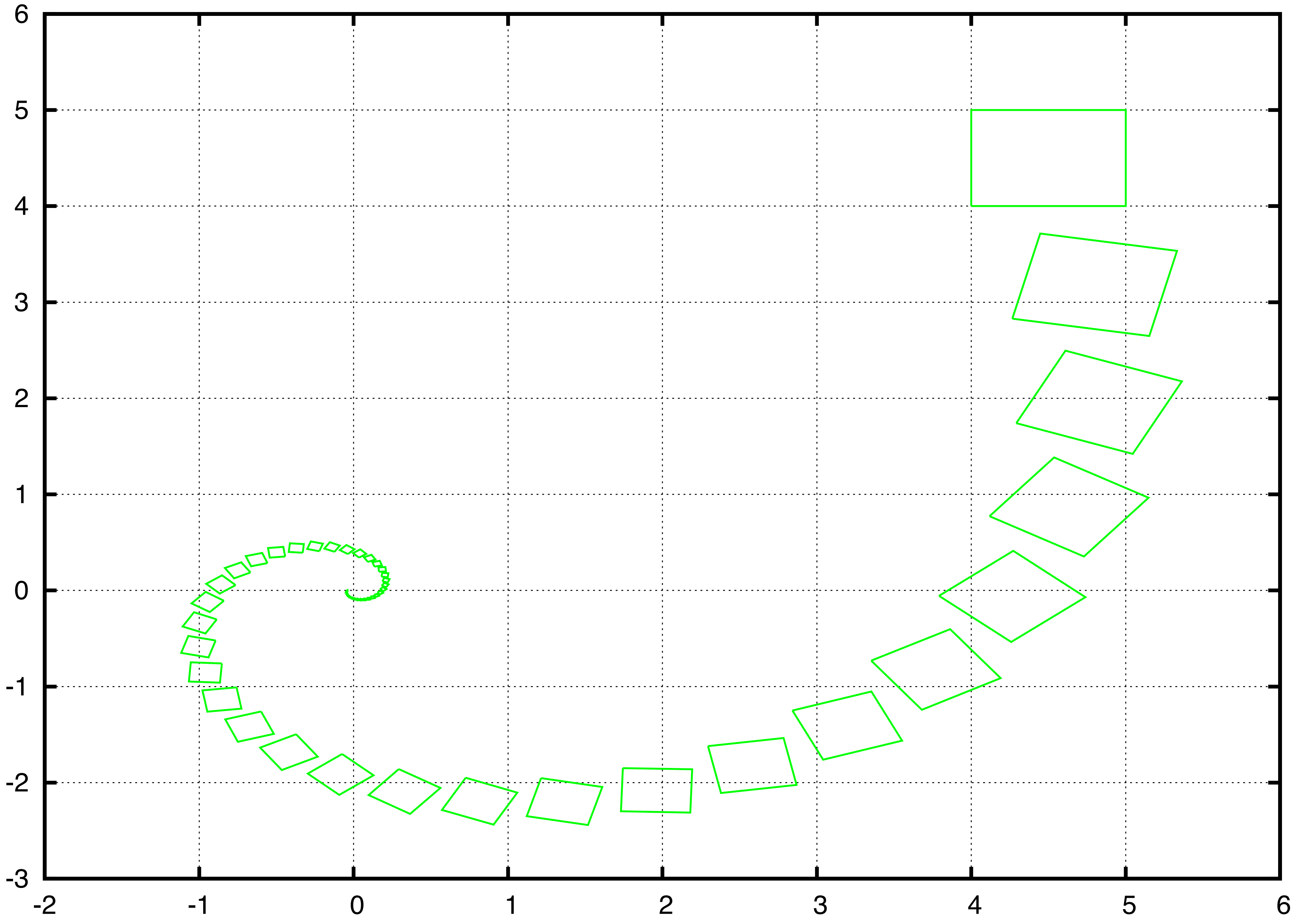
- **Guaranteed set integration with Taylor methods**
  - (Moore,66) (Lohner,88) (Rihm,94) (Berz,98) (Nedialkov,99)

- **Complexity**
  - *Work per step is of polynomial complexity*
    - Computing Taylor coefficients → o($\boldsymbol{k}^2$)
    - Linear algebra → o($\boldsymbol{n}^3$)

- **In practice :** Obtaining Taylor coefficients ...
  - **FADBAD++** (www.fadbad.com)

    Flexible Automatic differentiation using templates and operator overloading in C++

Nonlinear Set Integration



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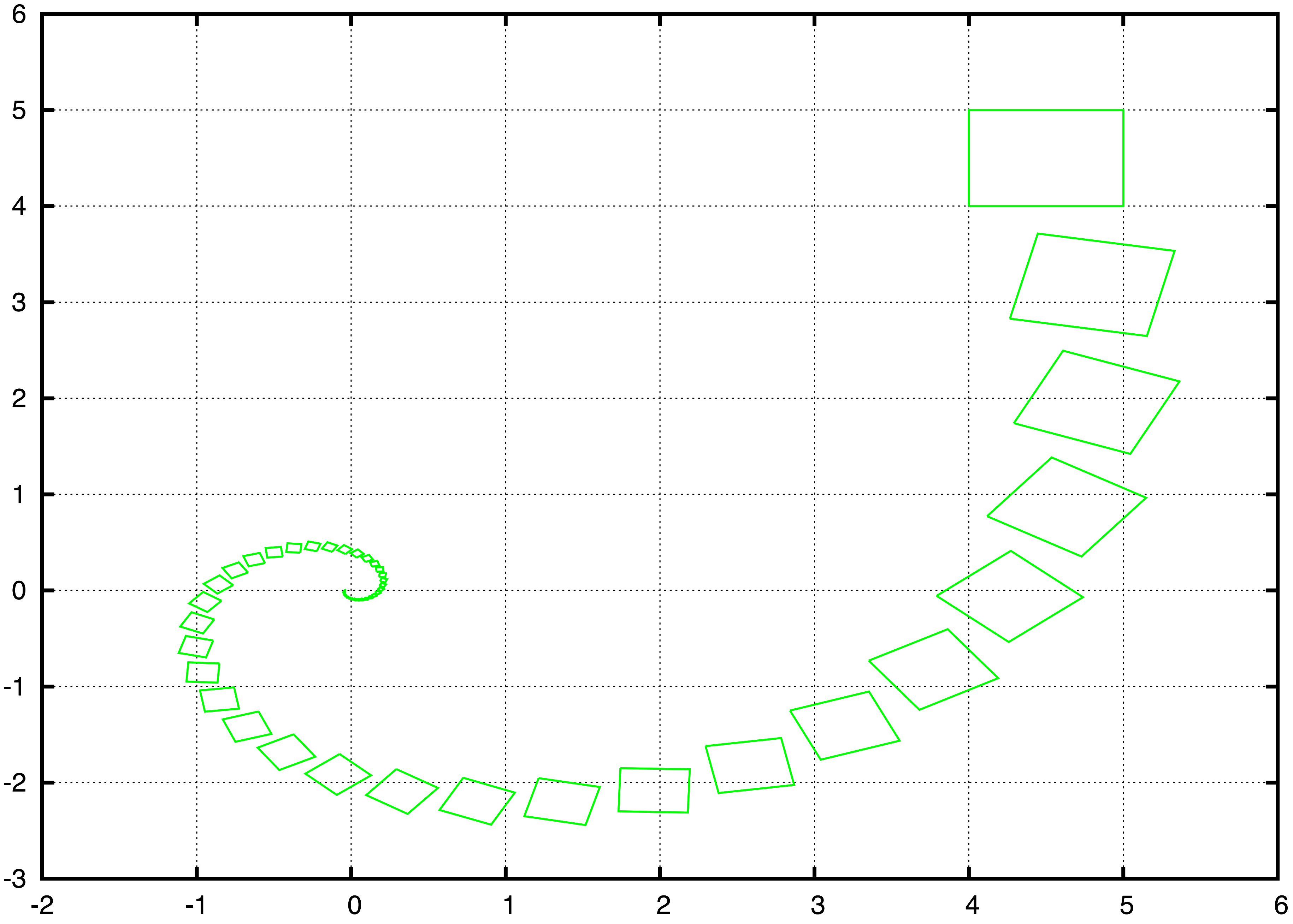

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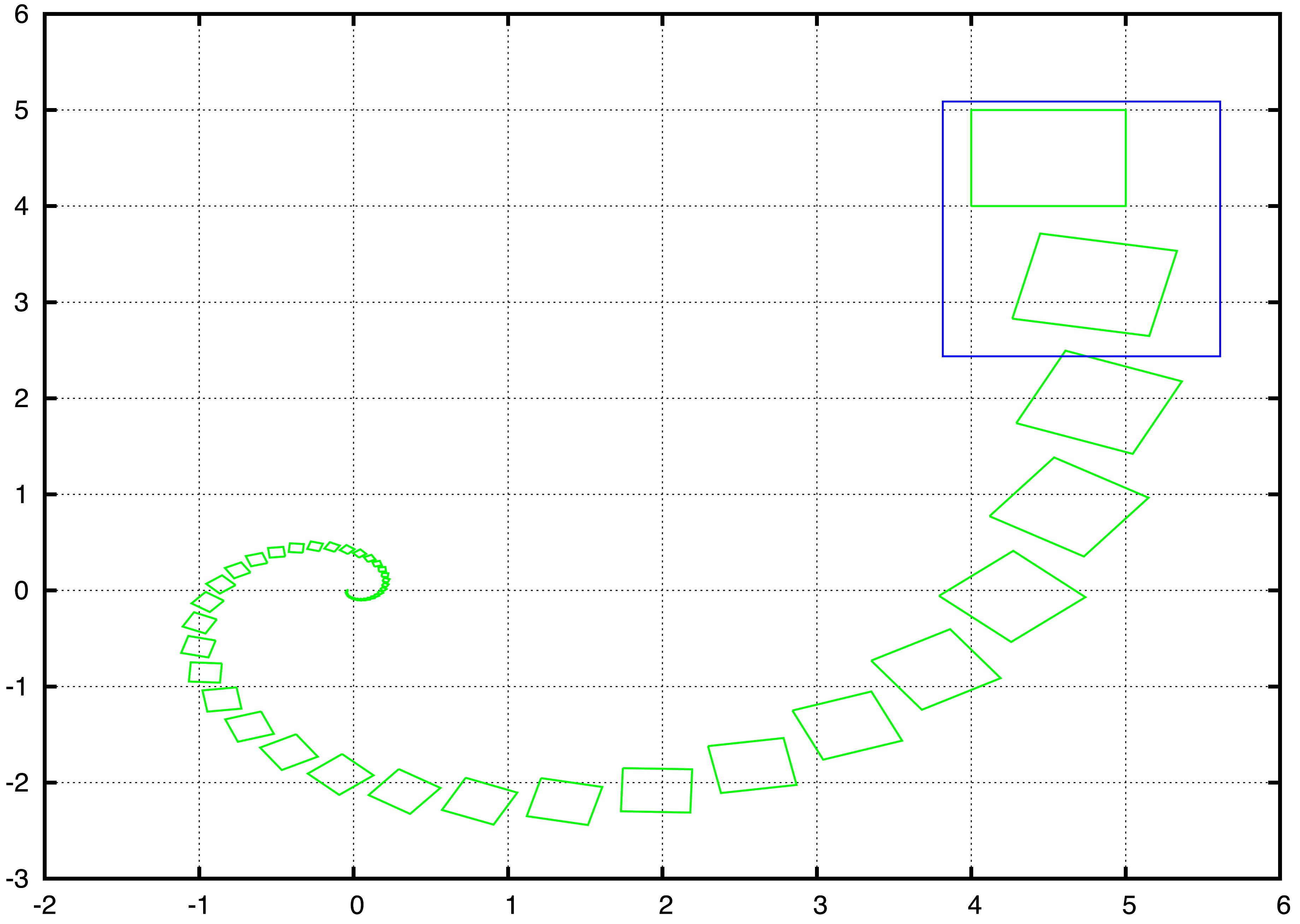

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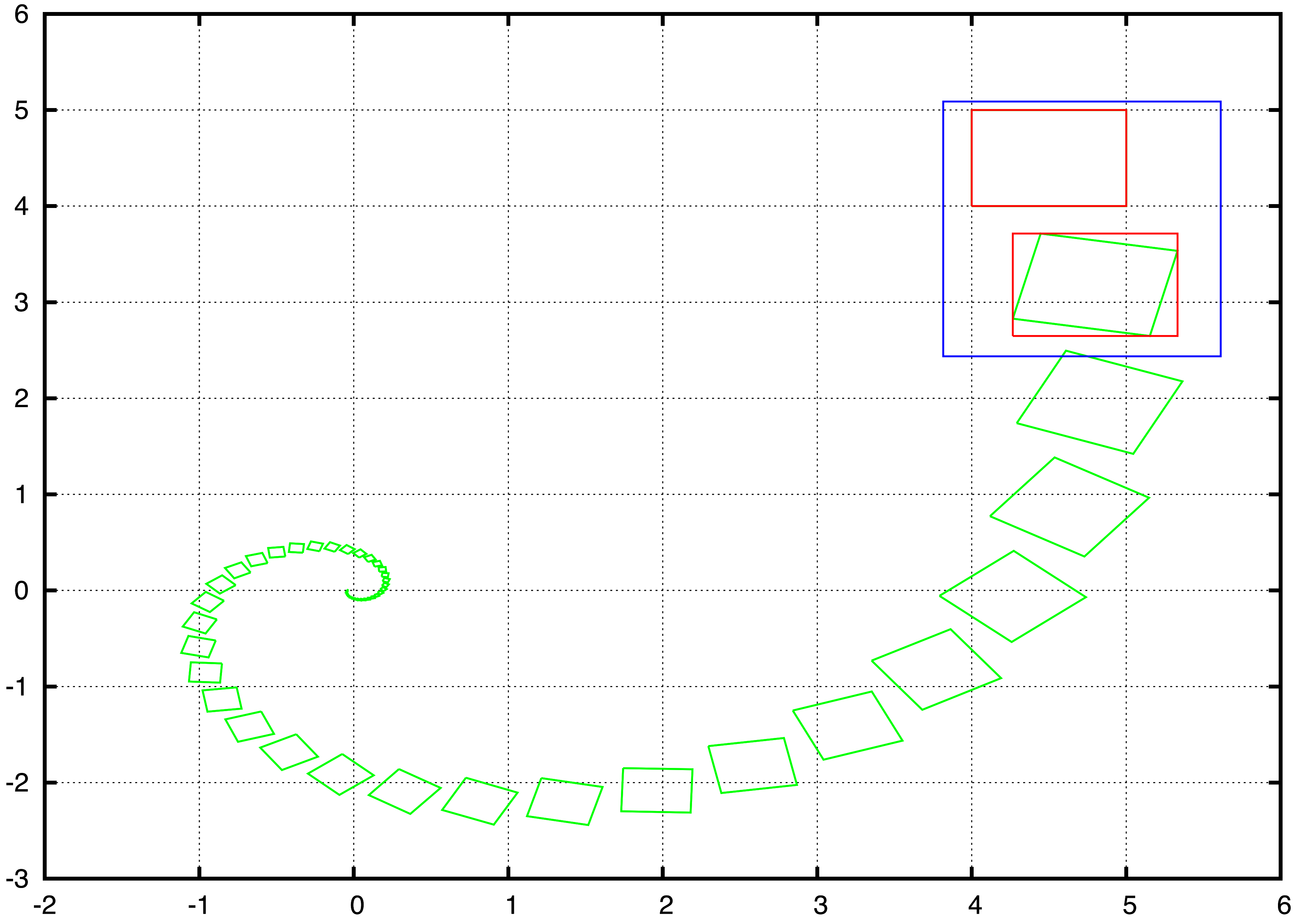

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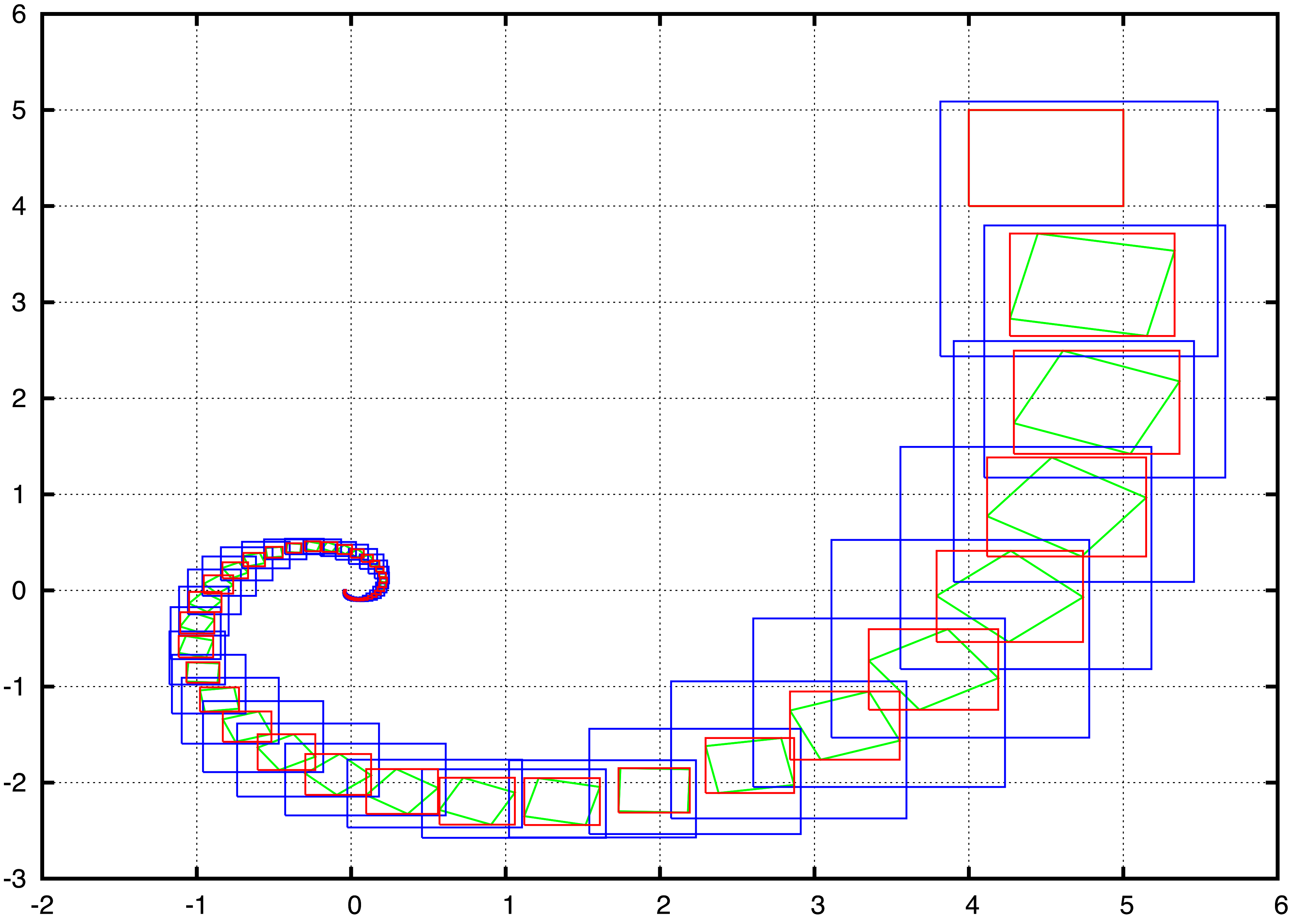

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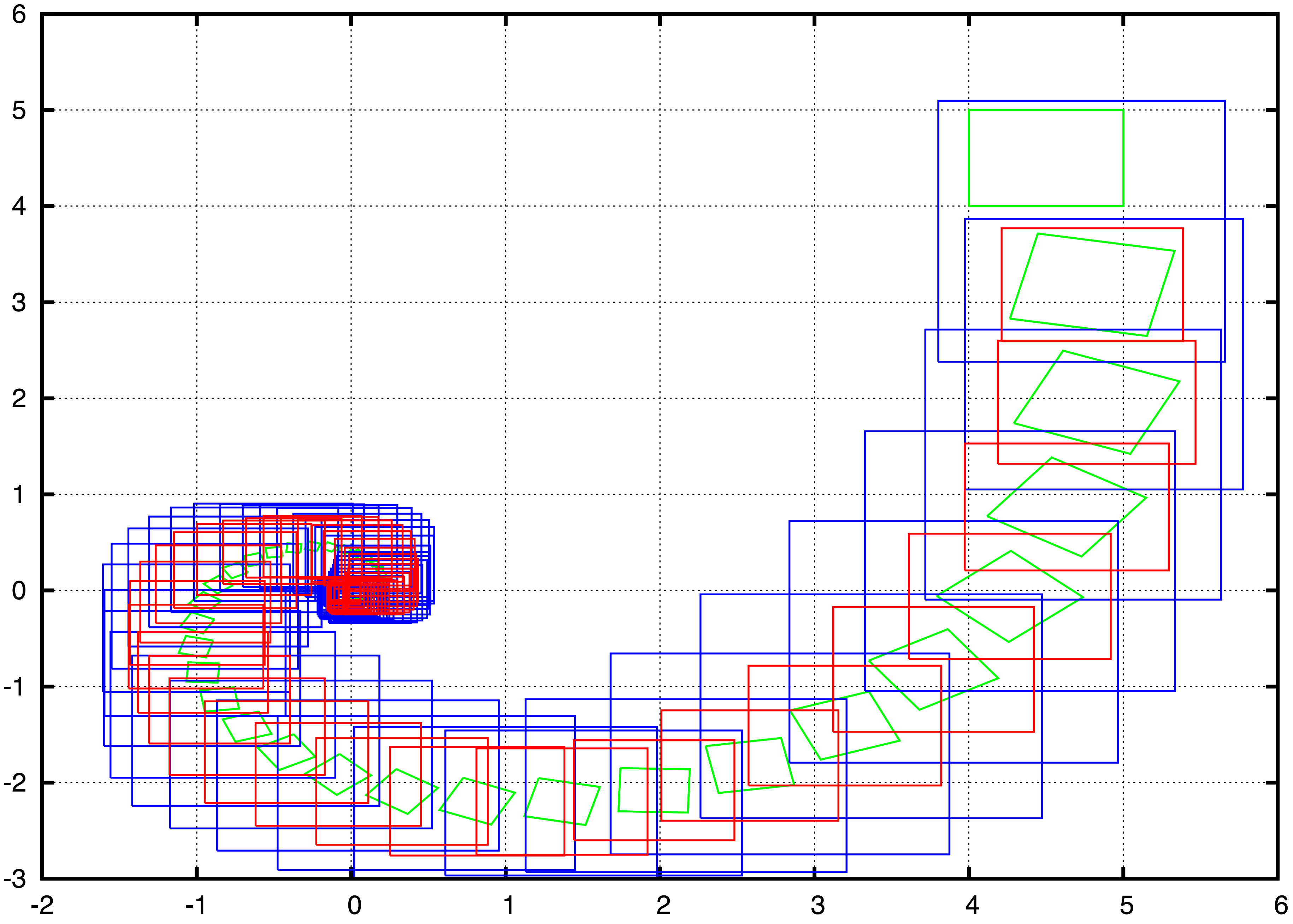


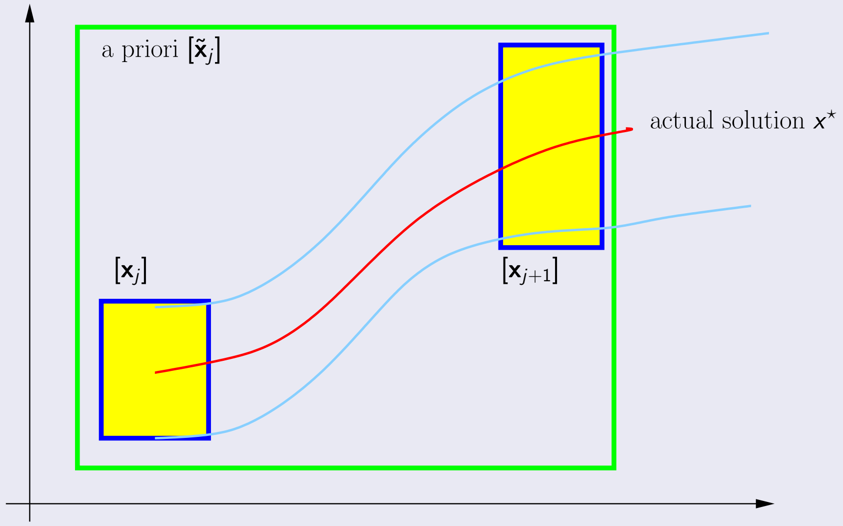
Nonlinear Set Integration

Piecewise analytical expressions for the solution tube

- (Ramdani et Nedialkov, 2011)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \le t \le t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \mathbf{p} \in [\mathbf{p}]$$

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



- Analytical solution for $[\mathbf{x}](t),\ t \in [t_j, t_{j+1}]$

Nonlinear Set Integration

Piecewise analytical expressions for the solution tube

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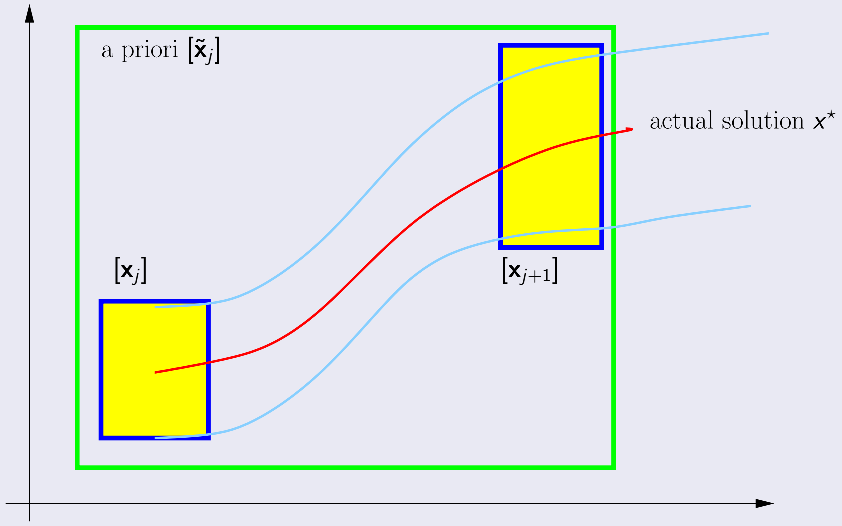

- Analytical solution for $[\mathbf{x}](t),\ t \in [t_j, t_{j+1}]$

$$[\mathbf{x}](t) = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t - t_j)^k \mathbf{f}^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

Nonlinear Set Integration

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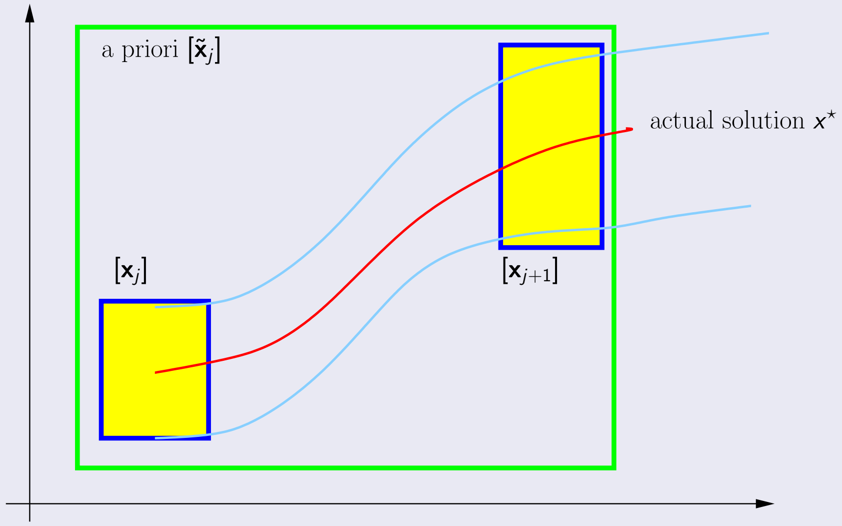
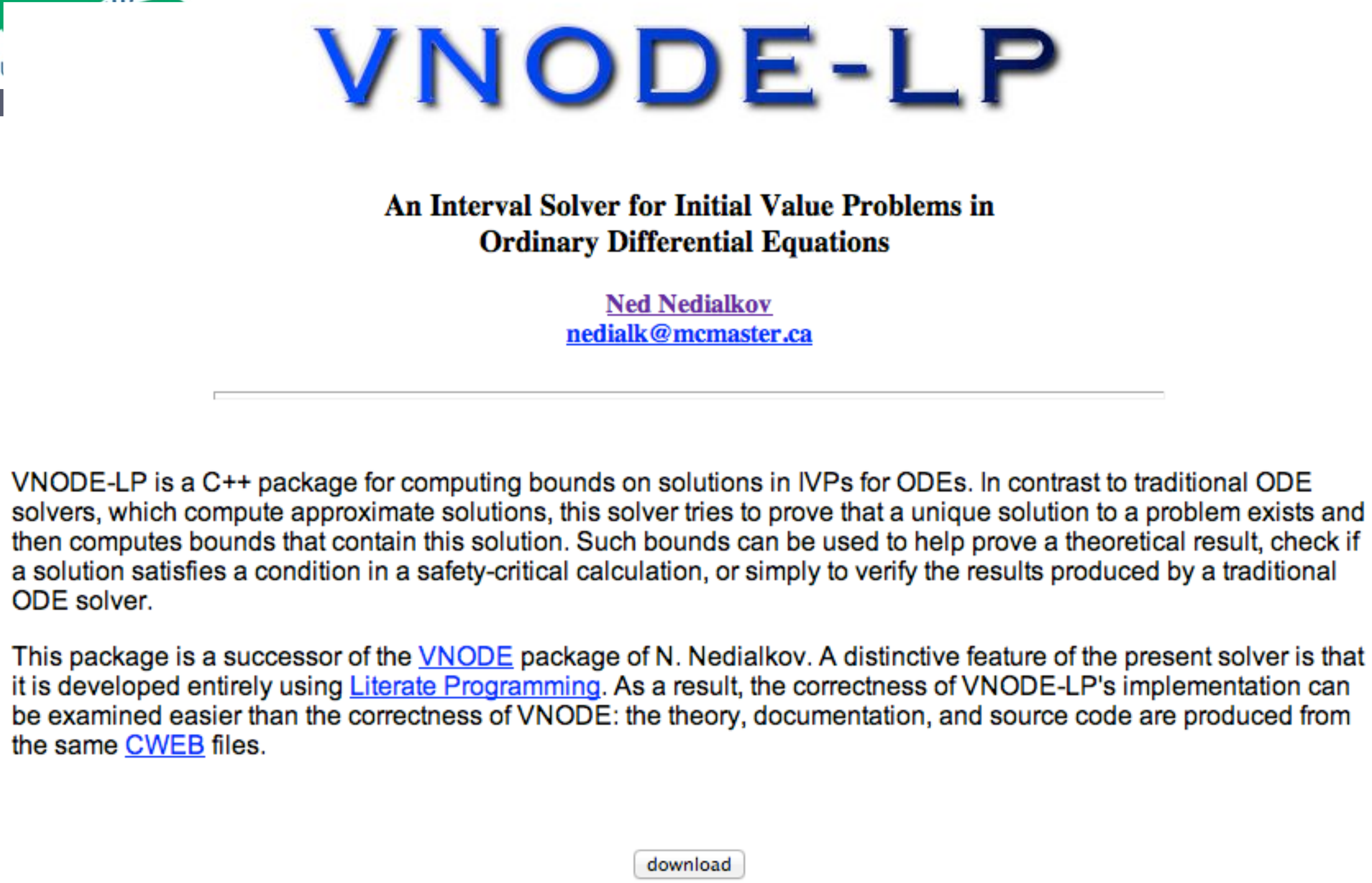


$$\forall \tau \in [t_j,\; t_j + h_j] \quad \mathbf{x}(\tau) \in [\mathbf{x}(\tau)]$$

$$[\mathbf{x}(\tau)] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (\tau - t_j)^i f^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (\tau - t_j)^k f^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

$$\mathbb{R}([t_0,\; t]\,; [\mathbf{x}_0]) \subseteq \cup_{\tau \in \{t_0,\, t\}} [\mathbf{x}(\tau)] \subseteq \cup_{j \in \{0,\, t\}} [\tilde{\mathbf{x}}_j]$$



VNODE-LP

An Interval Solver for Initial Value Problems in Ordinary Differential Equations

Ned Nedialkov
nedialk@mcmaster.ca

VNODE-LP is a C++ package for computing bounds on solutions in IVPs for ODEs. In contrast to traditional ODE solvers, which compute approximate solutions, this solver tries to prove that a unique solution to a problem exists and then computes bounds that contain this solution. Such bounds can be used to help prove a theoretical result, check if a solution satisfies a condition in a safety-critical calculation, or simply to verify the results produced by a traditional ODE solver.

This package is a successor of the VNODE package of N. Nedialkov. A distinctive feature of the present solver is that it is developed entirely using Literate Programming. As a result, the correctness of VNODE-LP's implementation can be examined easier than the correctness of VNODE: the theory, documentation, and source code are produced from the same CWEB files.

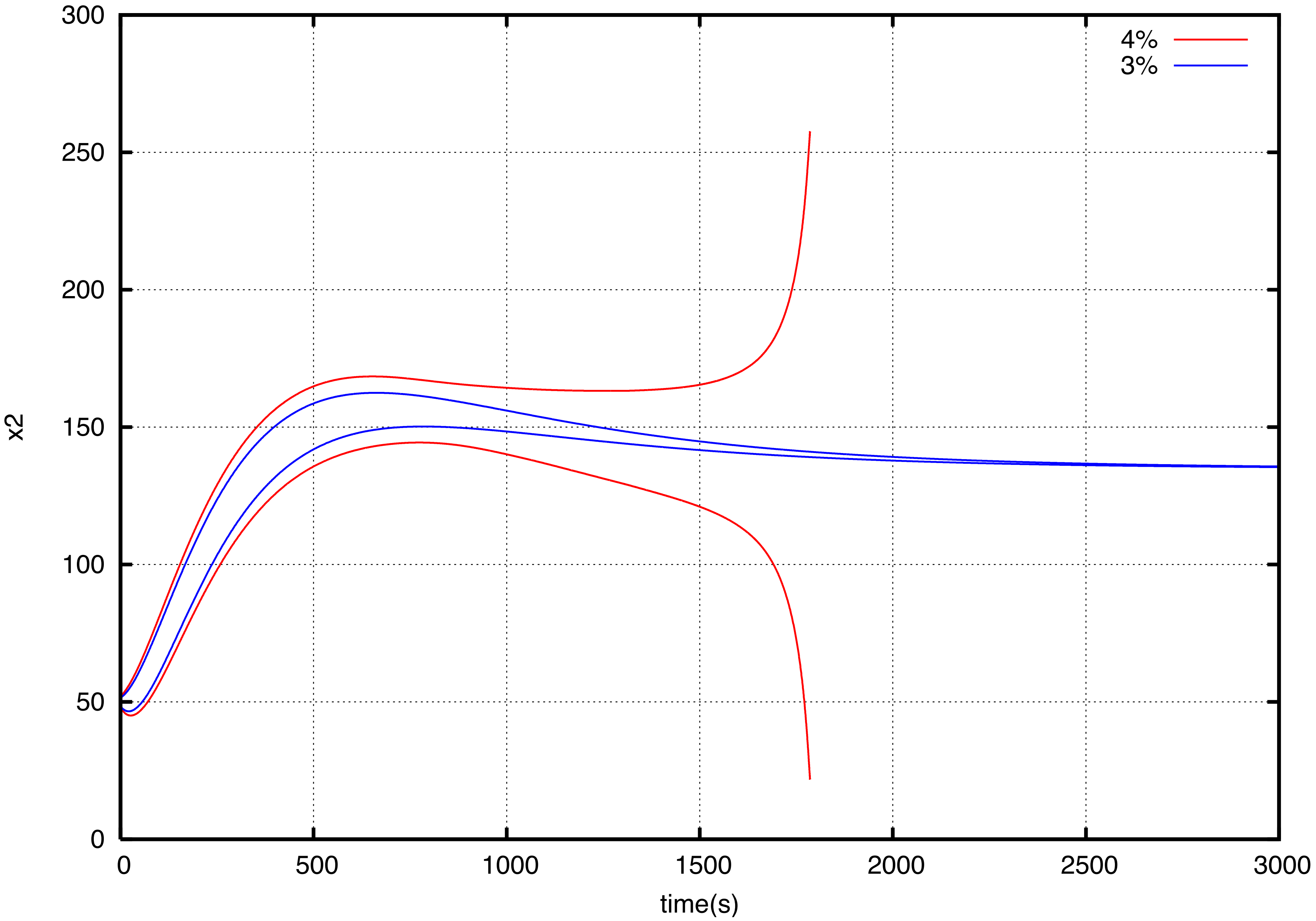
Nonlinear Set Integration

Nonlinear Set Integration

- **Guaranteed set integration**
  - **… with interval Taylor methods. (VNODE, VSPODE)**
    - (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)
  - **… with interval Taylor models. (Flow*, VSPODE)**
    - (Berz & Makino, 1996) (Chen, 2012)
  - **… with validated Runge Kutta. (DynIbex)**
    - (Alexandre dit Sandretto & Chapoutot, 2015)

Nonlinear Set Integration

Interval Taylor approach not always successful !!



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    - (Alexandre dit Sandretto & Chapoutot, 2015)
- **Use comparison theorems for differential inequalities**
  - **Monotone systems**
    - (Ramdani et al., 2010)
  - **Muller's theorem**
    - (Ramdani, et al. 2006) (Kieffer et al. 2006) (Ramdani, et al. 2009)

Nonlinear Set Integration

Comparison theorems for differential inequalities

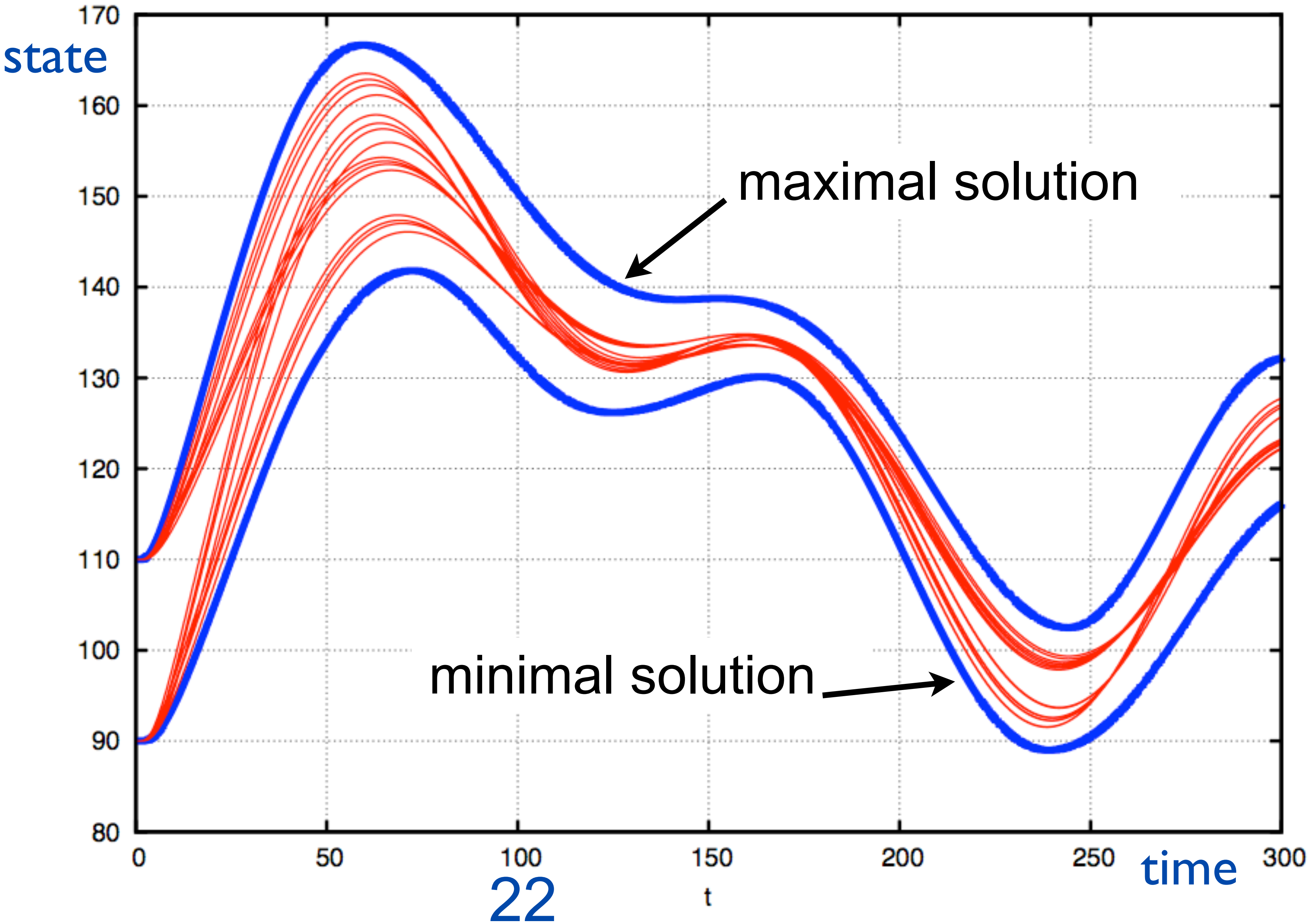
- Müller’s existence theorem (1936)

$$\text{If}\quad \left\{\begin{array}{c} \forall t \in [t_0,\ t_N],\ \forall \mathbf{x} \in \mathbb{D},\ \forall \mathbf{p} \in \mathbb{P} \\ \forall i\ \min_{x_i=\omega_i} f_i(\mathbf{x}, \mathbf{p}, t) \geq D^{\pm}\omega_i(t) \\ \forall i\ \max_{x_i=\Omega_i} f_i(\mathbf{x}, \mathbf{p}, t) \leq D^{\pm}\Omega_i(t) \\ \omega(t_0) \leq \mathbf{x}(t_0) \leq \Omega(t_0) \end{array}\right. \Rightarrow \left\{\begin{array}{c} \text{solution exists} \\ \\ \forall t \in [t_0,\ t_N], \\ \omega(t) \leq \mathbf{x}(t) \leq \Omega(t) \end{array}\right.$$

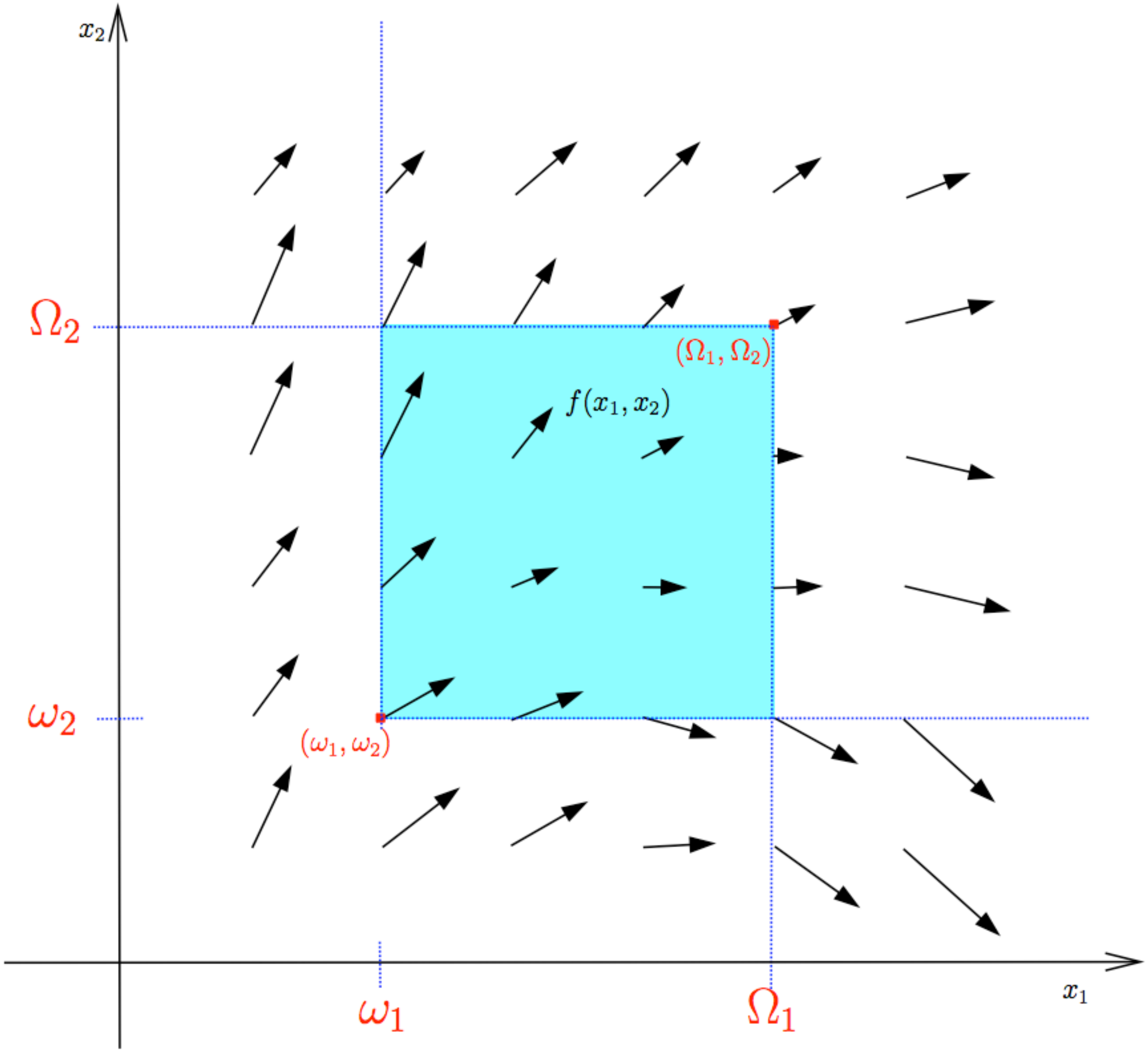
- **Bracketing systems**
  - (Ramdani, et al., IEEE Trans. Automatic Control 2009)

Nonlinear Set Integration

- Comparison theorems for differential inequalities
  - Bracketing systems



Müller’s theorem



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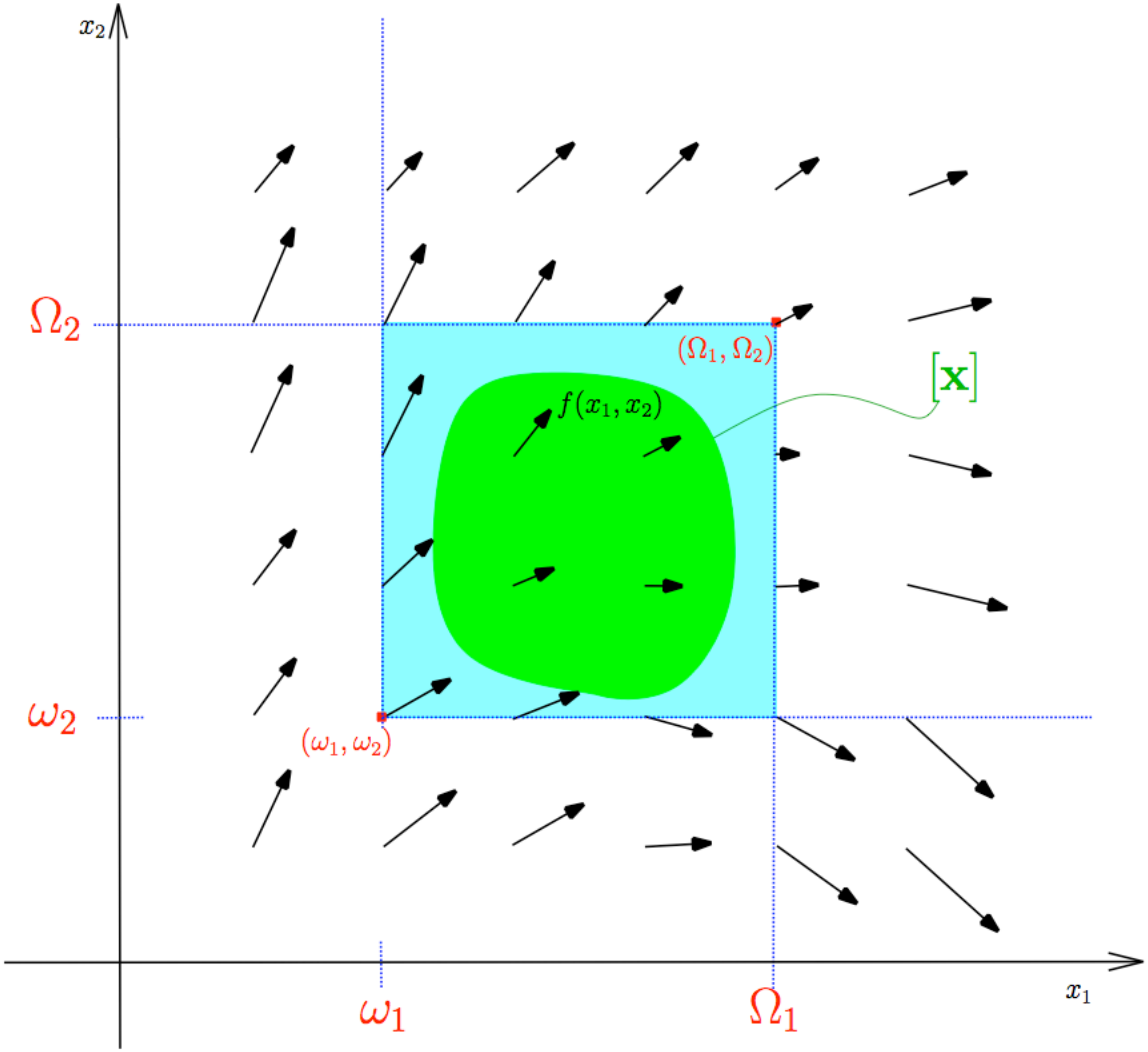

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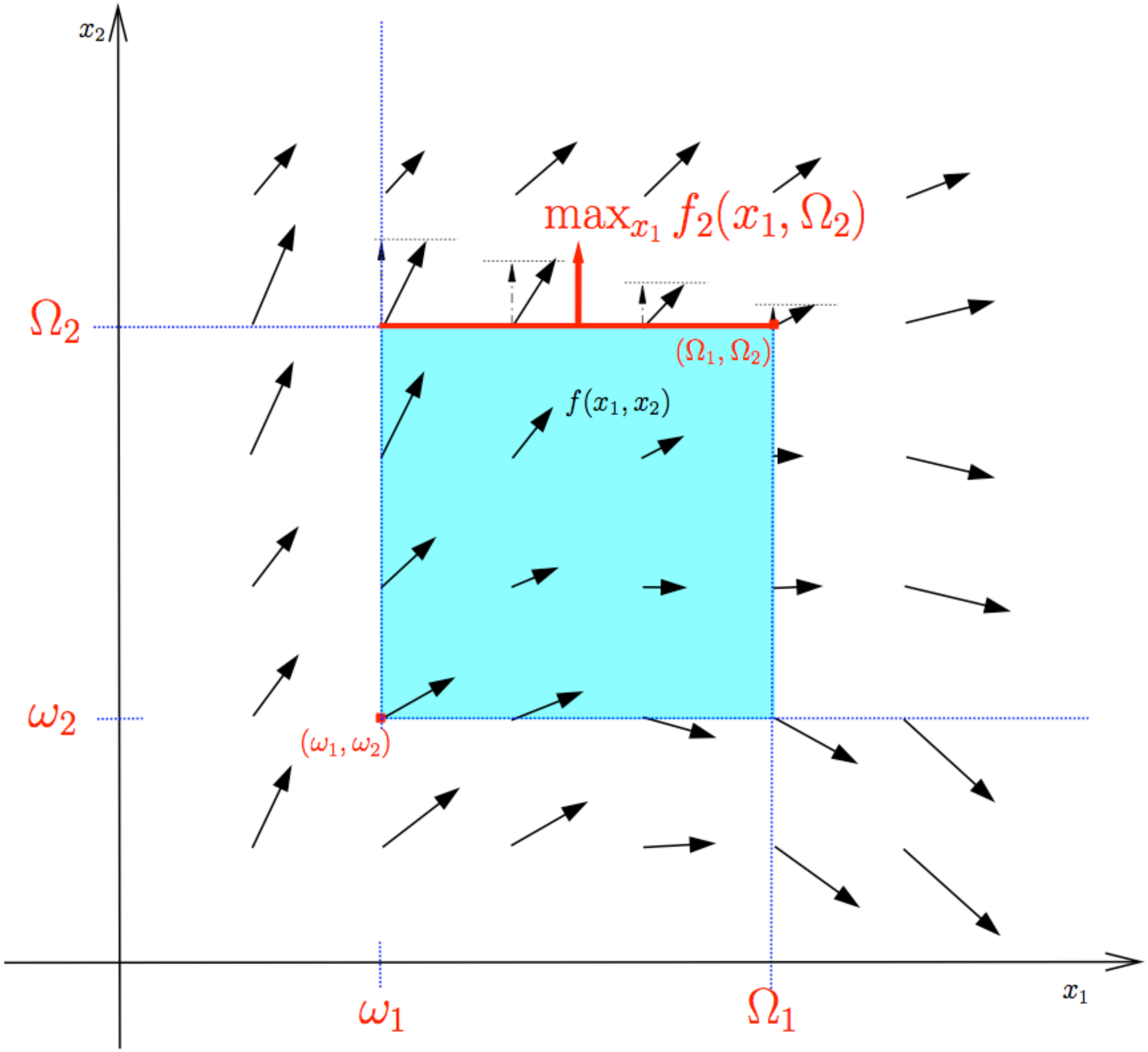

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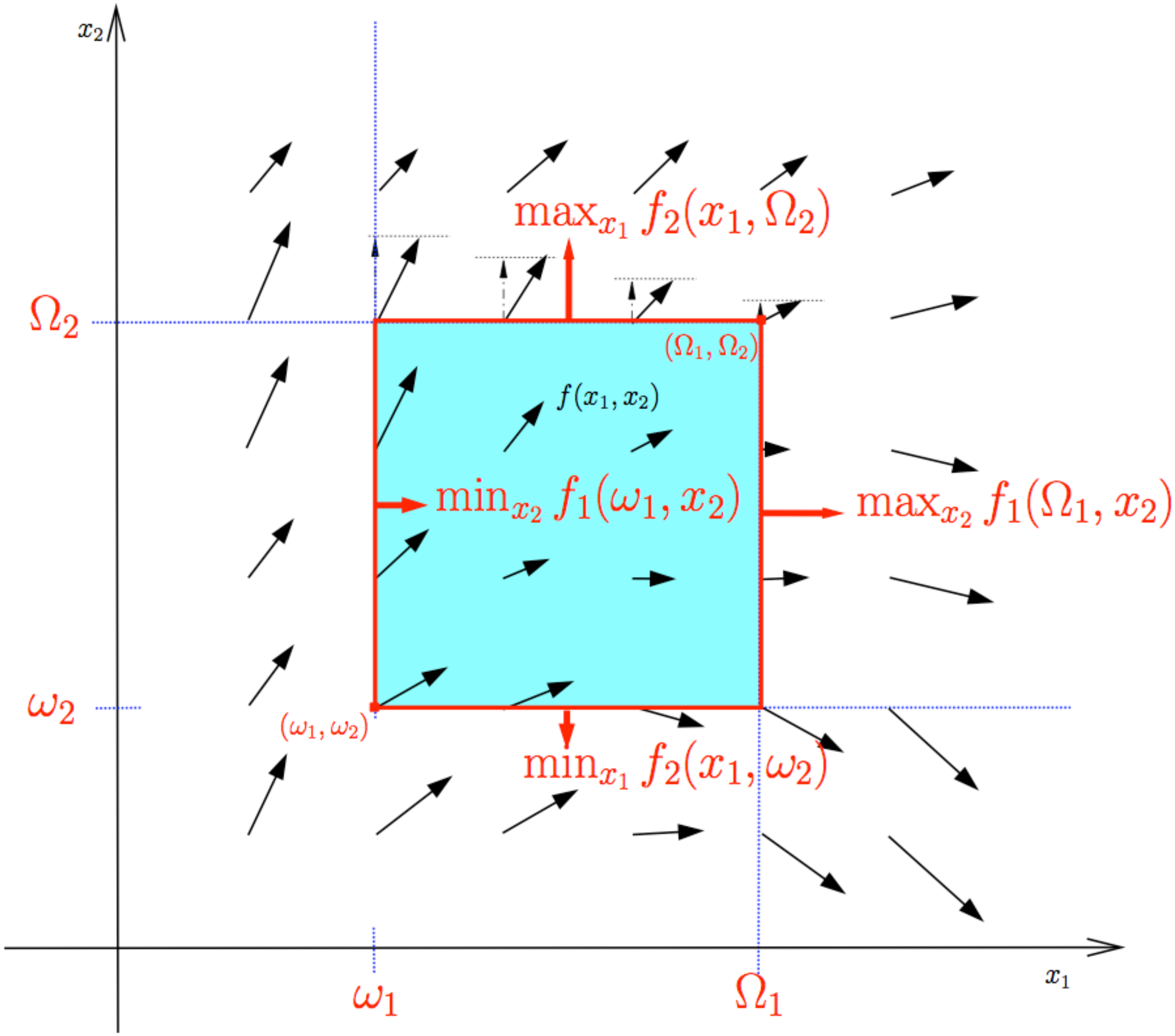

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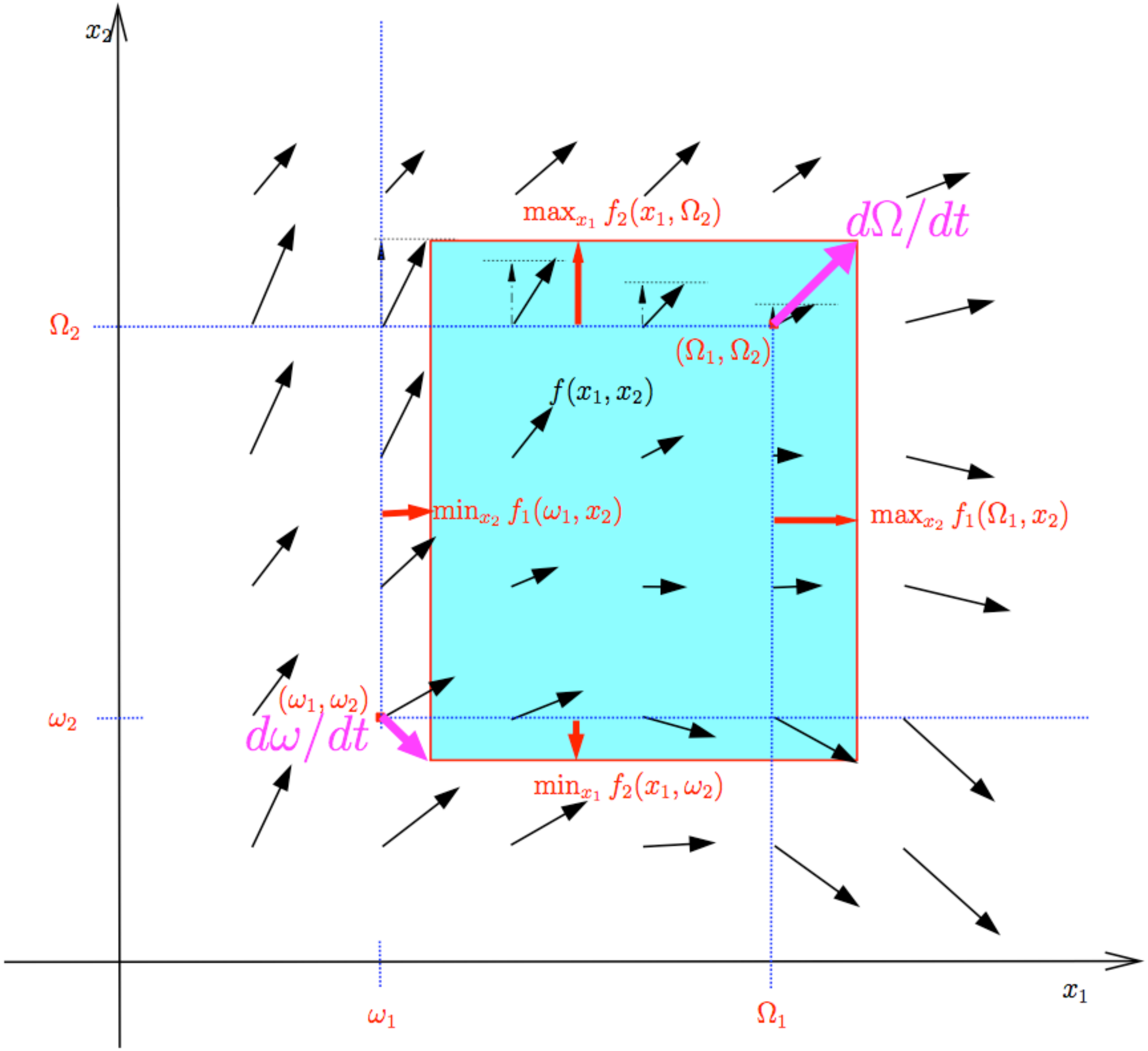

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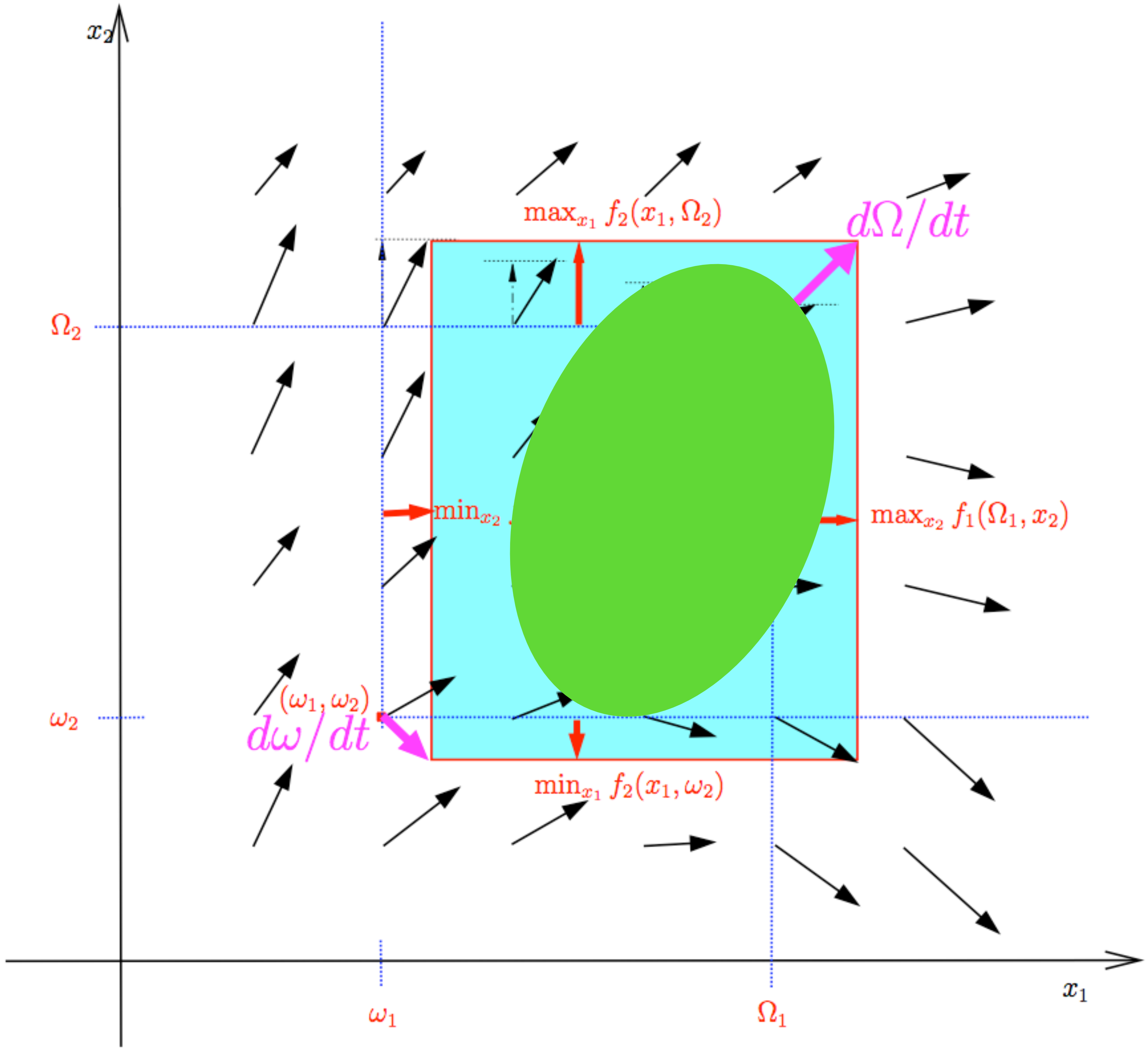

Nonlinear Set Integration

Bracketing systems

- Dynamics of ...

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, p, t), & x_1(t_0) \in [\underline{x}_{1,0}, \overline{x}_{1,0}] \subset \mathbb{R}, \quad p \in [\underline{p}, \overline{p}] \quad t \geq t_0 \\ \dot{x}_2 = f_2(x_1, x_2, p, t), & x_2(t_0) \in [\underline{x}_{2,0}, \overline{x}_{2,0}] \subset \mathbb{R}, \end{cases}$$

If $\forall t \geq t_0$, $\forall \mathbf{x}(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2$, $\forall p \in [\underline{p}, \overline{p}]$,

$$\frac{\partial f_1}{\partial x_2} > 0 \wedge \frac{\partial f_1}{\partial p} > 0$$

then $f_1(\omega_1, \omega_2, \underline{p}) \leq f_1(\omega_1, x_2, p, t)$ and $f_1(\Omega_1, x_2, p, t) \leq f_1(\Omega_1, \Omega_2, \overline{p})$

$$\dot{\omega}_1(t) \equiv f_1(\omega_1, \omega_2, \underline{p}) \quad \text{and} \quad f_1(\Omega_1, \Omega_2, \overline{p}) \equiv \dot{\Omega}_1(t)$$

Nonlinear Set Integration

- **Bracketing systems**
  - Dynamics of ...

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, p, t), & x_1(t_0) \in [\underline{x}_{1,0}, \overline{x}_{1,0}] \subset \mathbb{R}, \quad p \in [\underline{p}, \overline{p}] \quad t \geq t_0 \\ \dot{x}_2 = f_2(x_1, x_2, p, t), & x_2(t_0) \in [\underline{x}_{2,0}, \overline{x}_{2,0}] \subset \mathbb{R}, \end{cases}$$

If $\forall t \geq t_0$, $\forall \mathbf{x}(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2$, $\forall p \in [\underline{p}, \overline{p}]$,

$$\frac{\partial f_1}{\partial x_2} > 0 \wedge \frac{\partial f_1}{\partial p} > 0$$

then $f_1(\omega_1, \omega_2, \underline{p}) \leq f_1(\omega_1, x_2, p, t)$ and $f_1(\Omega_1, x_2, p, t) \leq f_1(\Omega_1, \Omega_2, \overline{p})$

$$\boxed{\dot{\omega}_1(t) \equiv f_1(\omega_1, \omega_2, \underline{p})} \quad \text{and} \quad \boxed{f_1(\Omega_1, \Omega_2, \overline{p}) \equiv \dot{\Omega}_1(t)}$$

Nonlinear Set Integration

Comparison theorems for differential inequalities

- Müller’s existence theorem (1936)

$$\text{If}\quad \left\{\begin{array}{c} \forall t \in [t_0,\ t_N],\ \forall \mathbf{x} \in \mathbb{D},\ \forall \mathbf{p} \in \mathbb{P} \\ \forall i\ \min_{x_i=\omega_i} f_i(\mathbf{x},\mathbf{p},t) \geq D^{\pm}\omega_i(t) \\ \forall i\ \max_{x_i=\Omega_i} f_i(\mathbf{x},\mathbf{p},t) \leq D^{\pm}\Omega_i(t) \\ \omega(t_0) \leq \mathbf{x}(t_0) \leq \Omega(t_0) \end{array}\right. \Rightarrow \left\{\begin{array}{c} \text{solution exists} \\ \\ \forall t \in [t_0,\ t_N], \\ \omega(t) \leq \mathbf{x}(t) \leq \Omega(t) \end{array}\right.$$

- **Bracketing systems : coupled ODEs**

$$\Rightarrow \left\{\begin{array}{ll} \dot{\boldsymbol{\omega}}(t) = \underline{f}(\boldsymbol{\omega},\boldsymbol{\Omega},\underline{\mathbf{p}},\overline{\mathbf{p}},t), & \boldsymbol{\omega}(t_0) = \underline{\mathbf{x}}_0 \\ \dot{\boldsymbol{\Omega}}(t) = \overline{f}(\boldsymbol{\omega},\boldsymbol{\Omega},\underline{\mathbf{p}},\overline{\mathbf{p}},t), & \boldsymbol{\Omega}(t_0) = \overline{\mathbf{x}}_0 \end{array}\right.$$

Nonlinear Set Integration

- **Bracketing systems**
  - Example : Mitogen- Activated Protein Kinase (Sontag, 2005)

Nonlinear Set Integration

- **Bracketing systems**
  - Example : Mitogen- Activated Protein Kinase (Sontag, 2005)

$$\begin{cases} \dot{x}_1 & = & -\frac{v_2 x_1}{k_2+x_1} + v_0 u + v_1 \\ \dot{x}_2 & = & \frac{v_6(y_{tot}-x_2-x_3)}{k_6+(y_{tot}-x_2-x_3)} - \frac{v_3 x_1 x_2}{k_3+x_2} \\ \dot{x}_3 & = & \frac{v_4 x_1 (y_{tot}-x_2-x_3)}{k_4+(y_{tot}-x_2-x_3)} - \frac{v_5 x_3}{k_5+x_3} \\ \dot{x}_4 & = & \frac{v_{10}(z_{tot}-x_4-x_5)}{k_{10}+(z_{tot}-x_4-x_5)} - \frac{v_7 x_3 x_4}{k_7+x_4} \\ \dot{x}_5 & = & \frac{v_8 x_3 (z_{tot}-x_4-x_5)}{k_8+(z_{tot}-x_4-x_5)} - \frac{v_9 x_5}{k_9+x_5} \\ u & = & g x_5 \end{cases}$$

Nonlinear Set Integration

Bracketing systems

- Example : Mitogen- Activated Protein Kinase (Sontag, 2005)

$$
\left\{
\begin{aligned}
\dot{\underline{x}}_1 &= -\frac{\overline{v}_2\underline{x}_1}{\underline{k}_2+\underline{x}_1} + \underline{v}_0\underline{u} + \underline{v}_1\\
\dot{\underline{x}}_2 &= \frac{\underline{v}_6(\underline{y}_{tot}-\underline{x}_2-\overline{x}_3)}{\overline{k}_6+(\underline{y}_{tot}-\underline{x}_2-\overline{x}_3)} - \frac{\overline{v}_3\overline{x}_1\underline{x}_2}{\underline{k}_3+\underline{x}_2}\\
\dot{\underline{x}}_3 &= \frac{\underline{v}_4\underline{x}_1(\underline{y}_{tot}-\overline{x}_2-\underline{x}_3)}{\overline{k}_4+(\underline{y}_{tot}-\overline{x}_2-\underline{x}_3)} - \frac{\overline{v}_5\underline{x}_3}{\underline{k}_5+\underline{x}_3}\\
\dot{\underline{x}}_4 &= \frac{\underline{v}_{10}(\underline{z}_{tot}-\underline{x}_4-\overline{x}_5)}{\overline{k}_{10}+(\underline{z}_{tot}-\underline{x}_4-\overline{x}_5)} - \frac{\overline{v}_7\overline{x}_3\underline{x}_4}{\underline{k}_7+\underline{x}_4}\\
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\dot{\overline{x}}_1 &= -\frac{\underline{v}_2\overline{x}_1}{\overline{k}_2+\overline{x}_1} + \overline{v}_0\overline{u} + \overline{v}_1\\
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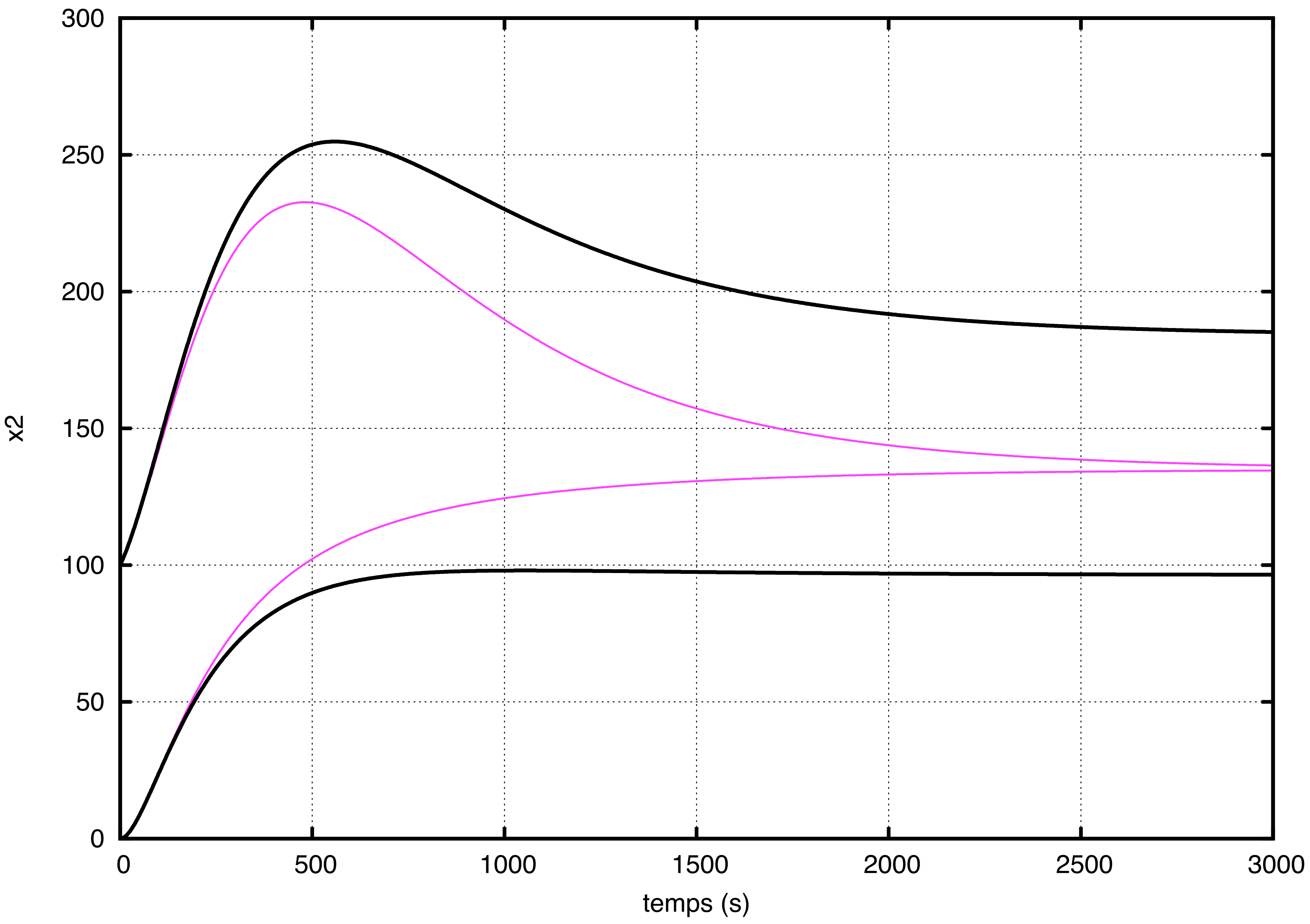
\underline{u} &= g\underline{x_5}\\
\overline{u} &= g\overline{x_5}
\end{aligned}
\right.
$$



Nonlinear Set Integration

- ***Convergence analysis* for bracketing systems enclosures.**
  - ***Practical stability analysis for a class of systems (Ramdani et al, IEEE TAC 2009)***

Nonlinear Set Integration

- ***Convergence analysis* for bracketing systems enclosures.**
  - ***Practical stability analysis for a class of systems (Ramdani et al, IEEE TAC 2009)***
  - ***Dual integration method (Meslem & Ramdani, IMA MCI 2017)***

$$\dot{\zeta} = f(\zeta),\, \zeta = [\zeta_1\ \zeta_2]^\top \Leftrightarrow \left\{ \begin{array}{rcl} \dot{\zeta}_1 & = & f_1(\zeta_1, \zeta_2) \\ \dot{\zeta}_2 & = & f_2(\zeta_1, \zeta_2) \end{array} \right. \quad \wedge \operatorname{diag}(f_2) < 0$$

$\Rightarrow$ Dual integration method (Bracketing on ζ_1 + Int. Taylor on ζ_2)

$\Rightarrow$ one can tune integration step size to obtain tight enclosures

Nonlinear Set Integration

- *Analysis of enclosures width for bracketing systems*

$$\dot{\zeta} = M\zeta + u(d), \qquad d \in [d] \Rightarrow u \in [\underline{u}, \overline{u}] = u([d]), \Rightarrow w = \overline{u} - \underline{u},$$

$$M = M_d + M_o^+ - M_o^-, \quad (M_o^+ \geq 0,\ M_o^- \geq 0)$$

Let us do it now as an exercise:

1. Apply the rule for building bracketing systems
2. Analyze the dynamics of the enclosure widths

Nonlinear Set Integration

- ***Analysis of enclosures width for bracketing systems***

$$\dot{\zeta} = M\zeta + u(d), \qquad d \in [d] \Rightarrow u \in [\underline{u}, \overline{u}] = u([d]), \Rightarrow w = \overline{u} - \underline{u},$$

$$M = M_d + M_o^+ - M_o^-, \quad (M_o^+ \geq 0,\ M_o^- \geq 0)$$

The bracketing systems by the Muller's theorem :

$$\begin{pmatrix} \dot{\Omega} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} M_d + M_o^+ & -M_o^- \\ -M_o^- & M_d + M_o^+ \end{pmatrix} \begin{pmatrix} \Omega \\ \omega \end{pmatrix} + \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

$$e = \Omega - \omega \quad \Rightarrow \quad \dot{e} = (M_d + M_o^+ + M_o^-)e + w$$

Nonlinear Set Integration

Monotone order-preserving systems

- Müller, Kamke, Krasnoselskii, Hirsch, Smith, Angeli and Sontag.
- **Preserve ordering on initial conditions.**

$$\mathbf{x}(t_0) \prec \mathbf{y}(t_0) \Rightarrow \forall \mathbf{t} \geqslant \mathbf{t_0} \quad \mathbf{x}(t) \prec \mathbf{y}(t) \qquad \prec \in \{<, \leq, \geq, >\}$$

- **Bracketing systems via Müller's theorem give tight enclosures !**

Nonlinear Set Integration

Monotone order-preserving systems

- Test based on graph theory : monotone wrt orthant cones.
  No negative cycle in the incidence graph (Kunze & Siegel, 1999)

$$\text{if } \exists\, \mathbf{D} = diag[(-1)^{\varepsilon_1}, ..., [(-1)^{\varepsilon_n}], \varepsilon_i \in \{0,1\}$$

s.t $\mathbf{x}(t, \mathbf{x}_0, t_0)$ and $\mathbf{y}(t, \mathbf{y}_0, t_0)$ satisfy

$$\mathbf{D}\mathbf{y}_0 \geq \mathbf{D}\mathbf{x}_0 \Rightarrow \mathbf{D}\mathbf{y}(t, \mathbf{y}_0, t_0) \geq \mathbf{D}\mathbf{x}(t, \mathbf{x}_0, t_0)\ \forall t \geq t_0.$$

Nonlinear Set Integration

Monotone order-preserving systems

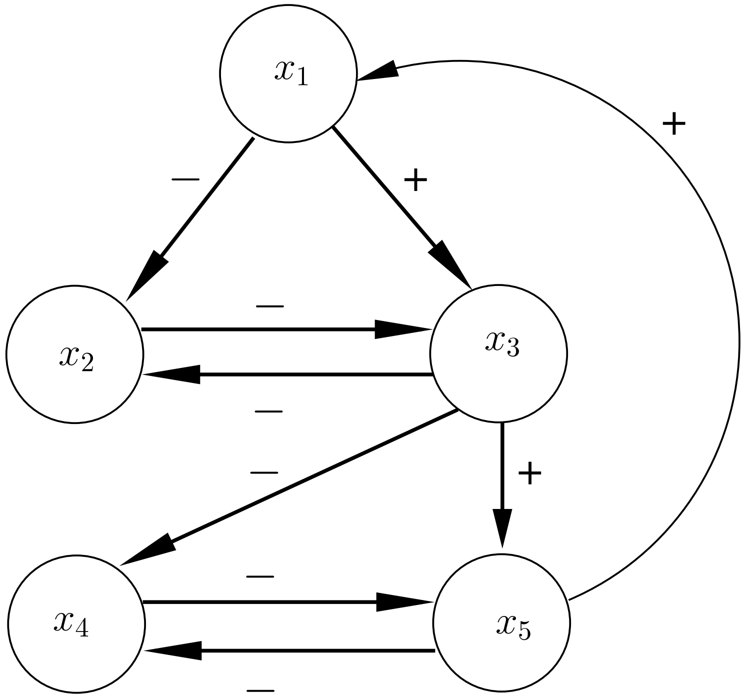
- **Test based on graph theory : monotone wrt orthant cones.**
  No negative cycle in the incidence graph (Kunze & Siegel, 1999)

$$\begin{cases} \dot{x}_1 = -(v_2 x_1)/(k_2 + x_1) + v_0 g x_5 + v_1 \\ \dot{x}_2 = (v_6(y_{tot} - x_2 - x_3))/(k_6 + (y_{tot} - x_2 - x_3)) - (v_3 x_1 x_2)/(k_3 + x_2) \\ \dot{x}_3 = (v_4 x_1 (y_{tot} - x_2 - x_3))/(k_4 + (y_{tot} - x_2 - x_3)) - (v_5 x_3)/(k_5 + x_3) \\ \dot{x}_4 = (v_{10}(z_{tot} - x_4 - x_5))/(k_{10} + (z_{tot} - x_4 - x_5)) - (v_7 x_3 x_4)/(k_7 + x_4) \\ \dot{x}_5 = (v_8 x_3 (z_{tot} - x_4 - x_5))/(k_8 + (z_{tot} - x_4 - x_5)) - (v_9 x_5)/(k_9 + x_5) \end{cases}$$

if $\exists\, \mathbf{D} = diag[(-1)^{\varepsilon_1}, ..., [(-1)^{\varepsilon_n}], \varepsilon_i \in \{0,1\}$

s.t $\mathbf{x}(t, \mathbf{x}_0, t_0)$ and $\mathbf{y}(t, \mathbf{y}_0, t_0)$ satisfy

$$\mathbf{Dy}_0 \geq \mathbf{Dx}_0 \Rightarrow \mathbf{Dy}(t, \mathbf{y}_0, t_0) \geq \mathbf{Dx}(t, \mathbf{x}_0, t_0)\ \forall t \geq t_0.$$



Nonlinear Set Integration

- **Monotone order-prese**
  - **Test based on graph theory**

    No negative cycle in the inci

$$\begin{cases}\dot{x}_1 = -(v_2x_1)/(k_2+x_1)+v_0gx_5+v_1\\ \dot{x}_2 = (v_6(y_{tot}-x_2-x_3))/(k_6+(y_{tot}-x_2\\ \dot{x}_3 = (v_4x_1(y_{tot}-x_2-x_3))/(k_4+(y_{tot}-x\\ \dot{x}_4 = (v_{10}(z_{tot}-x_4-x_5))/(k_{10}+(z_{tot}-x\\ \dot{x}_5 = (v_8x_3(z_{tot}-x_4-x_5))/(k_8+(z_{tot}-x\end{cases}$$

if $\exists\, \mathbf{D} = diag[(-1)^{\varepsilon_1}, \ldots$

s.t $\mathbf{x}(t,\mathbf{x}_0,t_0)$ and $\mathbf{y}(t,\mathbf{y}_0,t_0)$ satisfy

$\mathbf{D}\mathbf{y}_0 \geq \mathbf{D}\mathbf{x}_0 \Rightarrow \mathbf{D}\mathbf{y}(t,\mathbf{y}_0,t_0)$

$$\begin{cases}
\dot{\underline{x}}_1 = -\frac{\overline{v}_2\underline{x}_1}{\underline{k}_2+\underline{x}_1} + \underline{v}_0\underline{u} + \underline{v}_1\\
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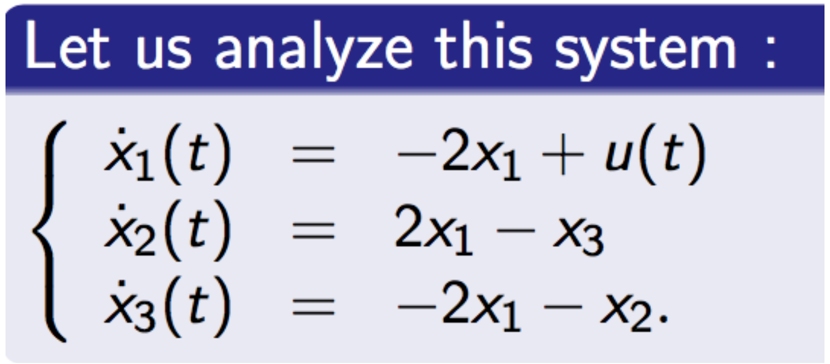
\end{cases}$$

- Ramdani, et al., Nonlinear Analysis Hybrid Systems 2010
- **Bracketing systems : decoupled ODEs**

Nonlinear Set Integration

Monotone order-preserving systems

- Test based on graph theory : monotone wrt orthant cones.
  No negative cycle in the incidence graph (Kunze & Siegel, 1999)



Let us analyze this system :

$$\begin{cases} \dot{x}_1(t) & = & -2x_1 + u(t) \\ \dot{x}_2(t) & = & 2x_1 - x_3 \\ \dot{x}_3(t) & = & -2x_1 - x_2. \end{cases}$$

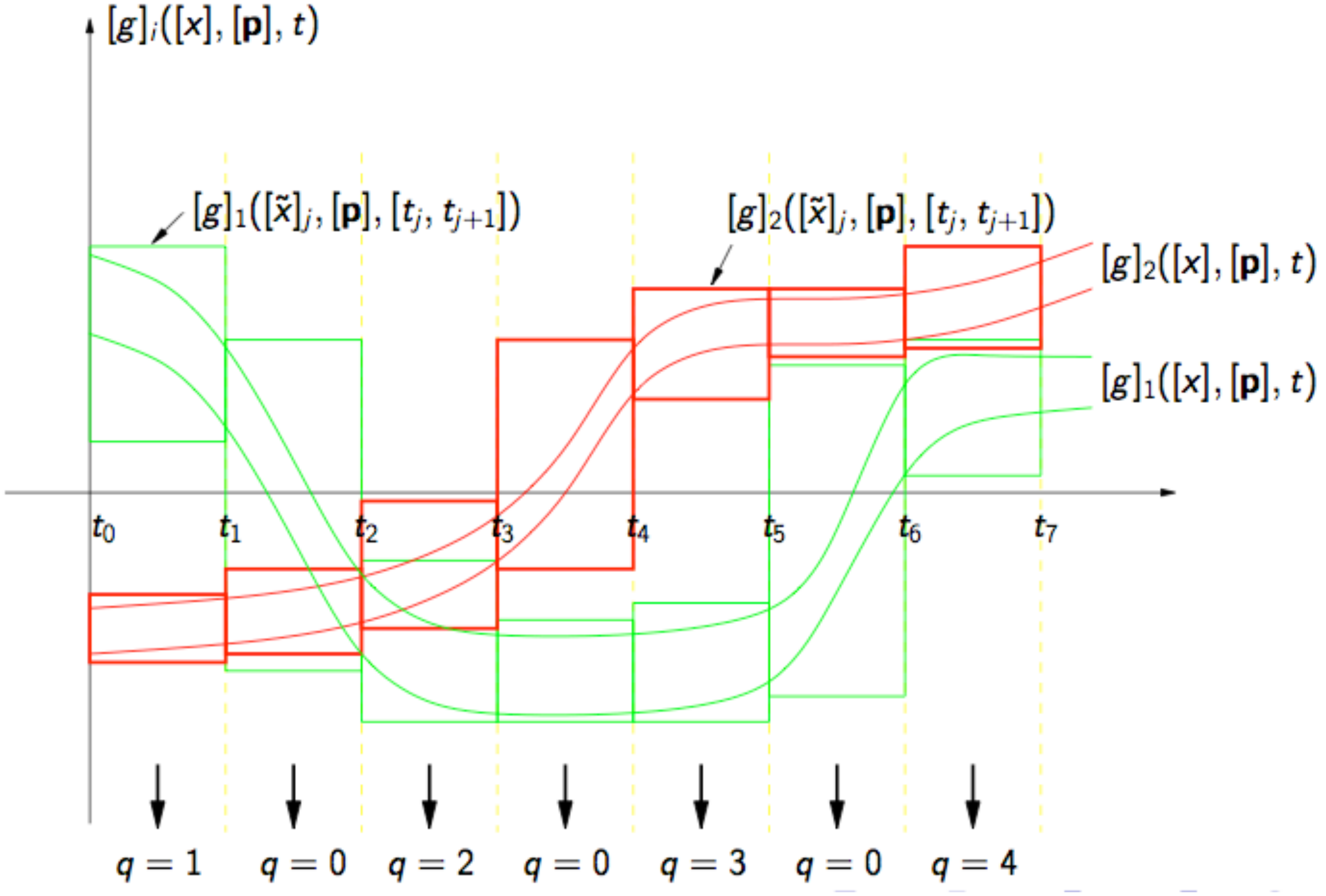
Nonlinear Set Integration

Nonlinear hybridization

- (Ramdani, et al., IEEE Trans. Automatic Control 2009)

$$\dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \overline{p}_i]$$

$$g_i(.) = \frac{\partial f}{\partial p_i}(.)$$



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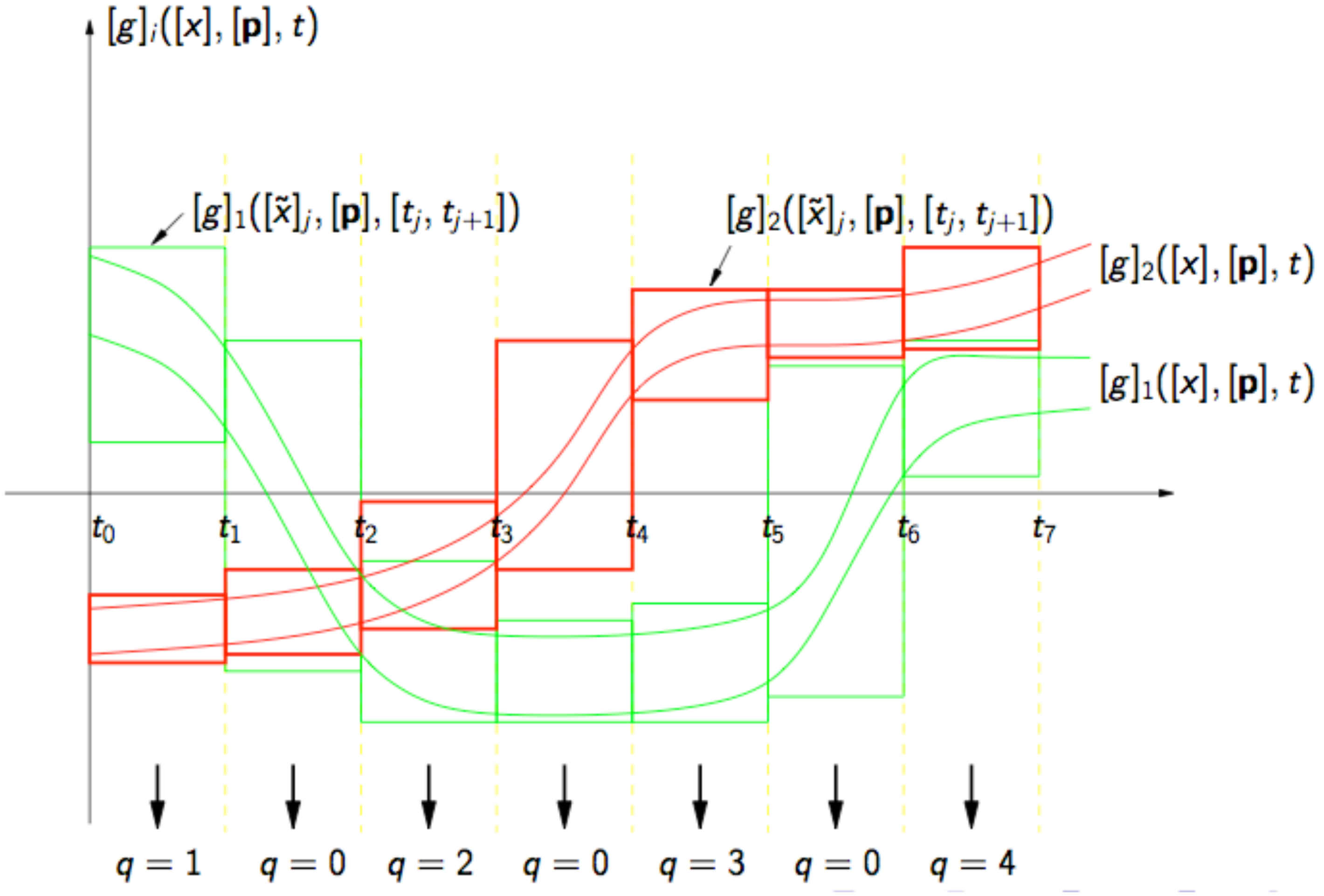

Nonlinear Set Integration

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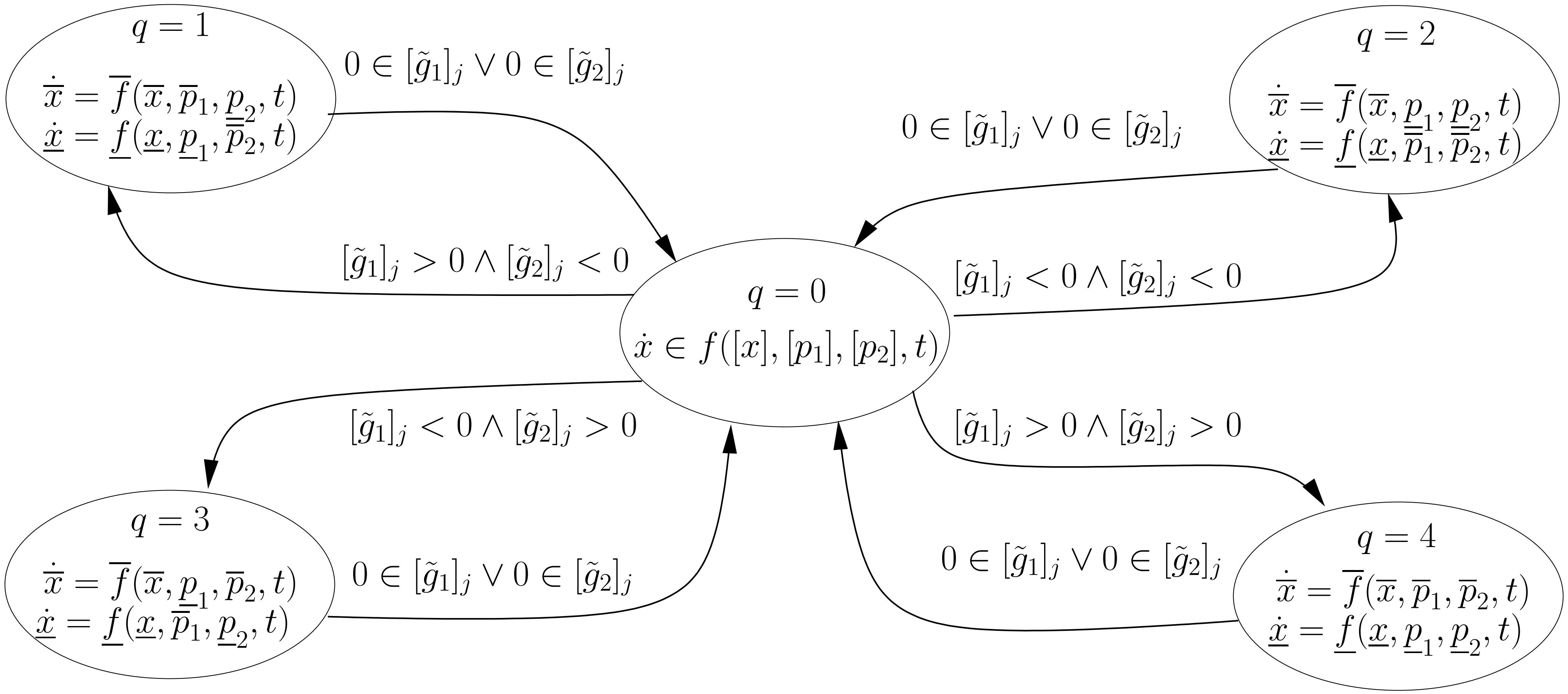

Nonlinear Set Integration

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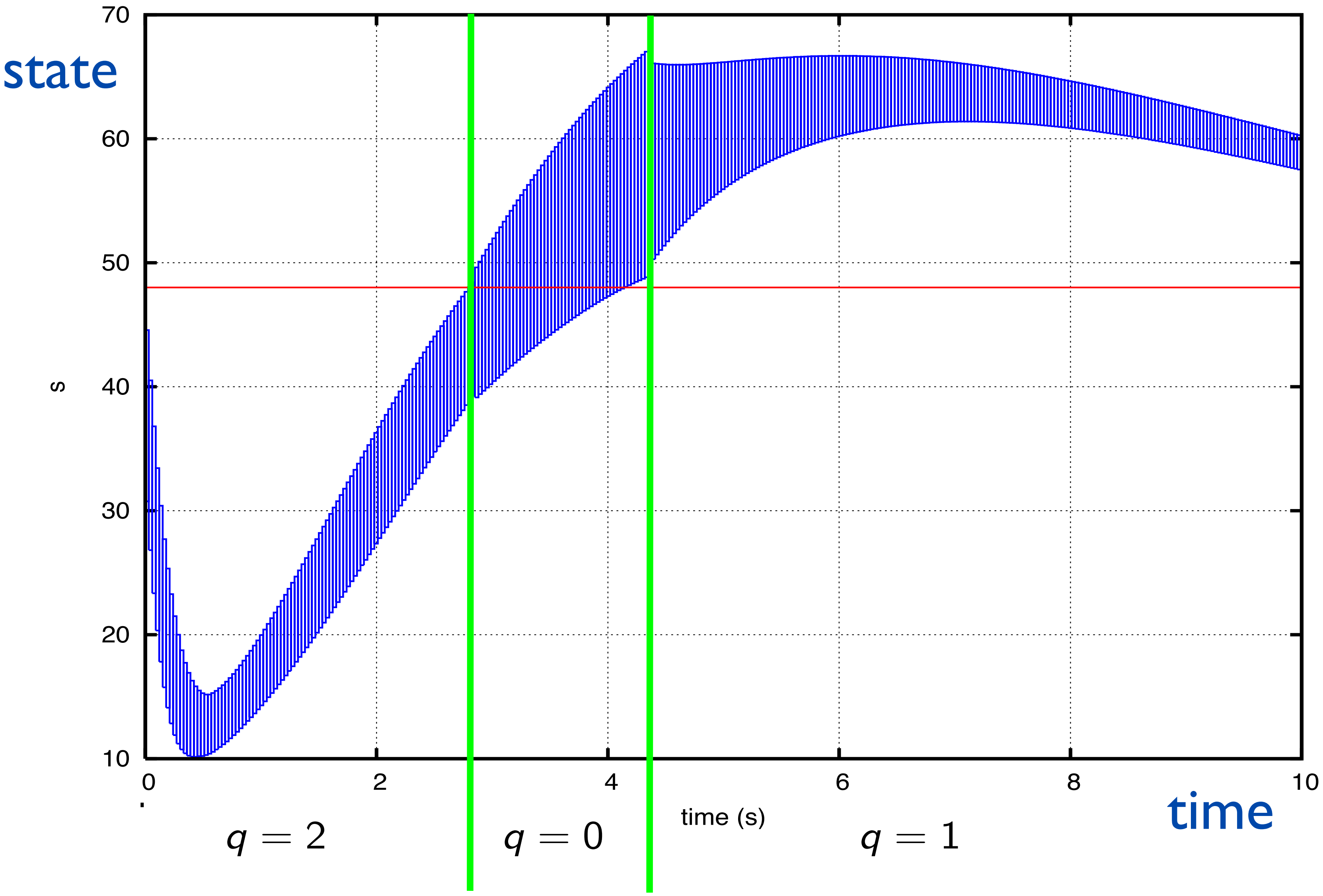
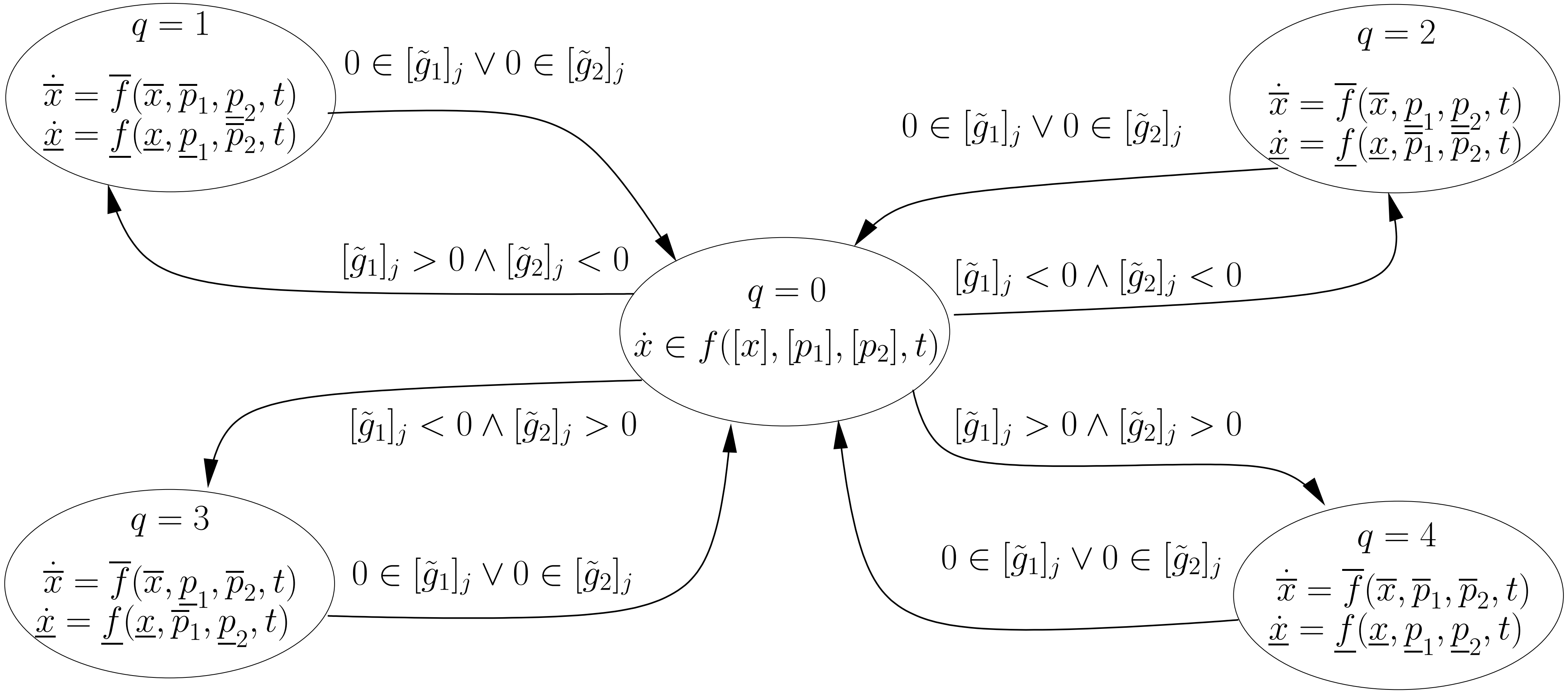


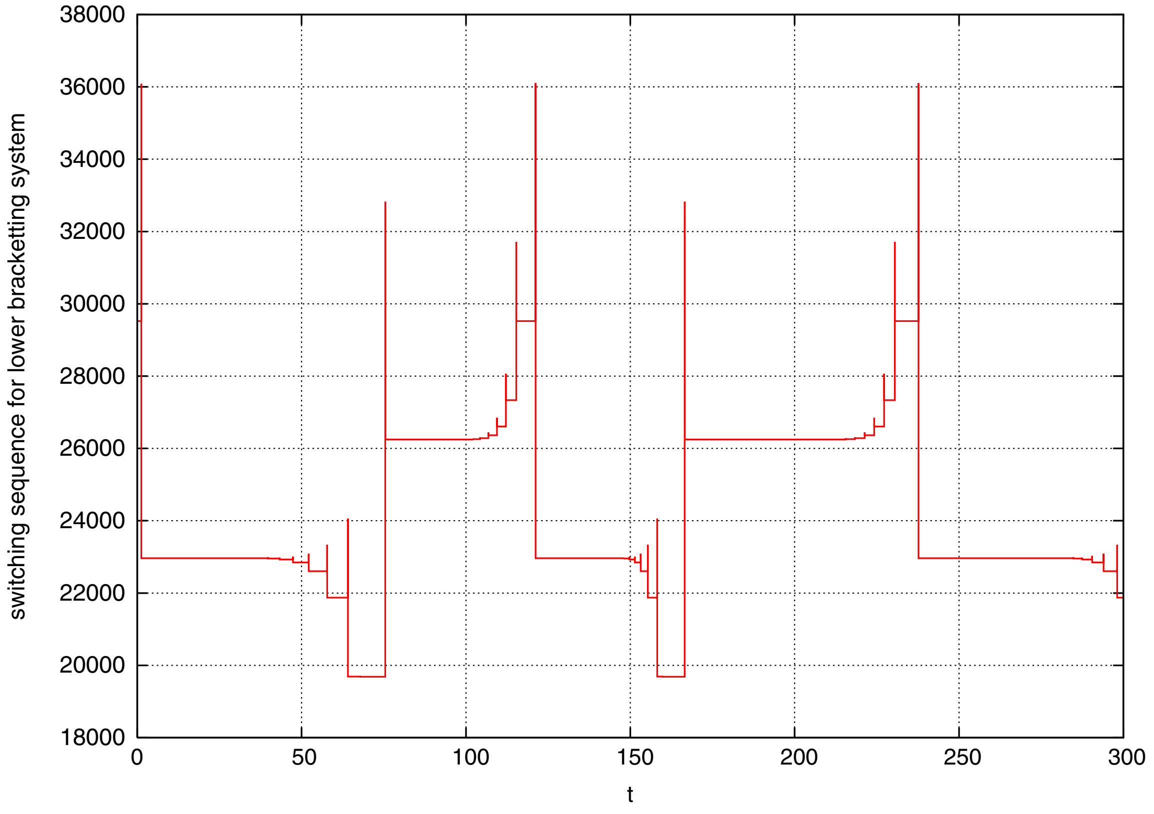
Nonlinear Set Integration

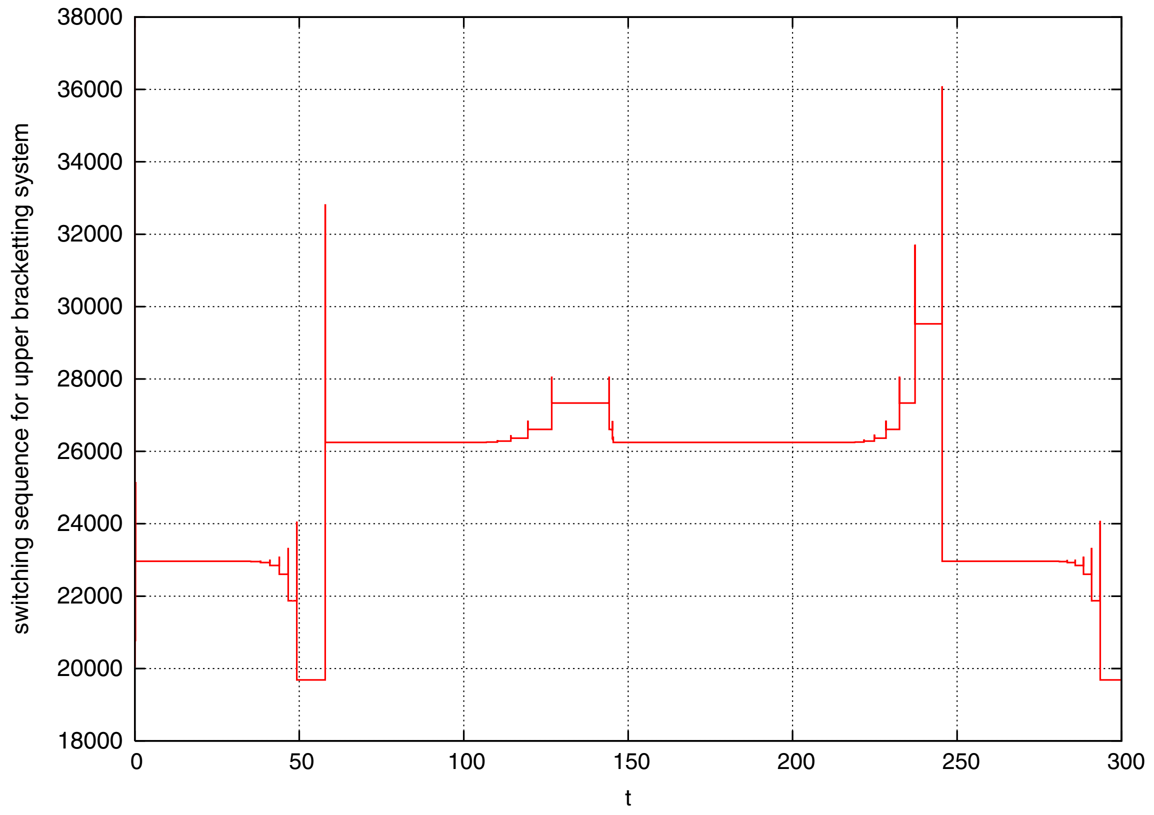
- **Nonlinear hybridization** & **Monotone systems**
  - (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2010)



Nonlinear Set Integration

- Nonlinear hybridization & Monotone systems
  - (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2010)
  - **decoupled hybrid systems as bracketing systems**





Nonlinear Set Integration

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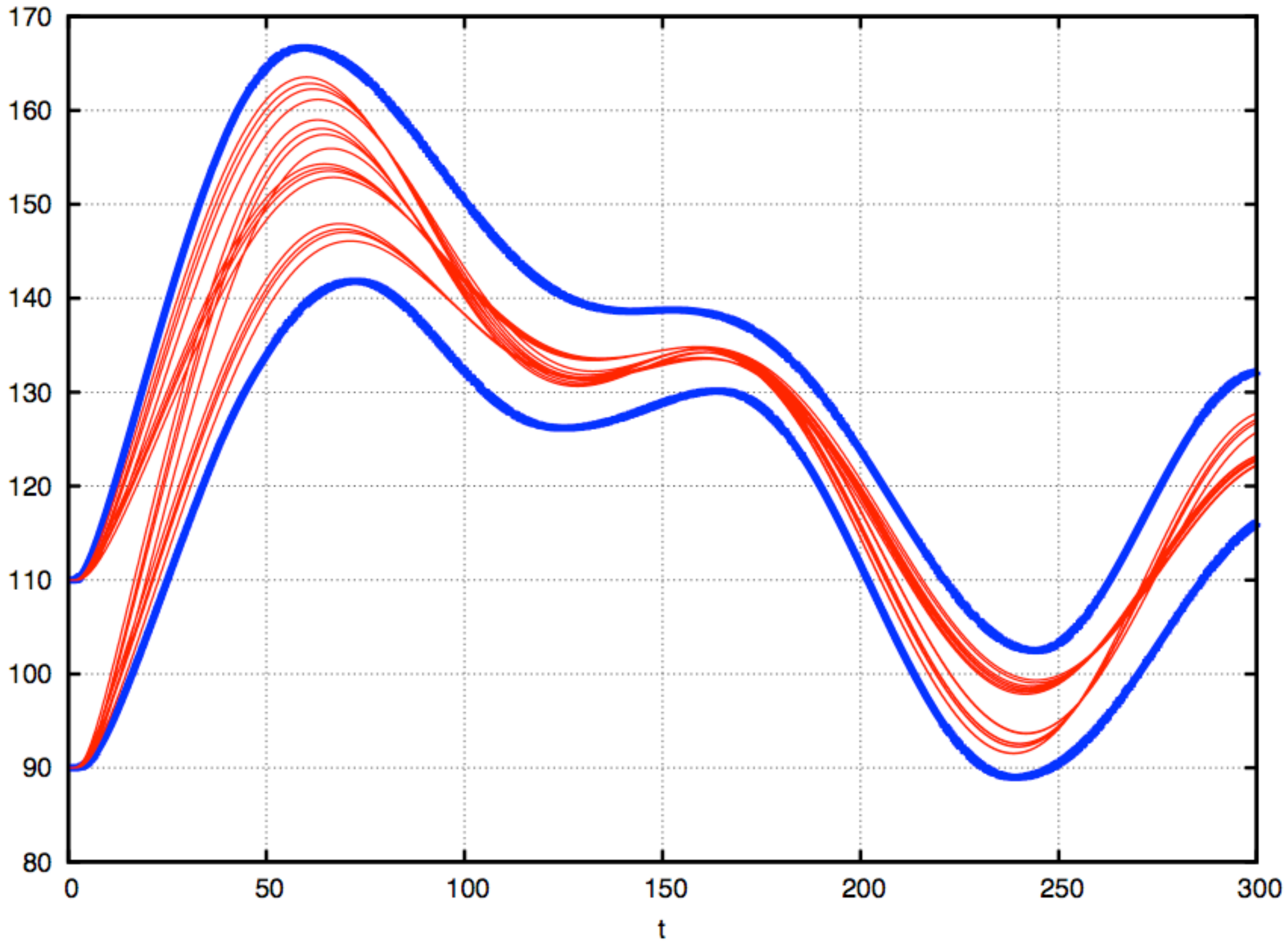
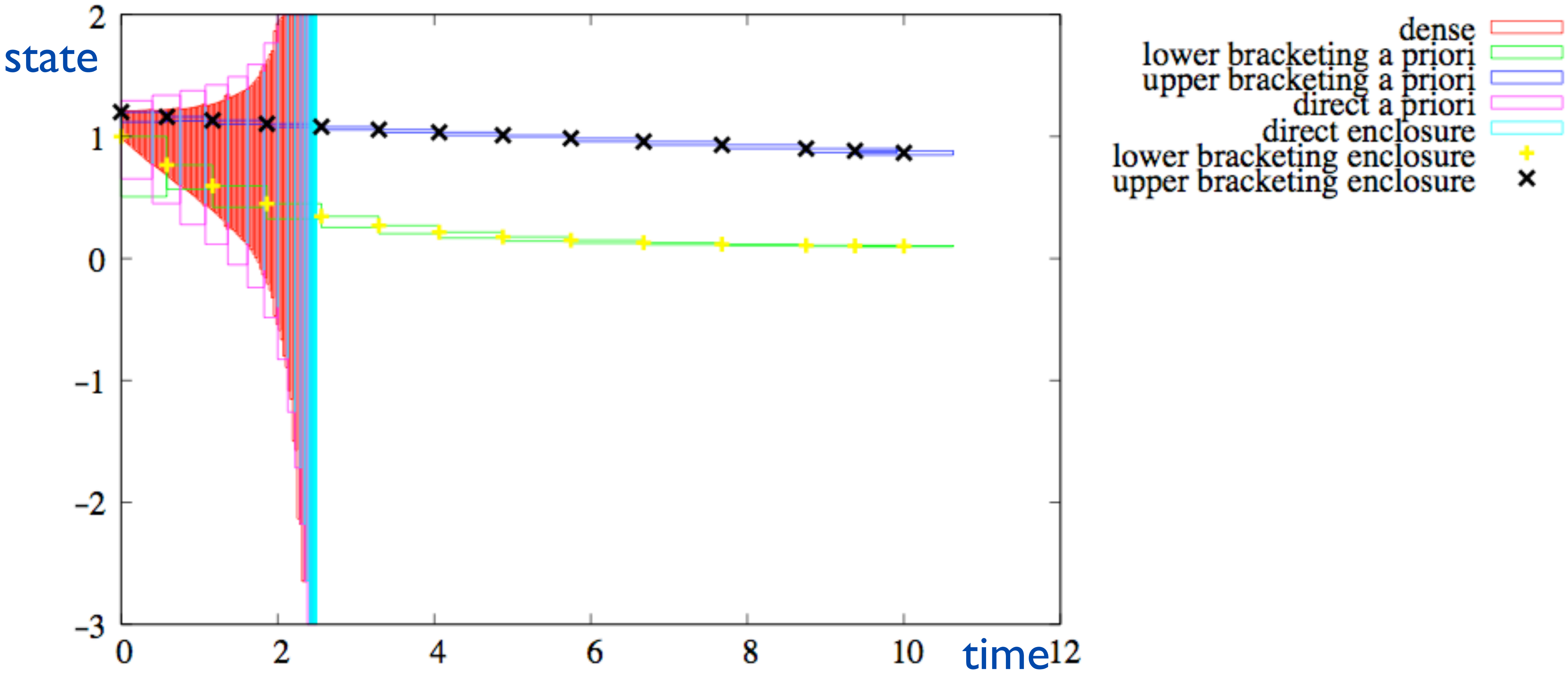


Nonlinear Set Integration

Interval Taylor methods vs Bracketing systems



$$\dot{x} = -p_4 x - \frac{p_1 x}{1 + p_2 y} + p_3 y + 0.1$$

$$\dot{y} = p_4 x - p_3 y$$

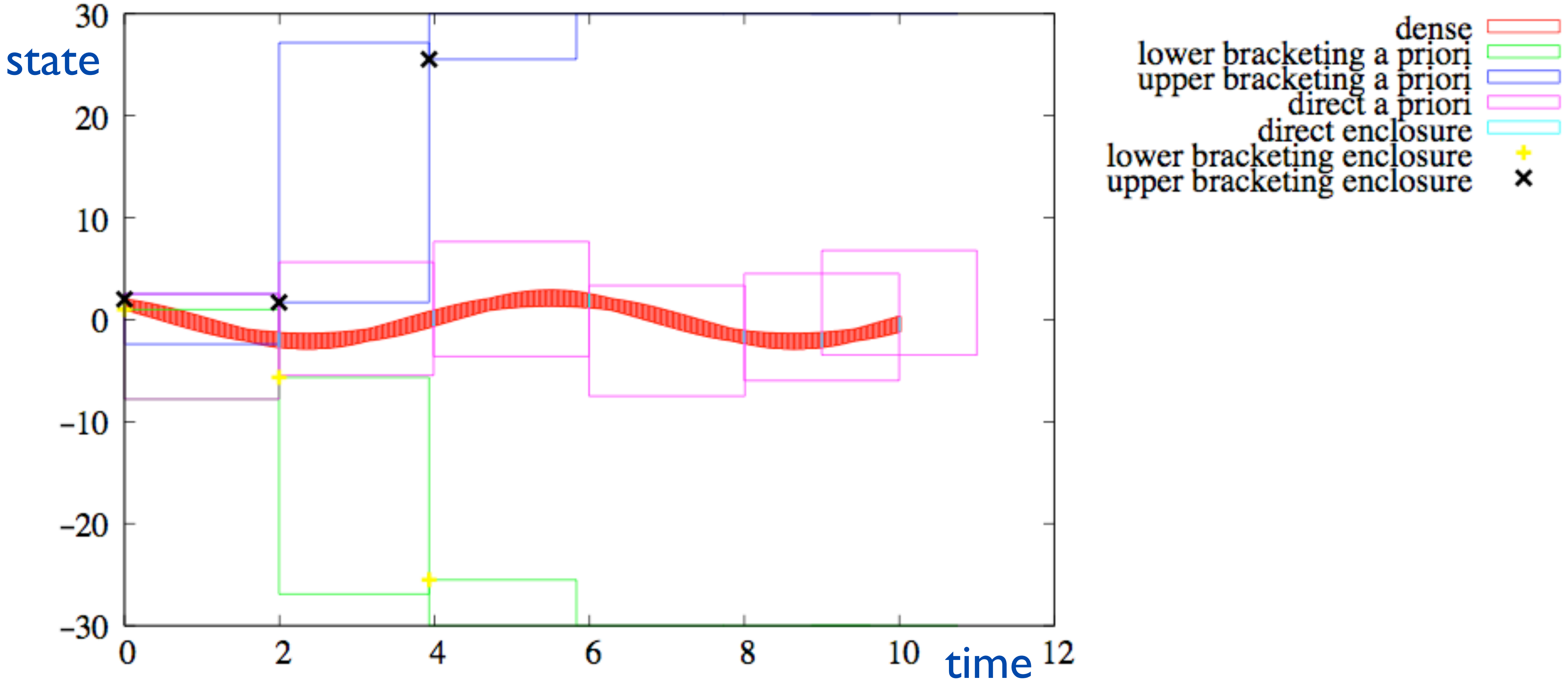
$x(0) \in [1, 1.2]$, $y(0) \in [0.8, 1]$

$p_1 \in [0.8, 1]$, $p_2 \in [1.0, 1.2]$, $p_3 \in [0.3, 0.5]$,

$p_4 \in [0.20, 0.25]$

Nonlinear Set Integration

Interval Taylor methods vs Bracketing systems



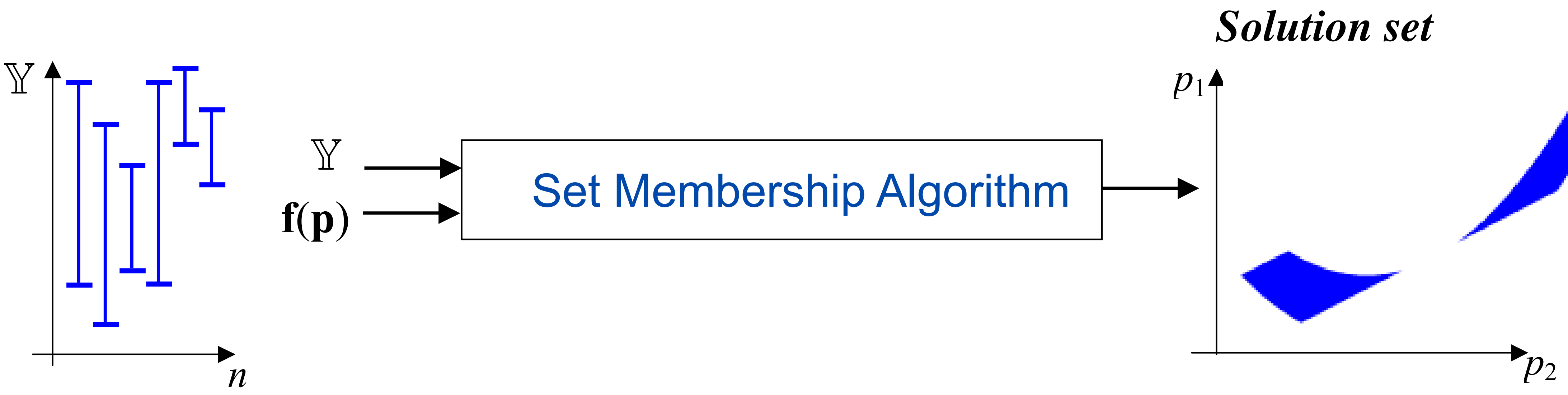
$$\dot{x} = y, \dot{y} = -x,$$

$$x(0), y(0) \in [1, 2]$$

- **Set-membership Estimation with Nonlinear Continuous Systems**

Set Membership Estimation

- Unknown but bounded-error framework



Solution set

Set Membership Algorithm

Hypothesis

Uncertainties and errors are bounded with known prior bounds

A set of feasible solutions

$$\mathbb{S} = \{\mathbf{p} \in \mathbb{P} | \mathbf{f}(\mathbf{p}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}) \cap \mathbb{P}$$

Set Membership Estimation

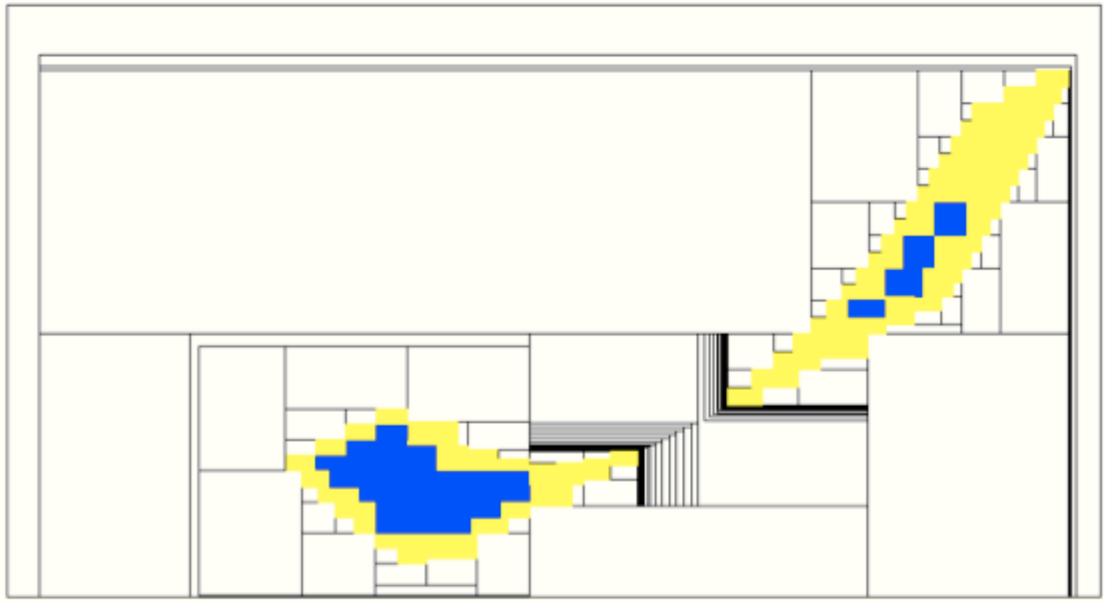
- **Parameter estimation. Set inversion.**
  - Branch-&-bound, branch-&-prune, interval contractors … (Jaulin, et al. 93) (Raïssi et al., 2004)
  - Separator Algebra …

$$\mathbb{S} = \{\mathbf{z} \in \mathcal{Z}, \mid f(\mathbf{z}) \in \mathcal{Y}\} \quad \rightarrow \underline{\mathbb{S}} \subseteq \mathbb{S} \subseteq \overline{\mathbb{S}}$$

$f([\mathbf{z}]) \subseteq \mathcal{Y}$ $\Rightarrow [\mathbf{z}] \subseteq \underline{\mathbb{S}}$: inner approximation

$f([\mathbf{z}])) \cap \mathcal{Y} = \emptyset$ $\Rightarrow [\mathbf{z}] \not\subseteq \overline{\mathbb{S}}$: outer approximation $\Rightarrow [\mathbf{z}] \subseteq \mathcal{Z} \backslash \overline{\mathbb{S}}$

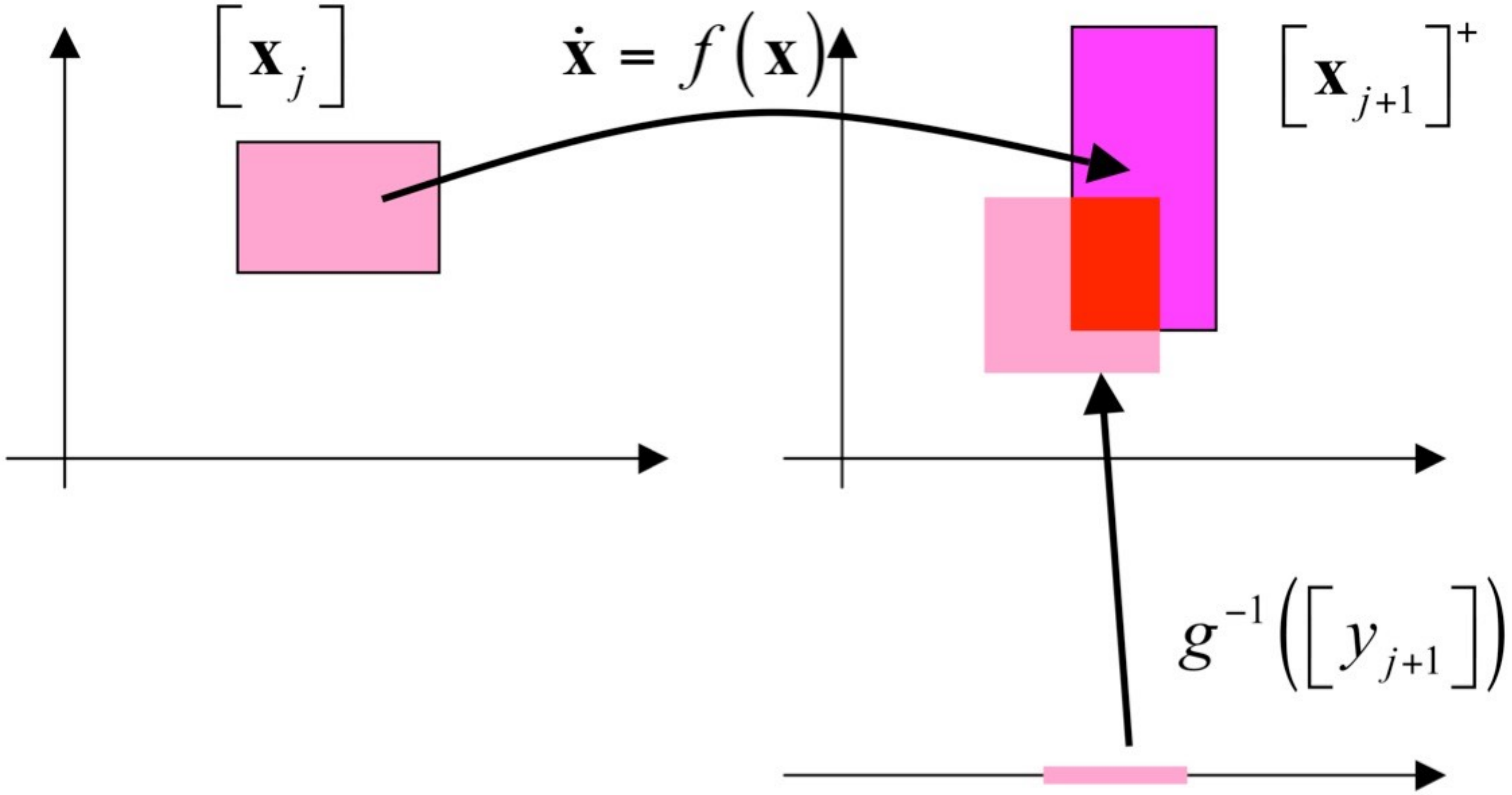
otherwise partition . . .



Set Membership Estimation

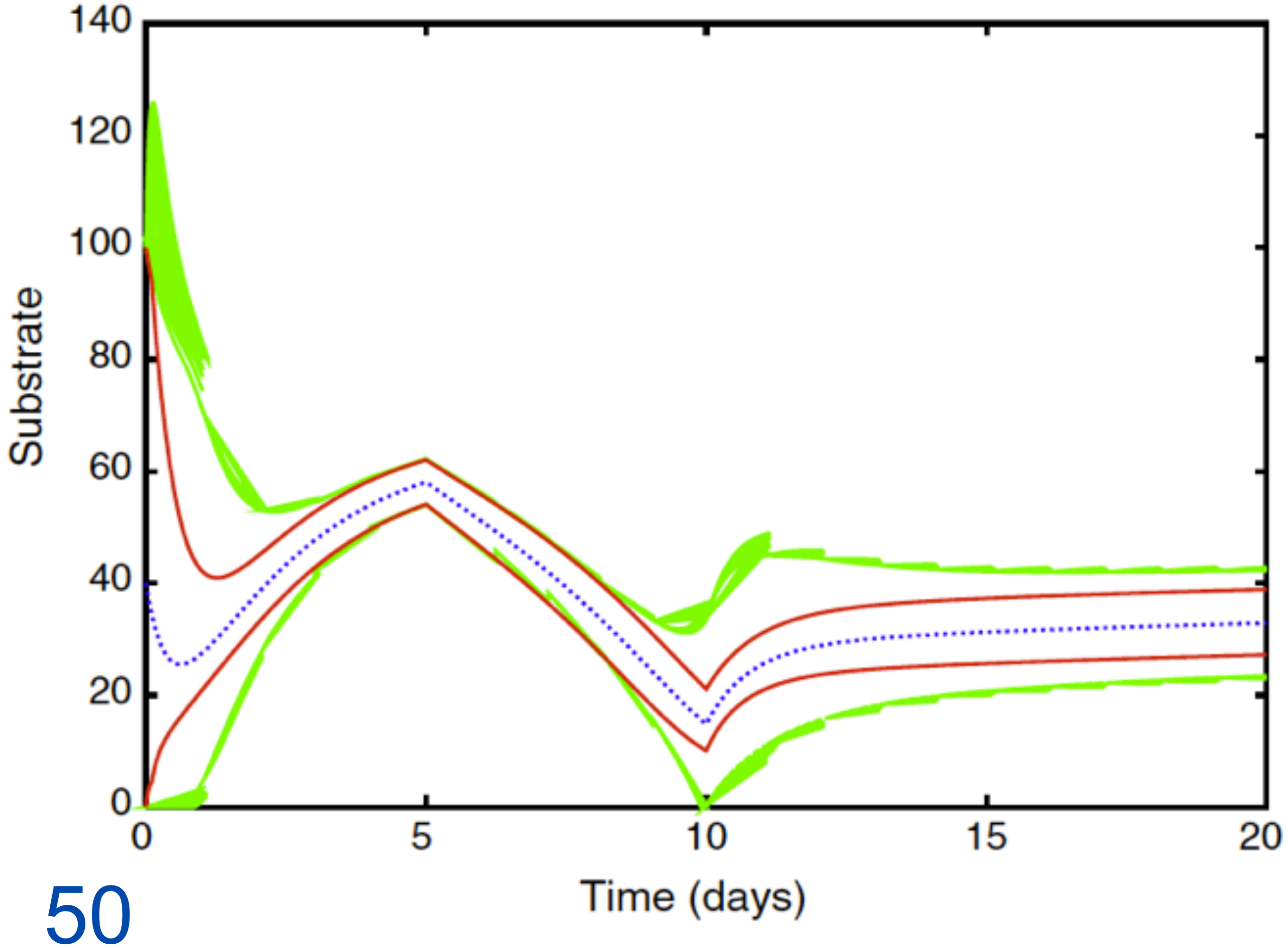
State estimation with continuous systems

- **Prediction - Correction / Filtering approaches**
  - (Jaulin, 02, Raïssi et al., 04, 05), (Meslem, et al, 10), (Milanese & Novara, 11), (Kieffer & Walter, 11) …
  - **Reachability**
    + Set inversion
  - Forward backward consistency



Set Membership Estimation

- **State estimation with continuous systems**
  - **Interval observers**
    - (Gouzé et al, 00), …. (Moisan, et al. 09), (Mazenc & Bernard, 10), (Meslem & Ramdani, 11), (Raïssi, et al., 12), (Combastel, 13), (El Thabet, et al. 14), (Efimov, et al. 15) …
    - Monotonicity
    - Change of coordinates
    - LMI ….
    - Ensure practical stability



- **Hybrid and Cyber-Physical Systems**

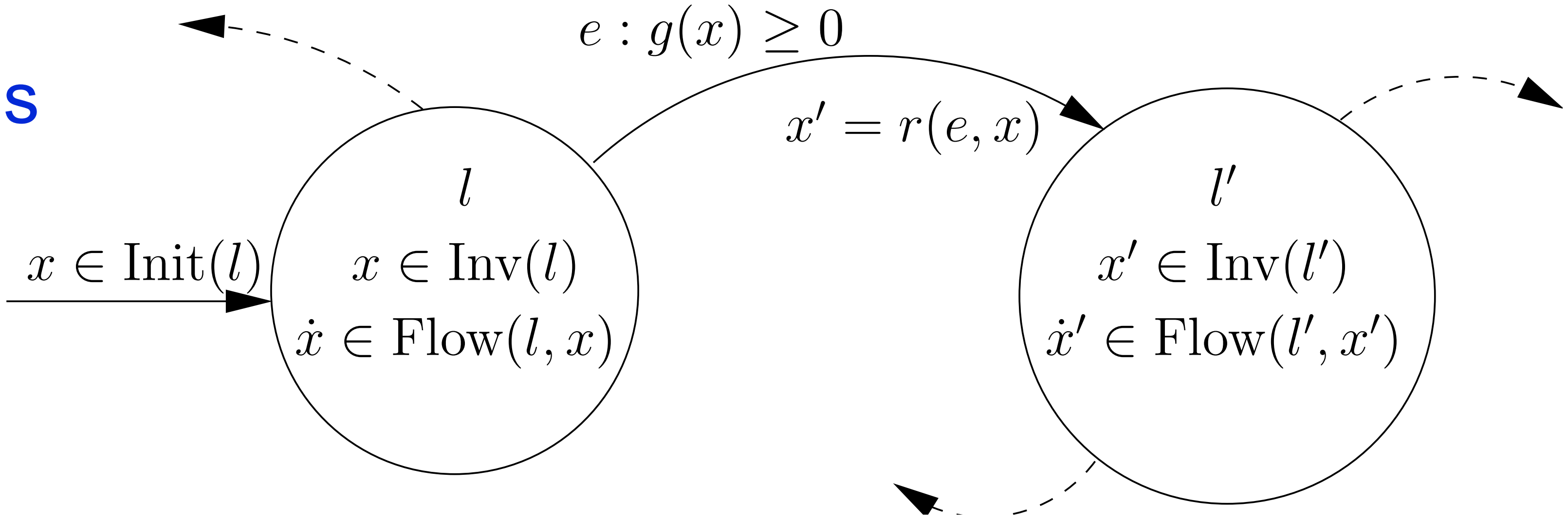
Hybrid Cyber-Physical Systems

- Interaction discrete
  + continuous dynamics
- Safety-critical
  embedded systems
- Networked
  autonomous systems

Hybrid Cyber-Physical Systems

Modelling → hybrid automaton (Alur, et al. 95)

- Non-linear continuous dynamics
- Nonlinear guards sets
- Nonlinear reset functions
- Bounded uncertainty



$$H = (\mathcal{Q}, \mathcal{D}, \mathcal{P}, \Sigma, \mathcal{A}, \mathsf{Inv}, \mathcal{F}),$$

Continuous dynamics

$$\begin{aligned} \mathsf{flow}(q): &\quad \dot{\mathbf{x}}(t) = f_q(\mathbf{x}, \mathbf{p}, t), \\ \mathsf{Inv}(q): &\quad \nu_q(\mathbf{x}(t), \mathbf{p}, t) < 0, \end{aligned}$$

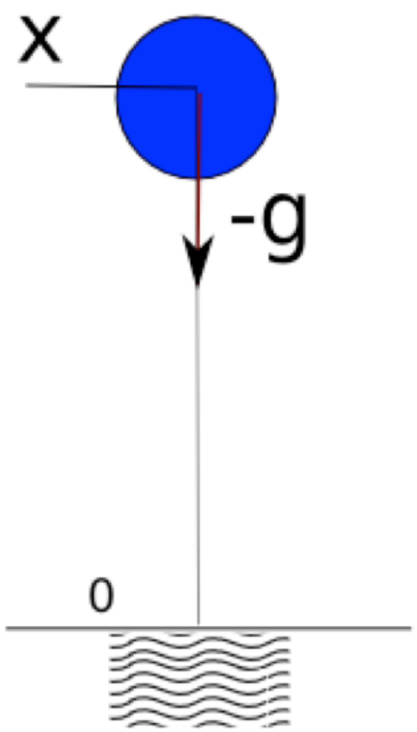
Discrete dynamics

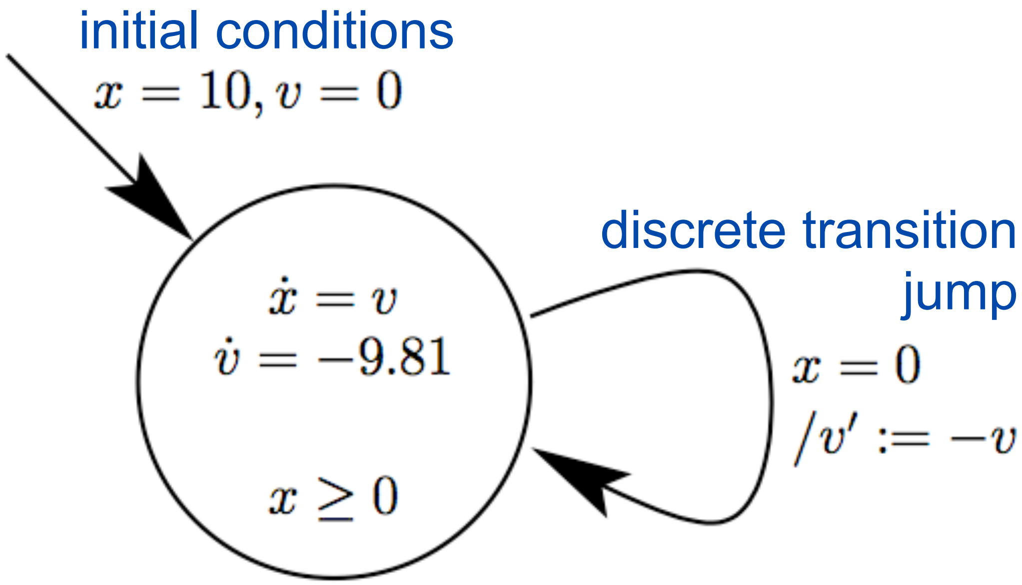
$$\begin{aligned} \mathcal{A} \ni e: &\quad (q \to q') = (q, \mathsf{guard}, \sigma, \rho, q'), \\ \mathsf{guard}(e): &\quad \gamma_e(\mathbf{x}(t), \mathbf{p}, t) = 0, \end{aligned}$$

$$t_0 \leq t \leq t_N, \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \subseteq \mathbb{R}^n, \quad \mathbf{p} \in \mathbb{P}$$

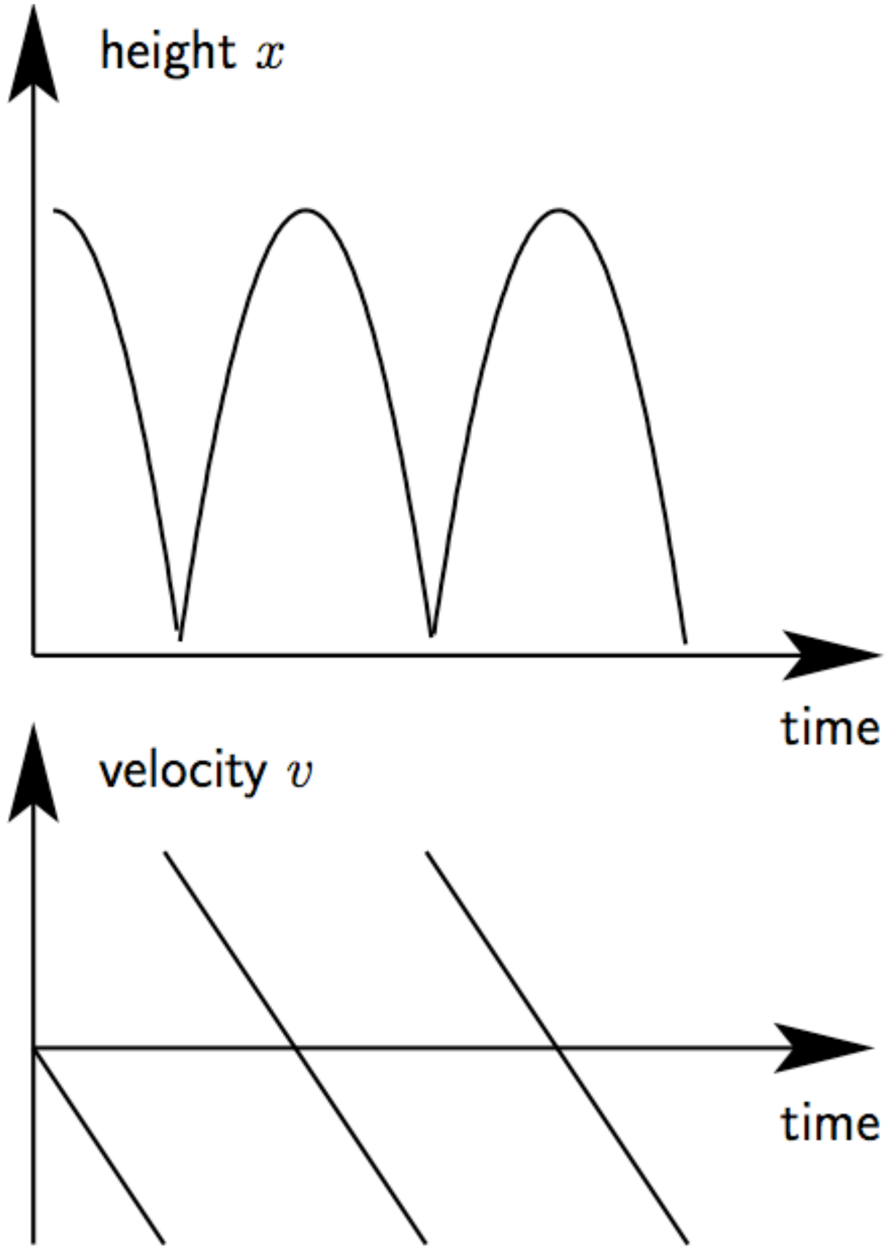
Hybrid Cyber-Physical Systems

- **Example :** the bouncing ball



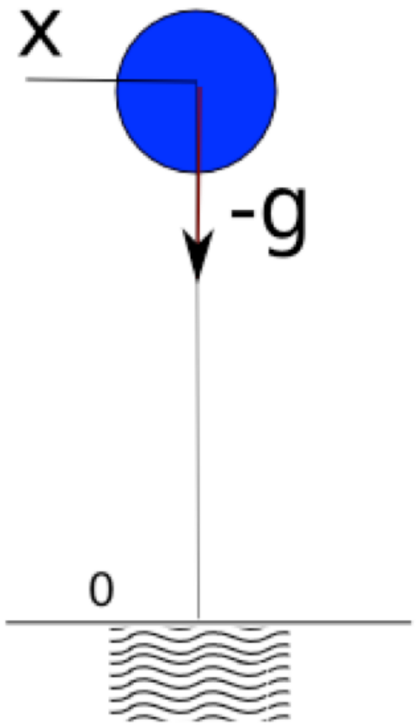

initial conditions
$x = 10, v = 0$

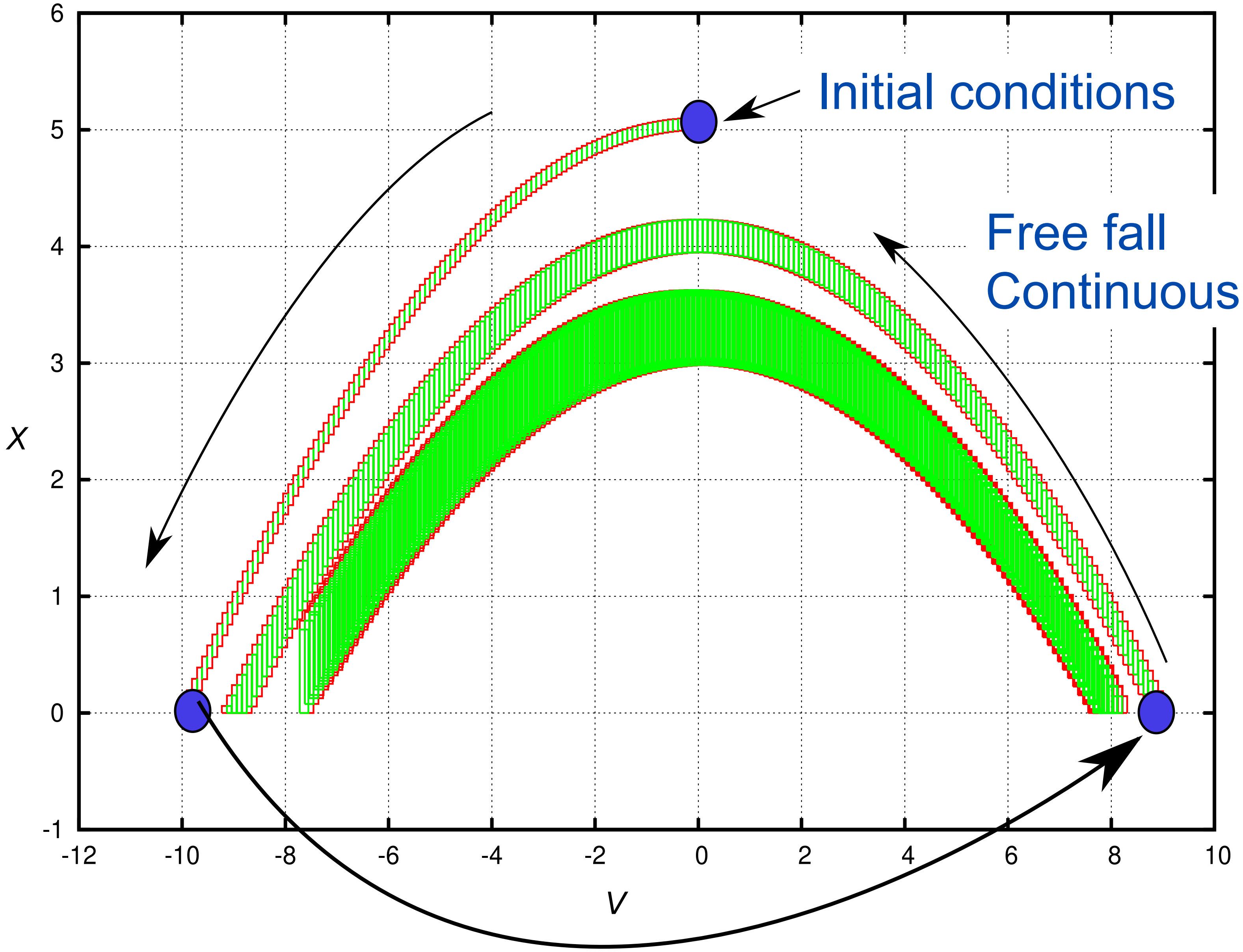
discrete transition
jump
$x = 0$
$/ v' := -v$



Hybrid Cyber-Physical Systems

- **Example :** the bouncing ball





Initial conditions

Free fall
Continuous transitions

Discrete transitions

Hybrid Cyber-Physical Systems

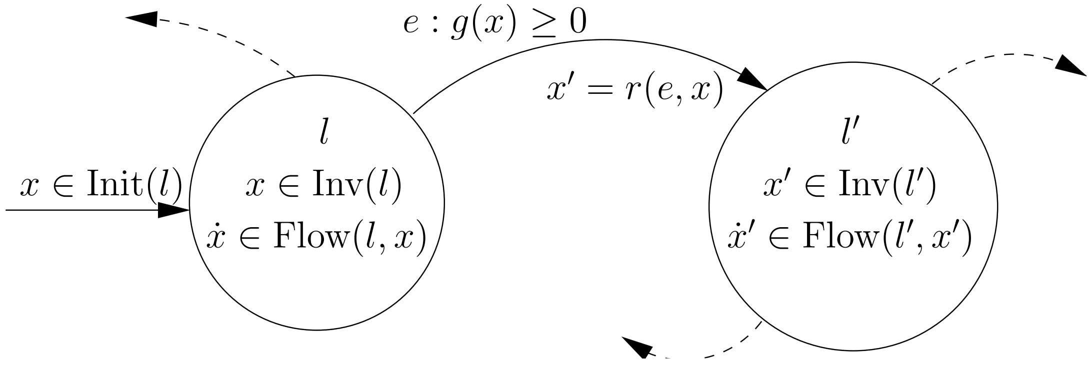
Operation in challenging environment, requires …

- Verification
  - Numerical proof, or
  - Falsification via counter-example
- Synthesis
  - « Correct by construction » …
- Monitoring, FDI
  - Complete state reconstruction
  - Worst-case scenario

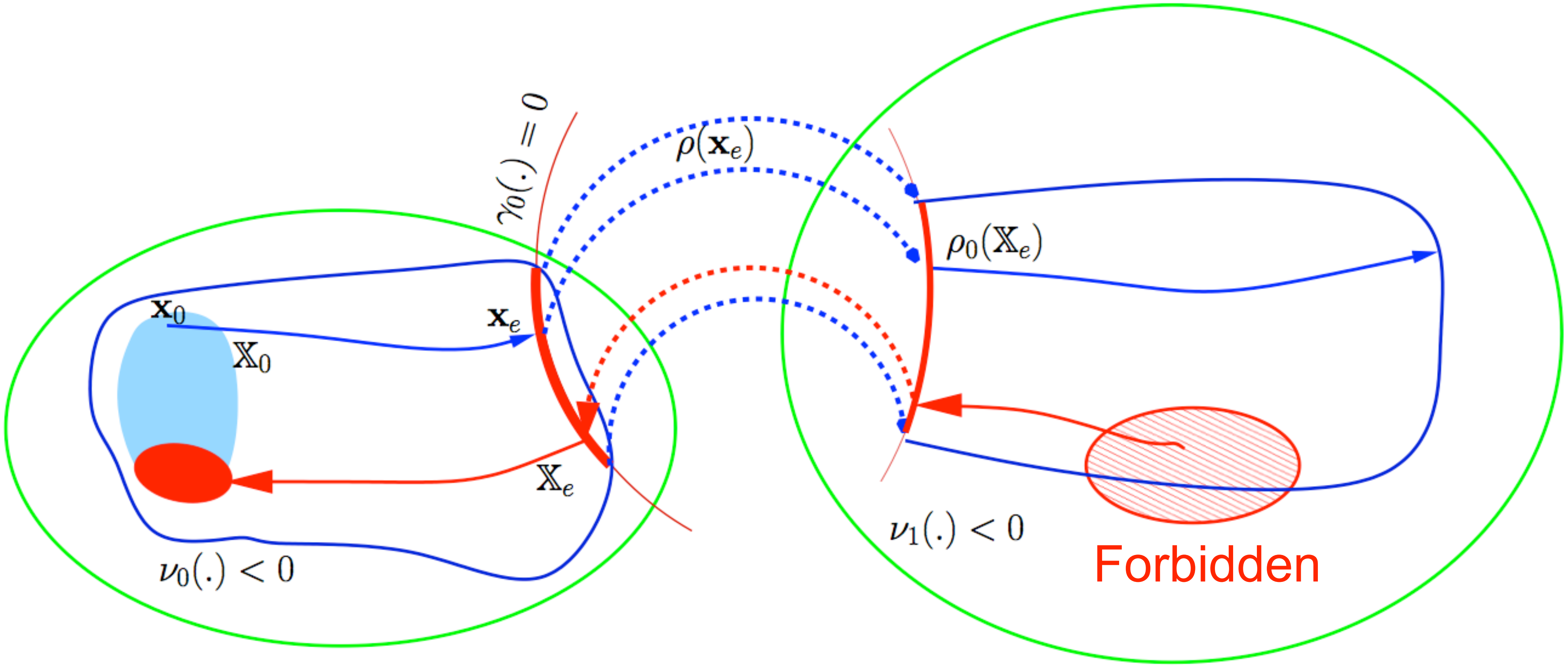
Verification of Hybrid Systems

Verification

- ***Modelling :***
- ***Property specification*** **:**

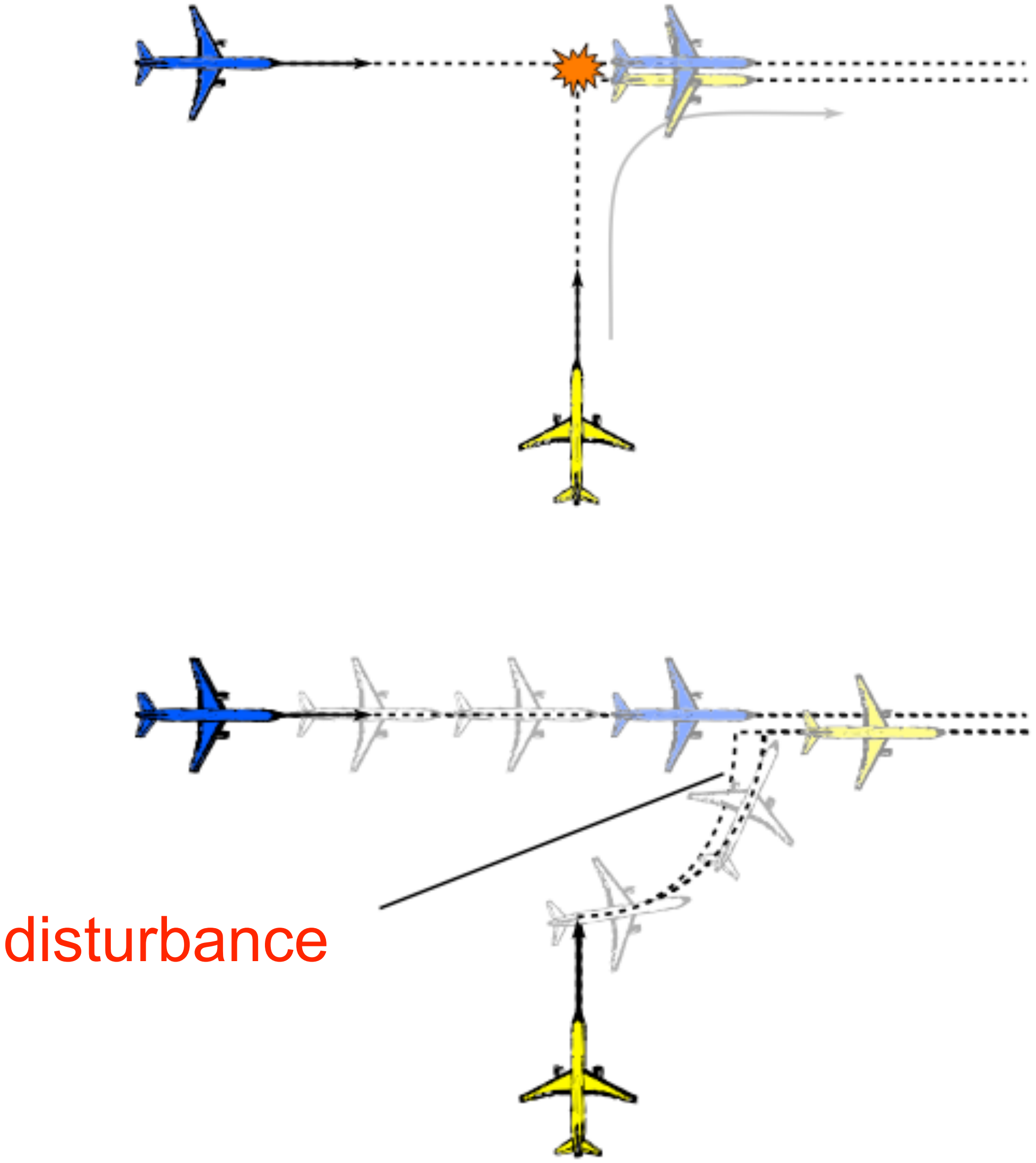


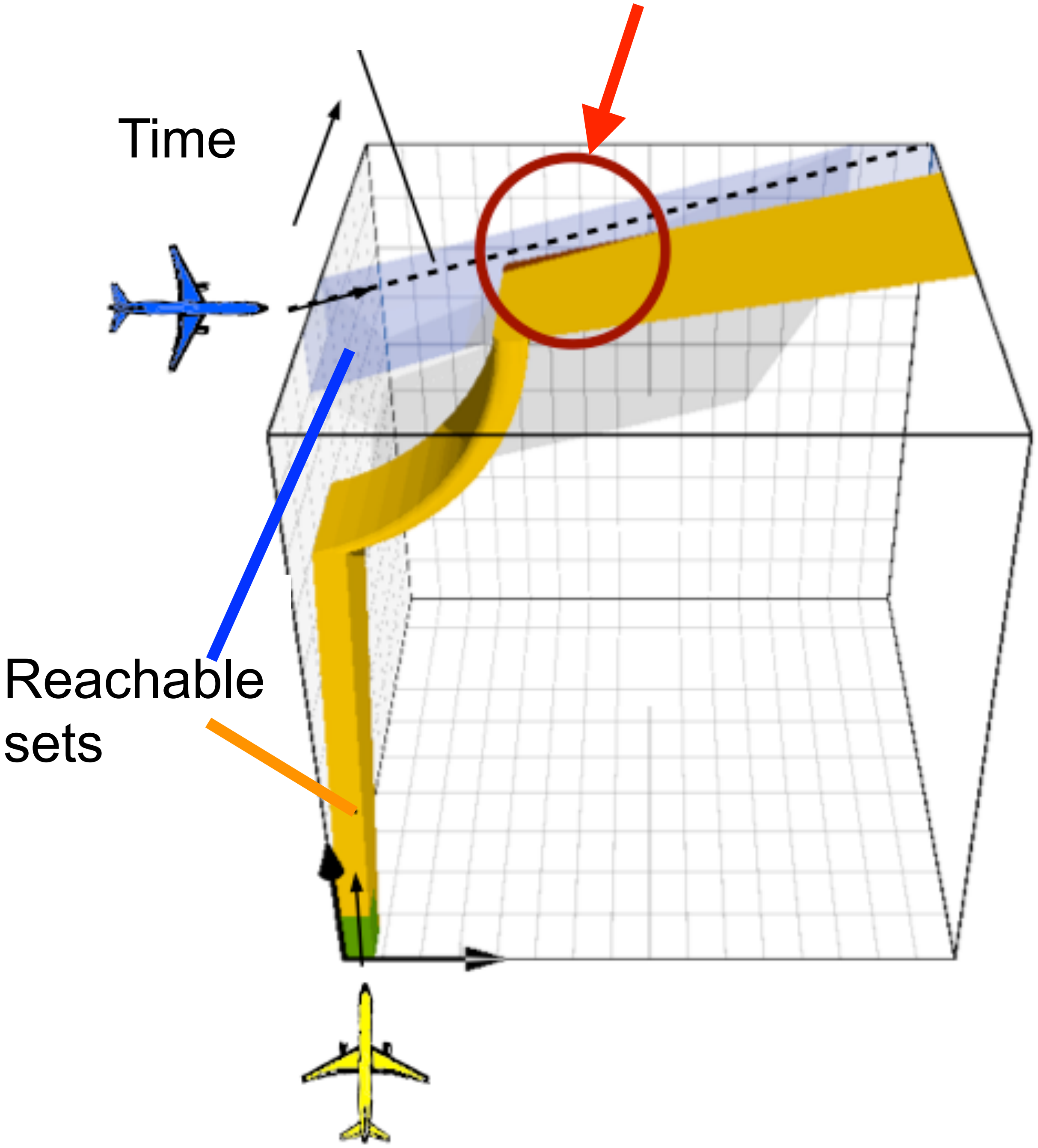
- **Verification algorithm : Reachability of unsafe regions**
  - **Hybrid / Continuous reachability**



Verification of Hybrid Systems

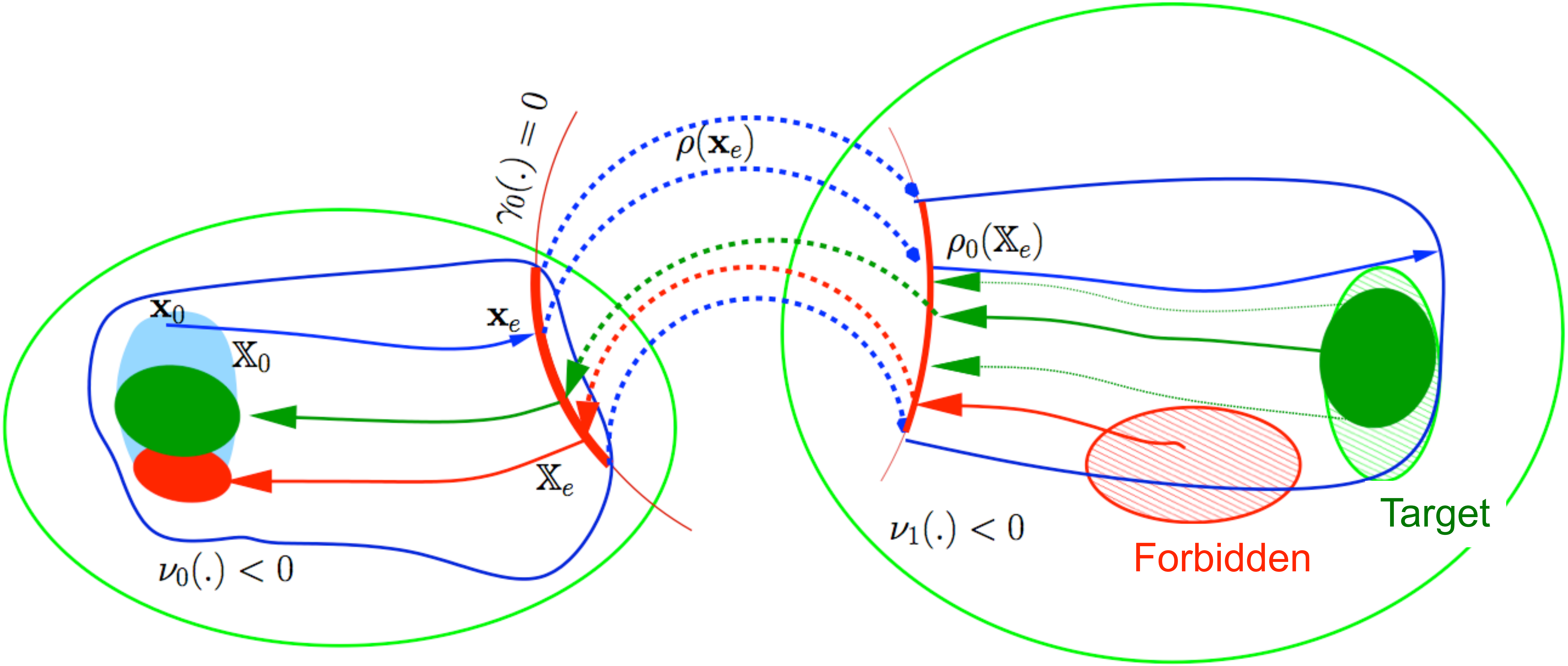
- **Aircraft trafic control (Tomlin, et al.)** **Collision possible!**





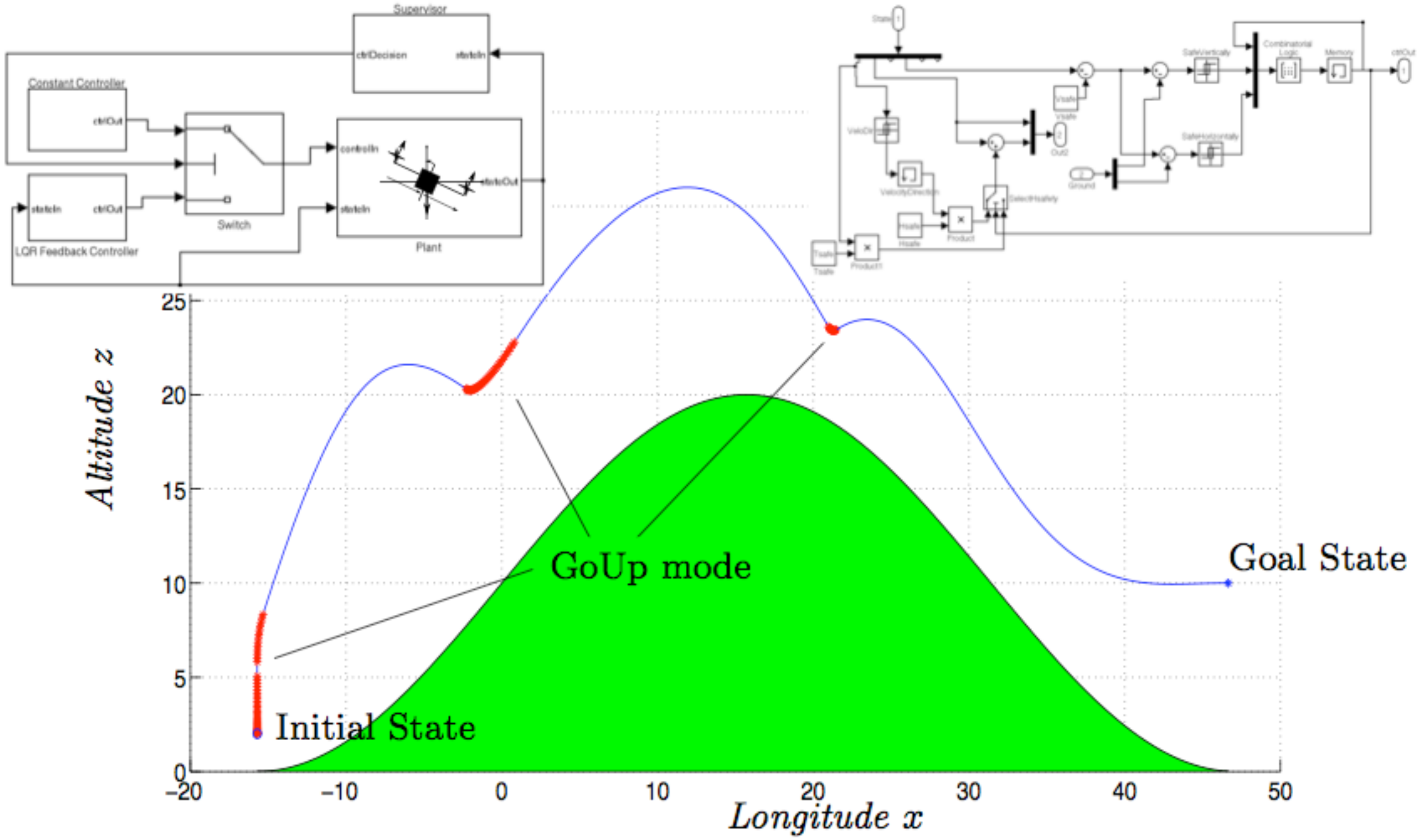
Hybrid Cyber-Physical Systems

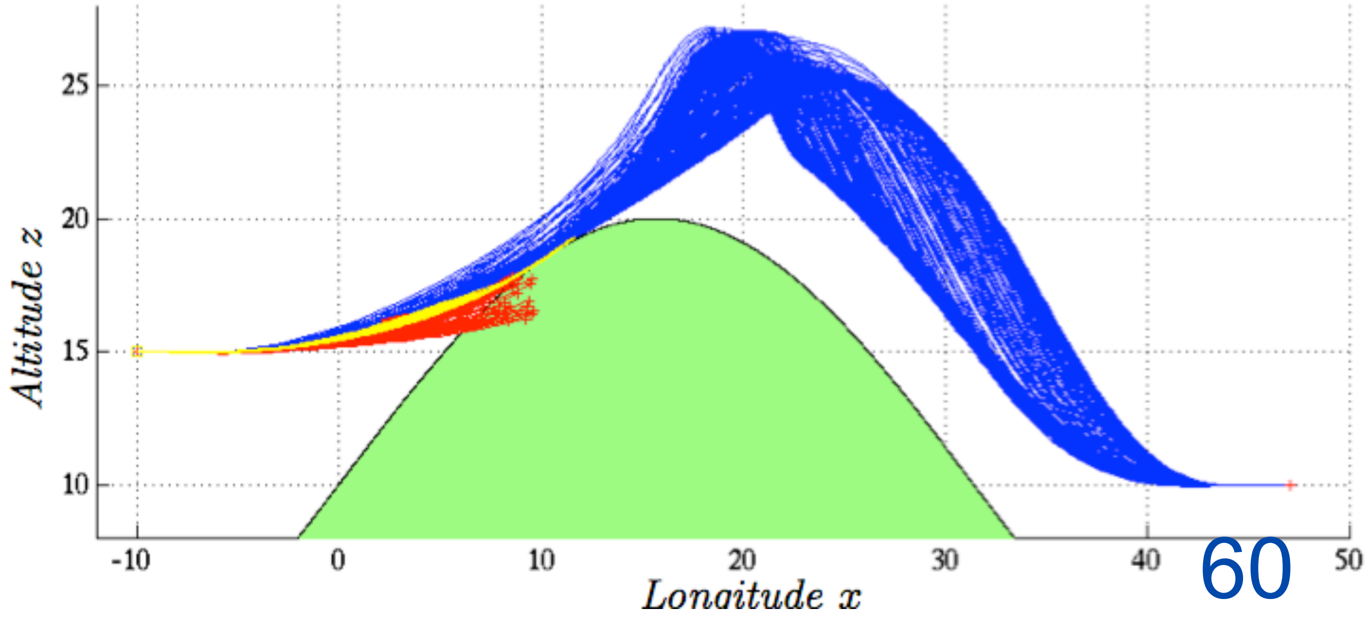
- **Parametric synthesis**
- **Set-membership estimation**

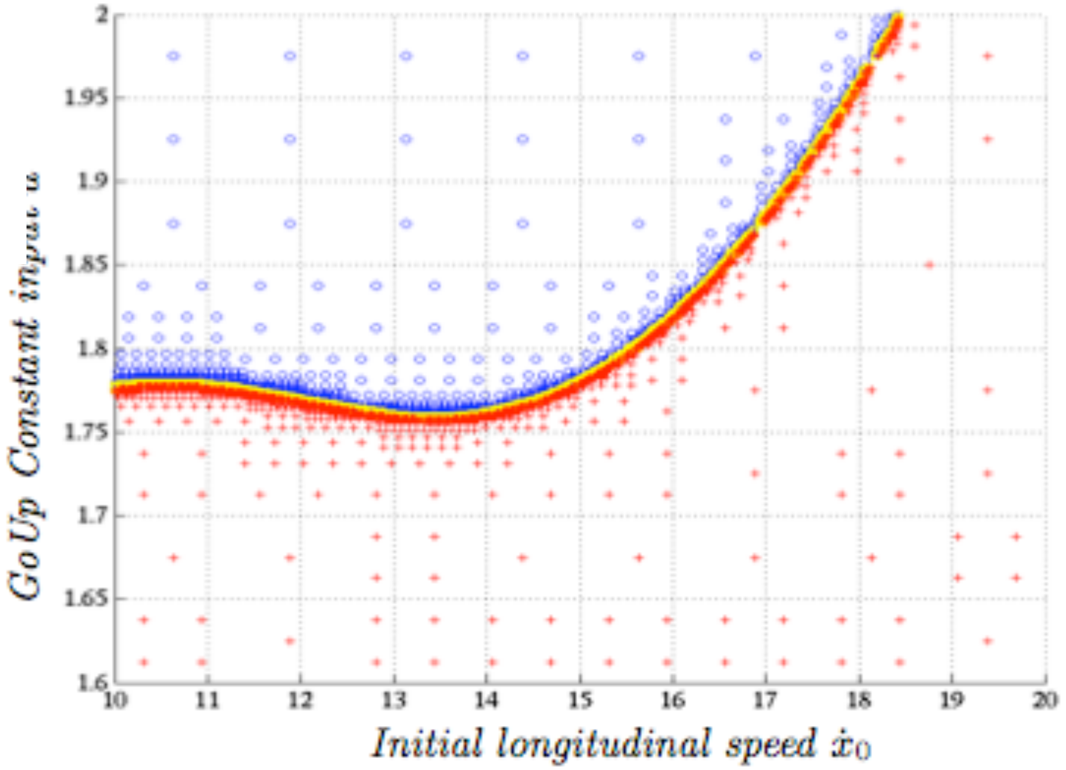


Synthesis of Hybrid Systems

A. Donzé, B. Krogh & A.Rajhans. *Parameter synthesis for hybrid systems with an application to simulink models.* ***HSCC 2009:165-179***.

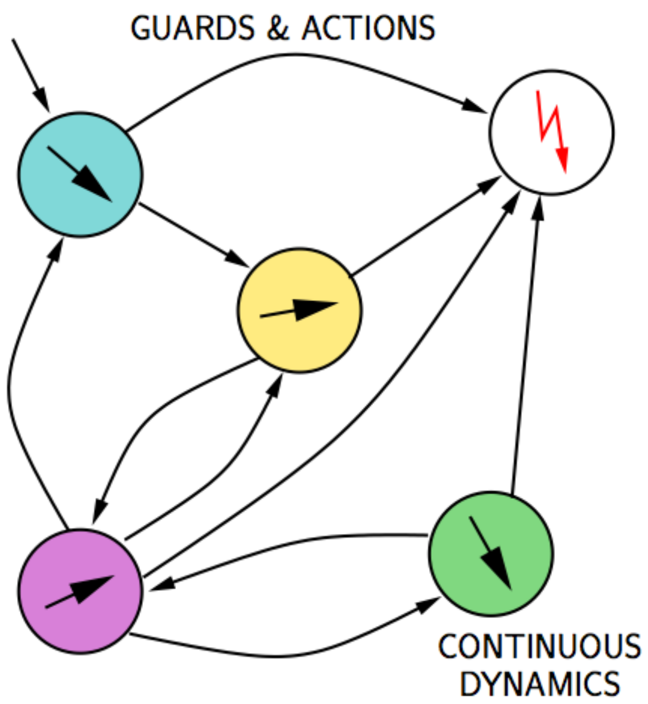




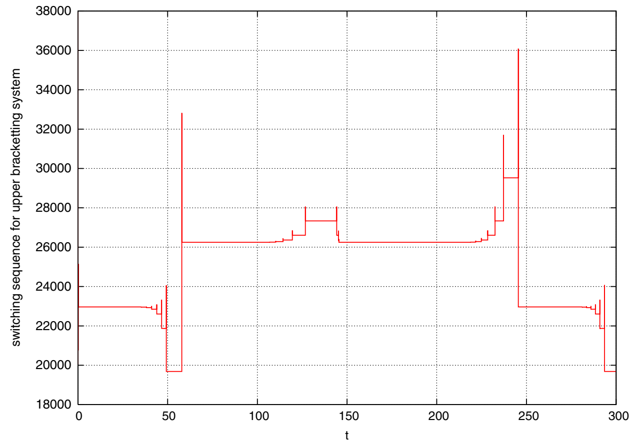


Monitoring of Hybrid Systems

- **Modelling → hybrid automaton**
  - Non-linear continuous dynamics
  - Bounded uncertainty



- **State Estimation → reconstruct system state variables**
  - switching sequence
  - continuous variables
- **Important issue**
  - Control & Diagnosis ...

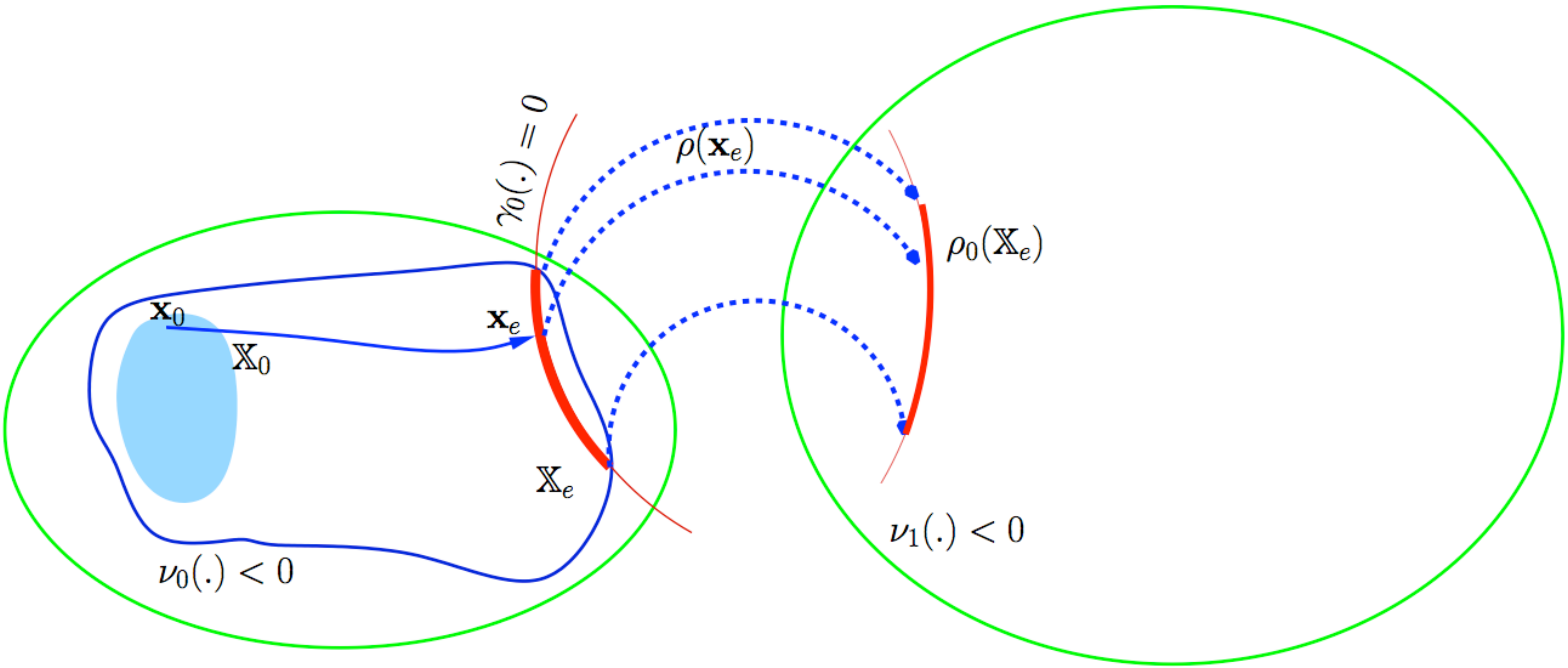


- **Nonlinear Hybrid Reachability**

Hybrid Cyber-Physical Systems

Hybrid reachability

- Continuous reachability
- Event detection, jump & reset



Hybrid Reachability Computation

- **Guaranteed event detection & localization**
  - **An interval constraint propagation approach**
    - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

Hybrid Reachability Computation

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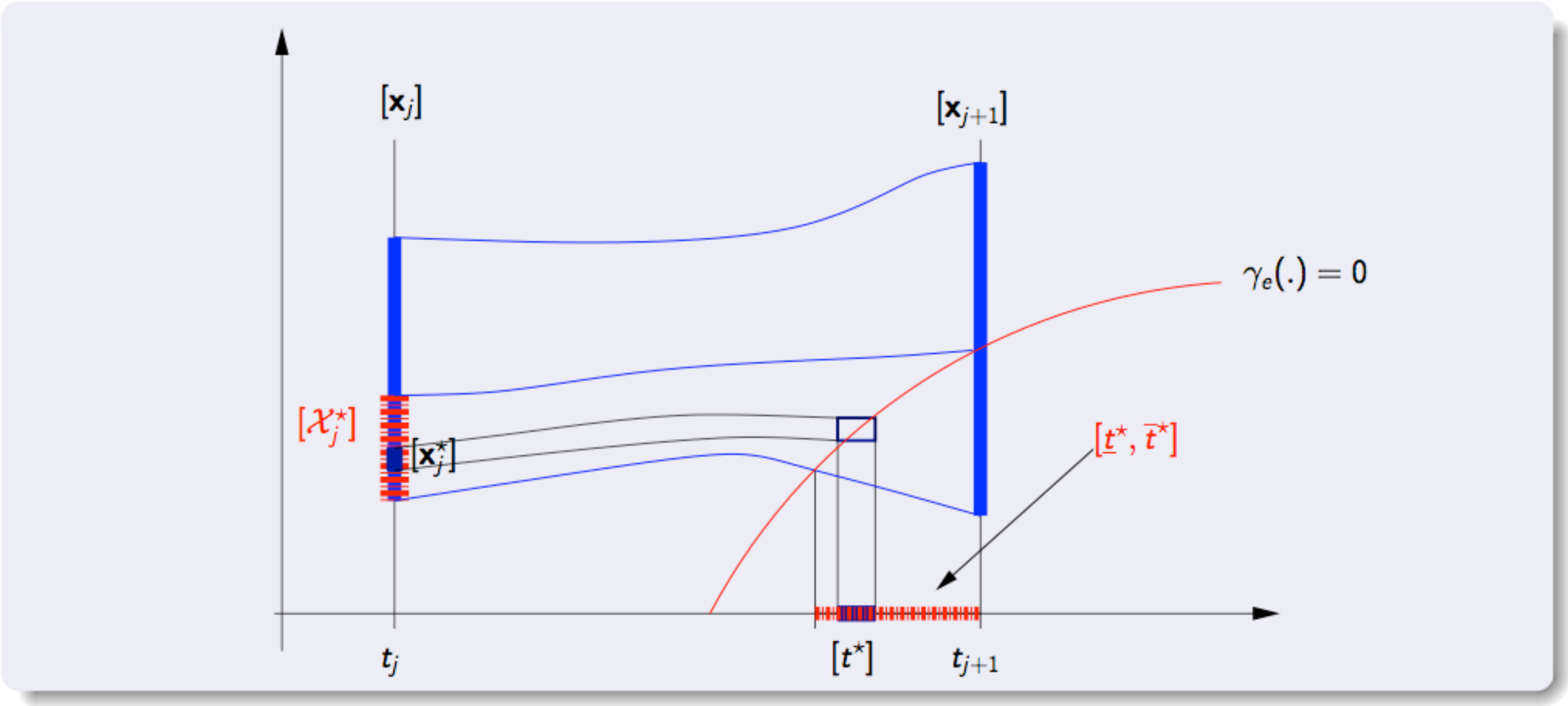
  - **An interval constraint propagation approach**
    - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



Compute $[\underline{t}^\star, \overline{t}^\star] \times [\mathcal{X}_j^\star]$

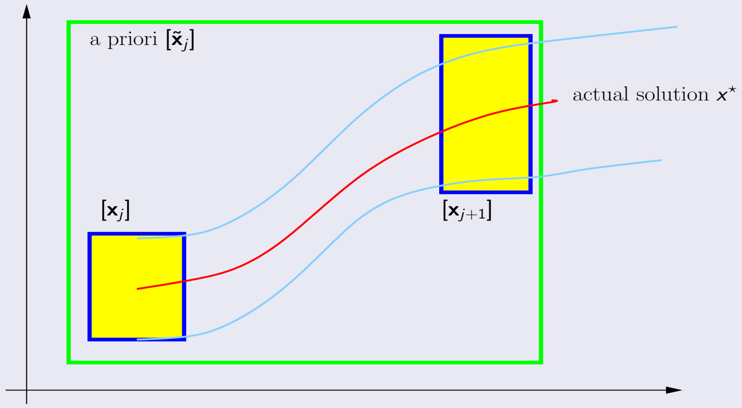
Hybrid Reachability Computation

Guaranteed event detection & localization

- An interval constraint propagation approach
  - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0]\,,\ \mathbf{p} \in [\mathbf{p}]$$

Time grid $\longrightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$



- Analytical solution for $[\mathbf{x}](t),\ t \in [t_j, t_{j+1}]$

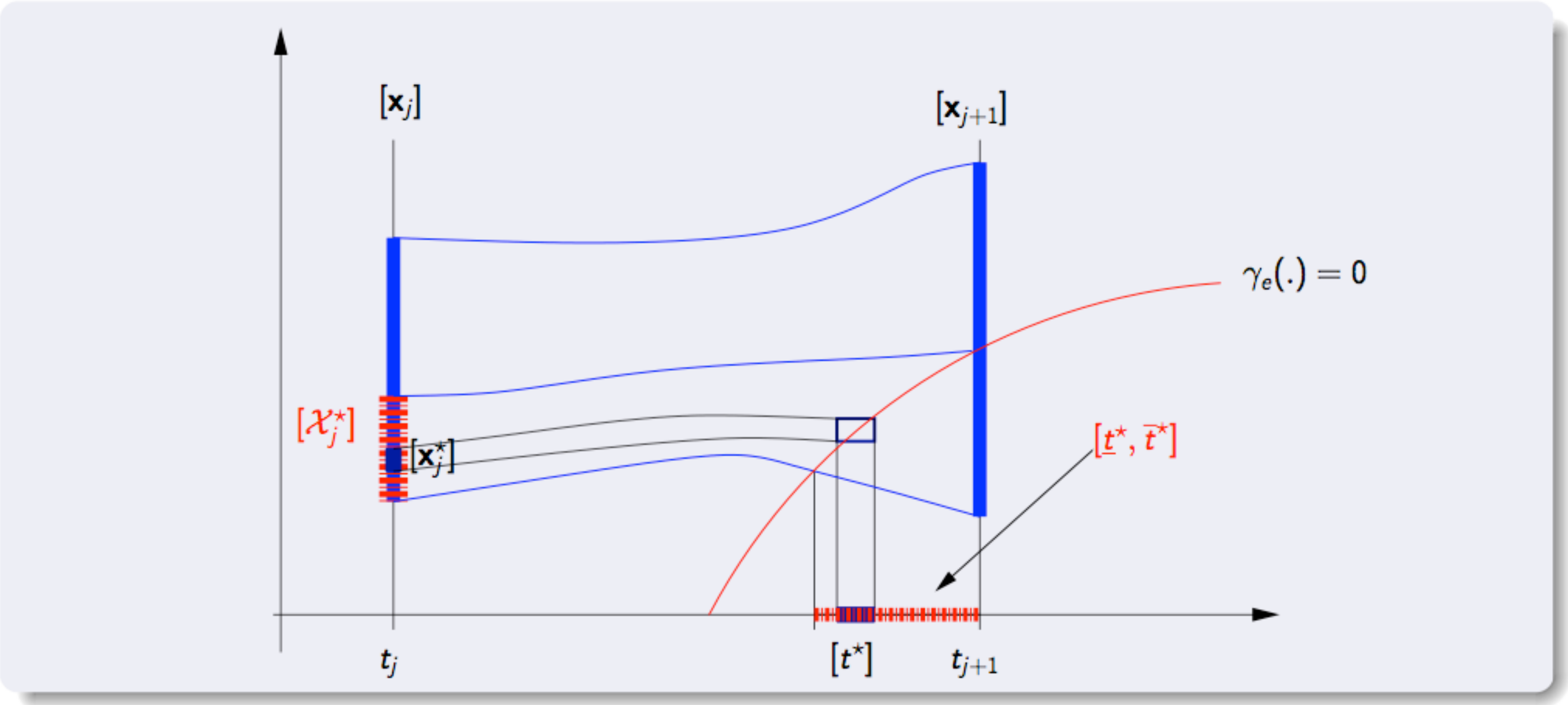
$$[\mathbf{x}](t) = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t - t_j)^k \mathbf{f}^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

Hybrid Reachability Computation

Guaranteed event detection & localization

- **An interval constraint propagation approach**
  - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



Compute $[\underline{t}^\star, \overline{t}^\star] \times [\mathcal{X}_j^\star]$

Hybrid Reachability Computation

Guaranteed event detection & localization

- An interval constraint propagation approach
  - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$

- $[\mathbf{x}](t) = \text{Interval Taylor Series (ITS)}(t, [\mathbf{x}_j], [\tilde{\mathbf{x}}_j])$
- $\gamma([\mathbf{x}](t)) = 0$

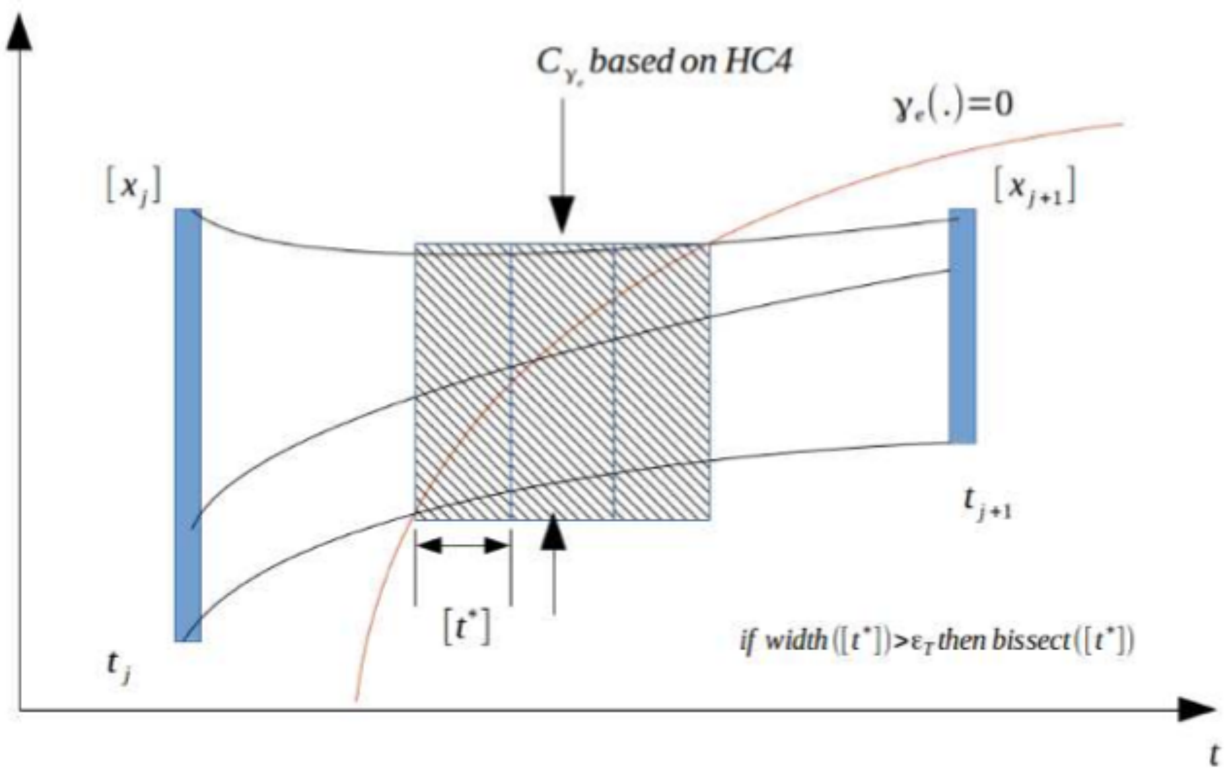
$\Rightarrow \gamma \circ \text{ITS}(t, \mathbf{x}_j, [\tilde{\mathbf{x}}_j]) \rightarrow \psi(t, \mathbf{x}_j)$

Solve CSP $([t_j, t_{j+1}] \times [\mathbf{x}_j], \psi(.,.) \ni 0)$

Hybrid Reachability Computation

Detecting and localizing events

- **Improved and enhanced version. A faster version.**
  - (Maïga, Ramdani, Travé-Massuyès, IEEE CDC 2013, ECC 2014)



Hybrid Reachability Computation

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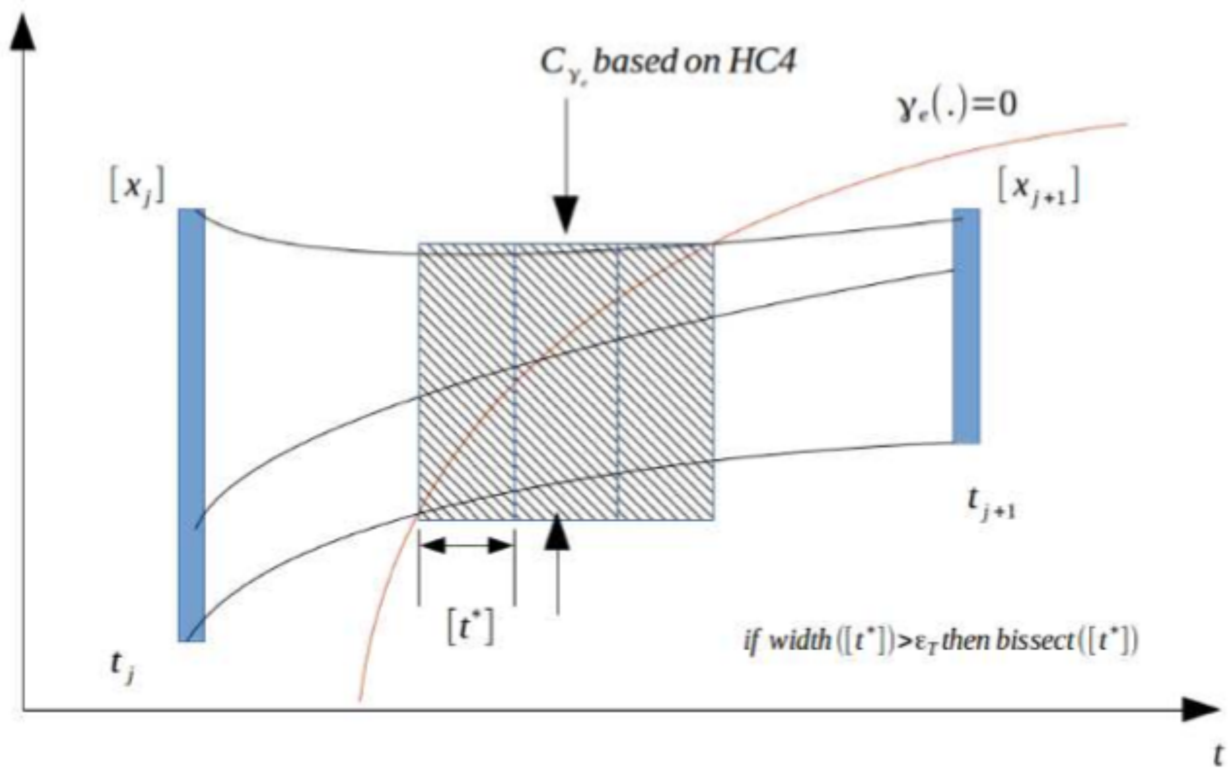
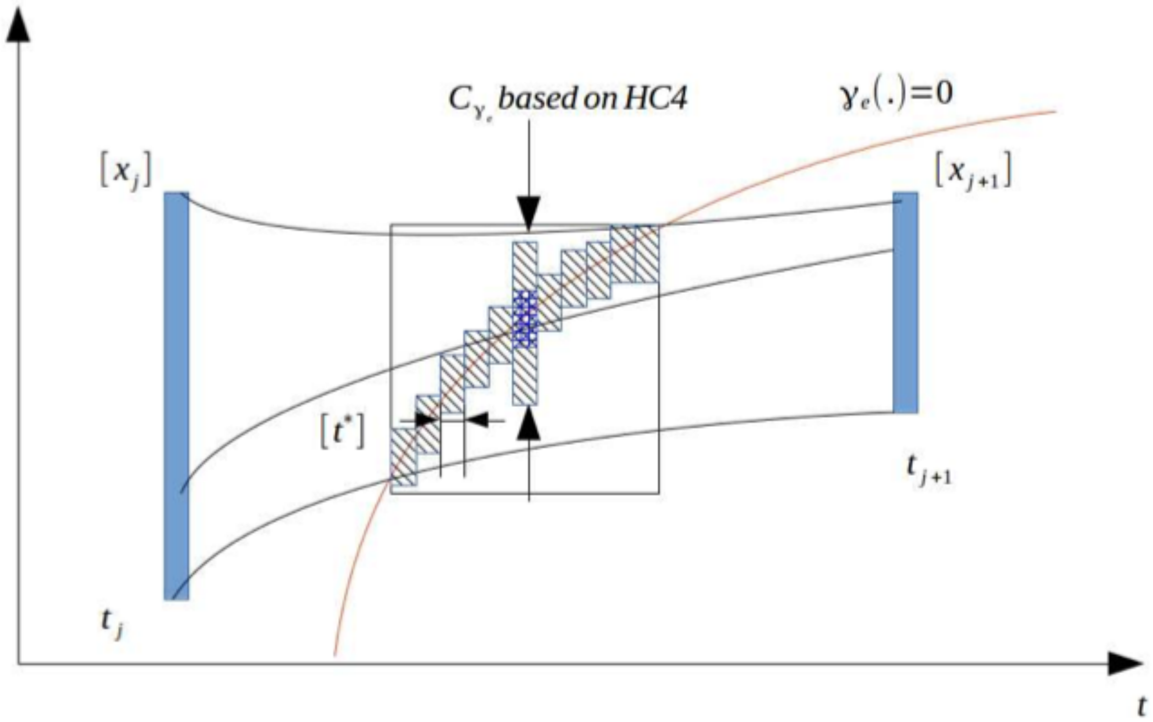


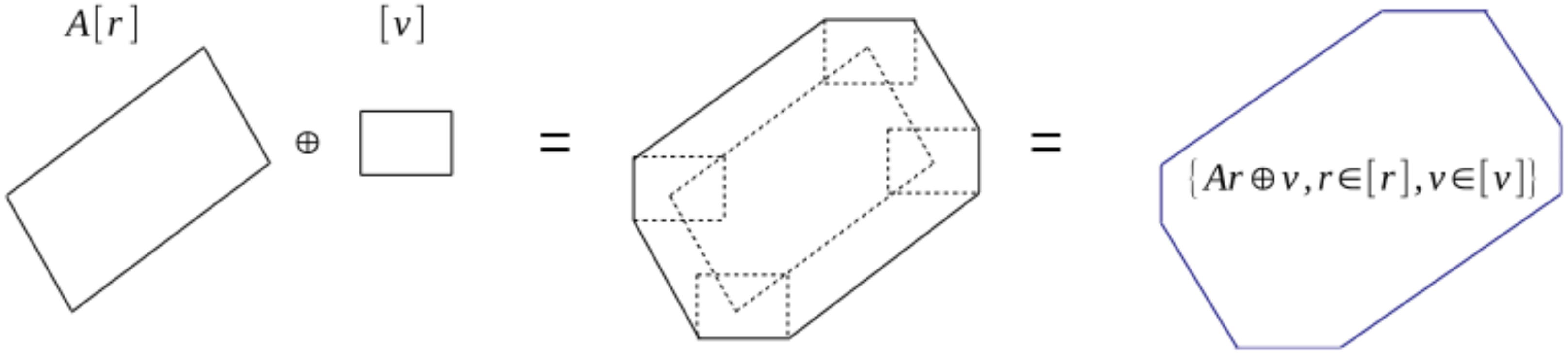


Hybrid Reachability Computation

Solution set: $$[\mathbf{x}](t) = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t - t_j)^k \mathbf{f}^{[k]}([\tilde{x}_j], [\mathbf{p}])$$

Mean value form + Lohner’s QR transformation method

$$\boxed{[\mathbf{x}](t) = A(t)[r](t) \oplus [v](t) \rightarrow \text{MSBP}}$$



$$\boxed{[\mathbf{x}](t) = c(t) \oplus R(t)\mathbf{B}^{2n}}$$ is a particular zonotope

$$\begin{aligned} c(t) &= A(t)mid([r](t)) + mid([v](t)), \\ R(t) &= (A(t)diagrad([r](t)) \mid diagrad([v](t))) . \end{aligned}$$

Hybrid Reachability Computation

Hybrid Reachability Computation

(Maïga, Ramdani, Travé-Massuyès, Combastel IEEE TAC 2016)

- **Reduce over-approximation in event-detection**
  - Solve Redundant constraints

$$\mathscr{C}_l^R := (\gamma_e\,(v + A_l^\star r) = 0) \wedge (\gamma_e(z) = 0) \wedge (z = v + A_l^\star r)$$

Hybrid Reachability Computation

(Maïga, Ramdani, Travé-Massuyès, Combastel IEEE TAC 2016)

- **Reduce over-approximation in event-detection**
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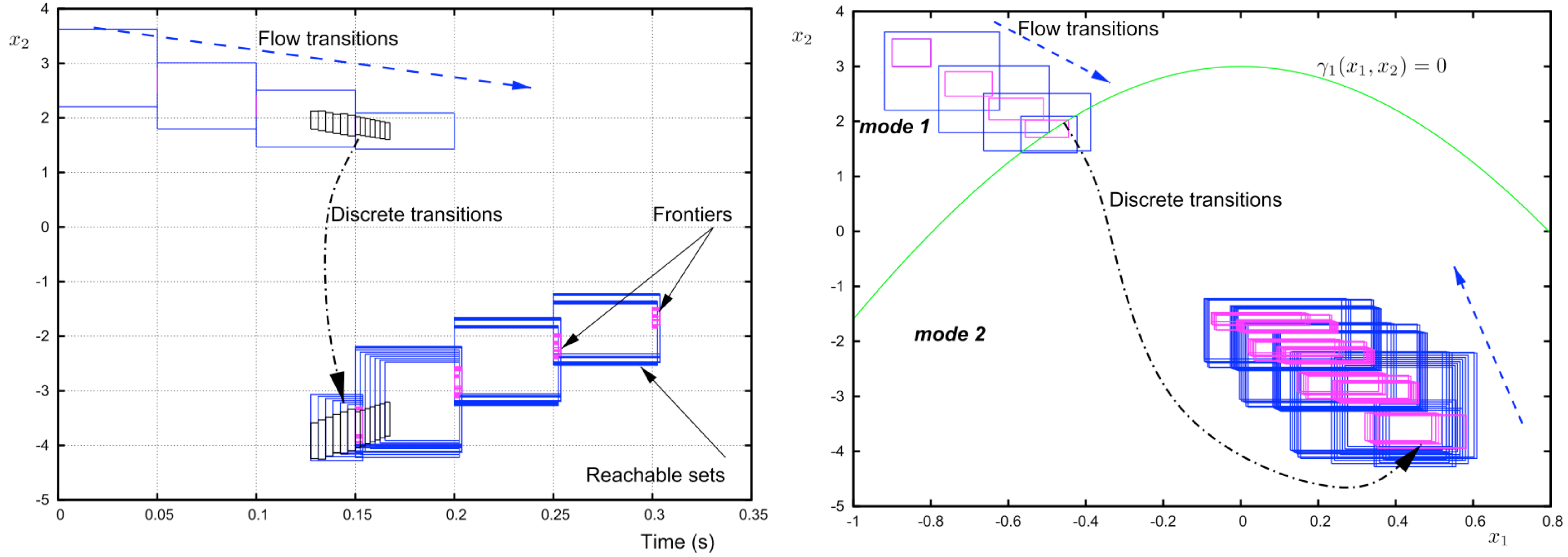
- **Change-of-coordinate-aware approach to discrete transitions with nonlinear guards**
  - Zonotope computation
  - Inclusion of family of zonotopes. Zonotope extension

Hybrid Reachability Computation

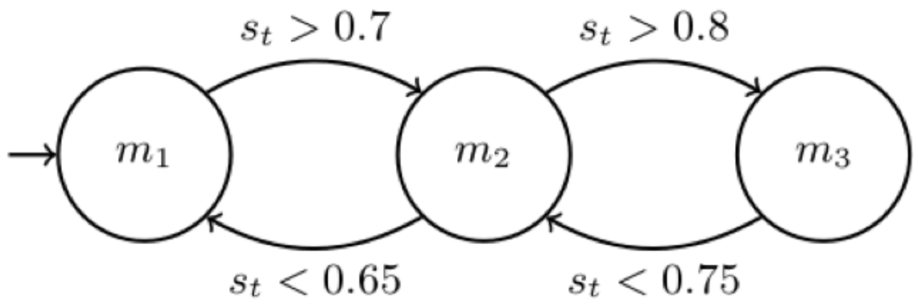
- **Detecting and localizing events**
  - **Improved and enhanced version**

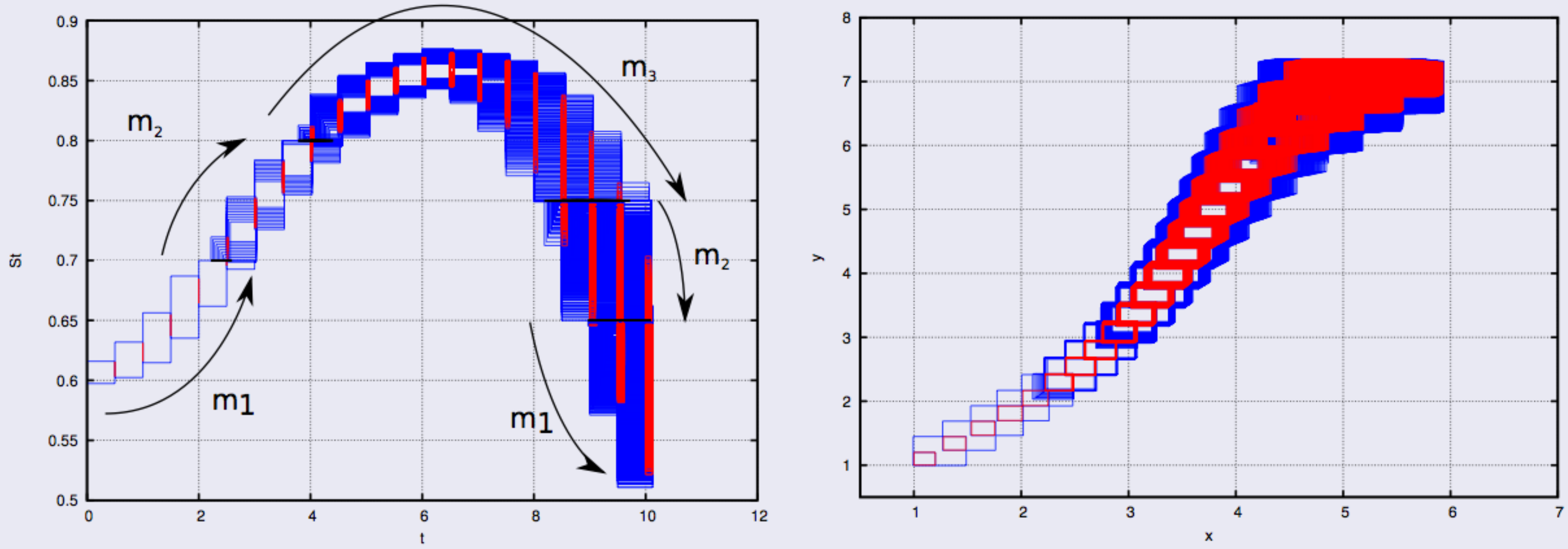


Hybrid Reachability Computation

Detecting and localizing events

- Improved and enhanced version





(e) St×t

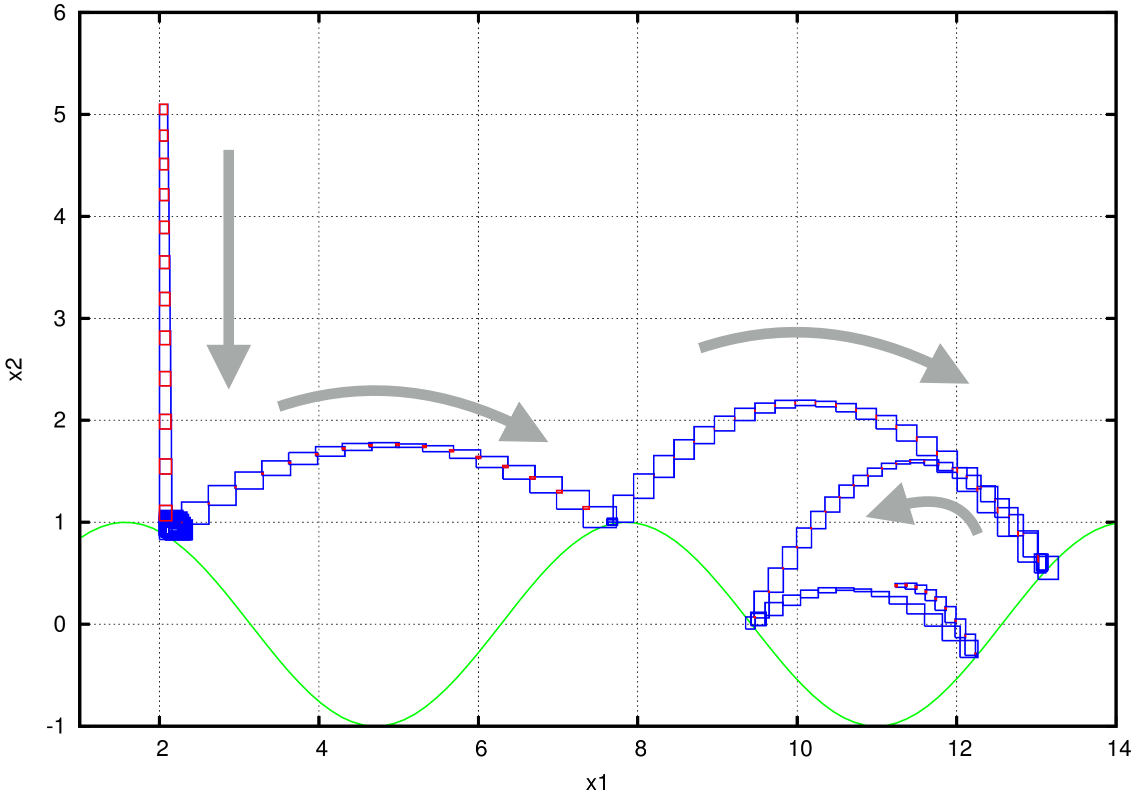
(f) Y×X space

Hybrid Reachability Computation

Detecting and localizing events

- Improved and enhanced version

Bouncing ball in 2D.



Hybrid Reachability Computation

Detecting and localizing events

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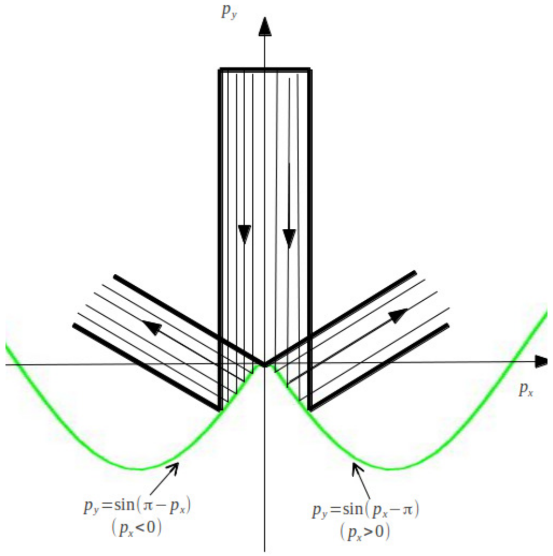
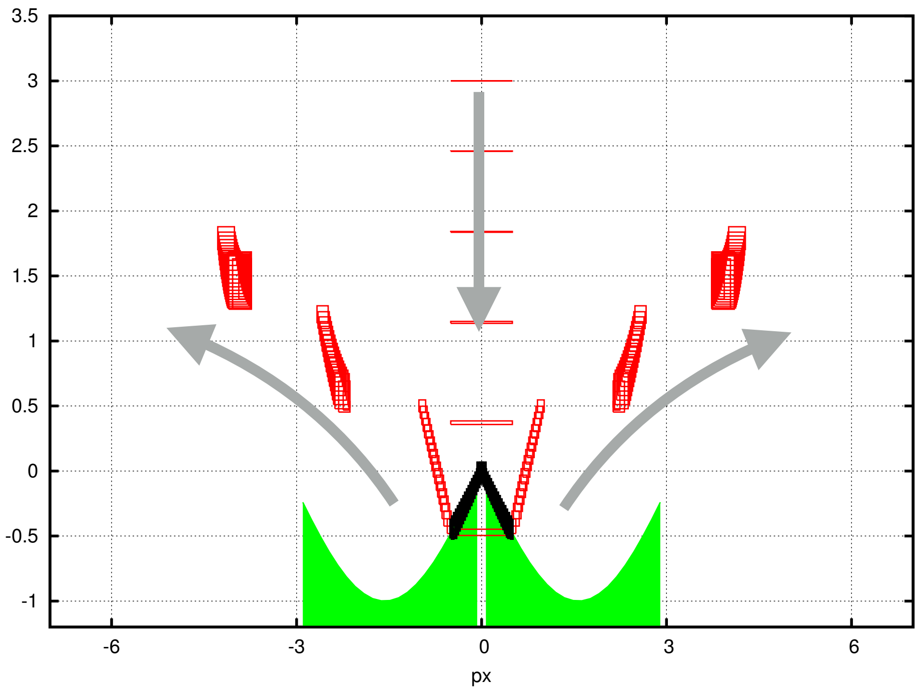
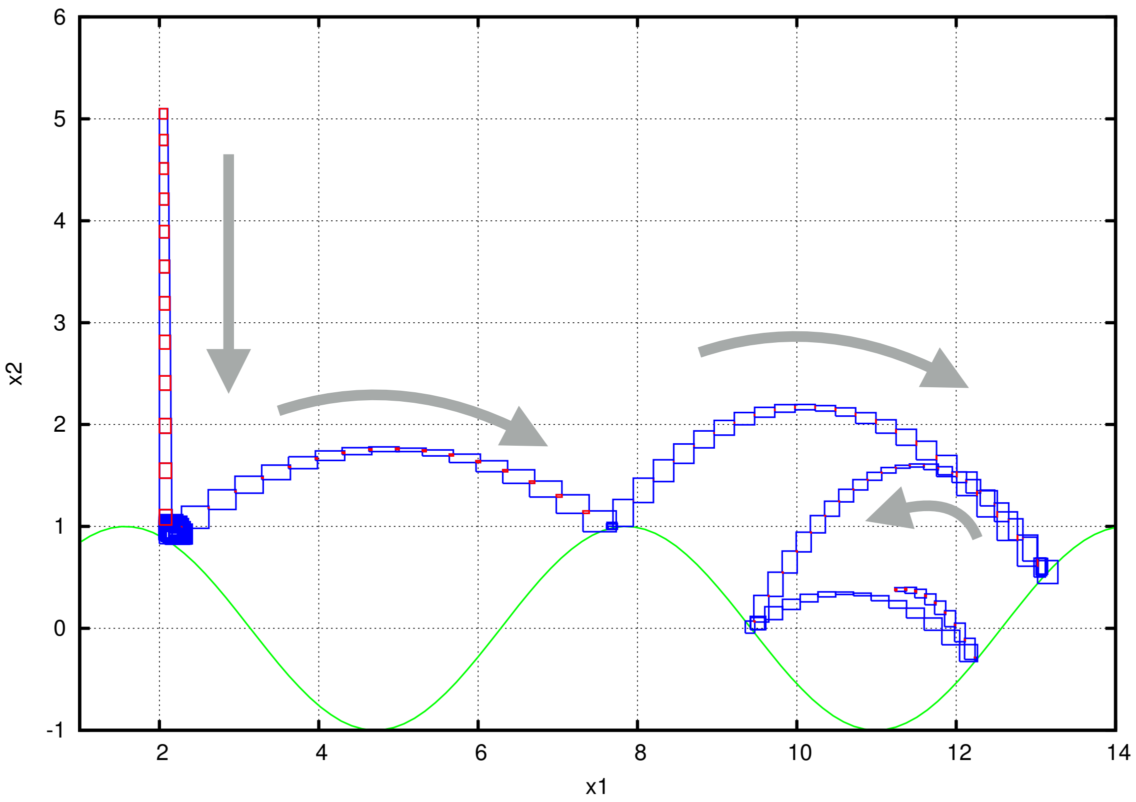


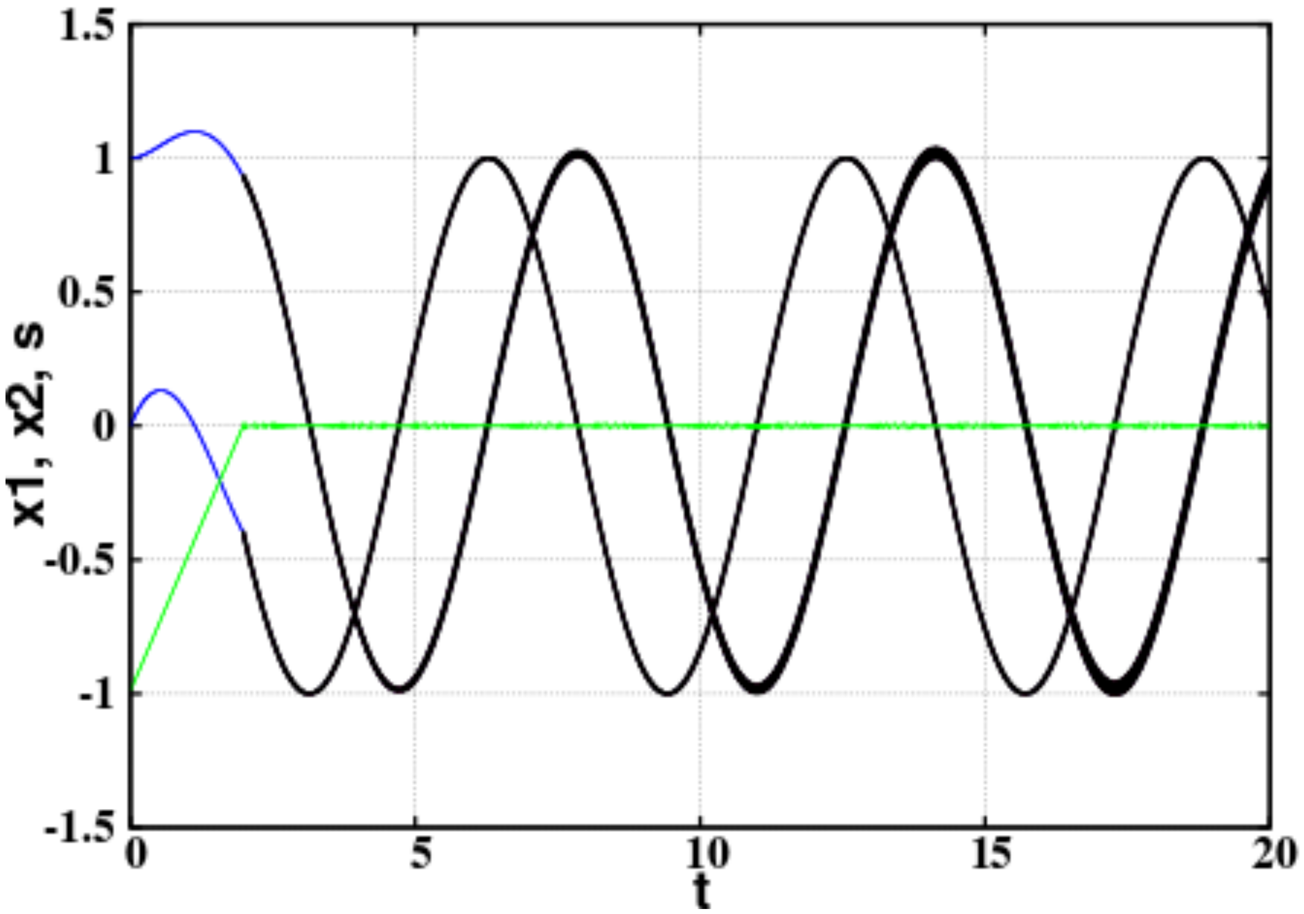




Hybrid Reachability Computation

Detecting and localizing events

- **Improved and enhanced version**
  - Impact of uncertainty on **sliding mode control** (Maïga, Ramdani, Travé-Massuyès, Combastel, IEEE TAC 2016)



Hybrid Reachability Computation

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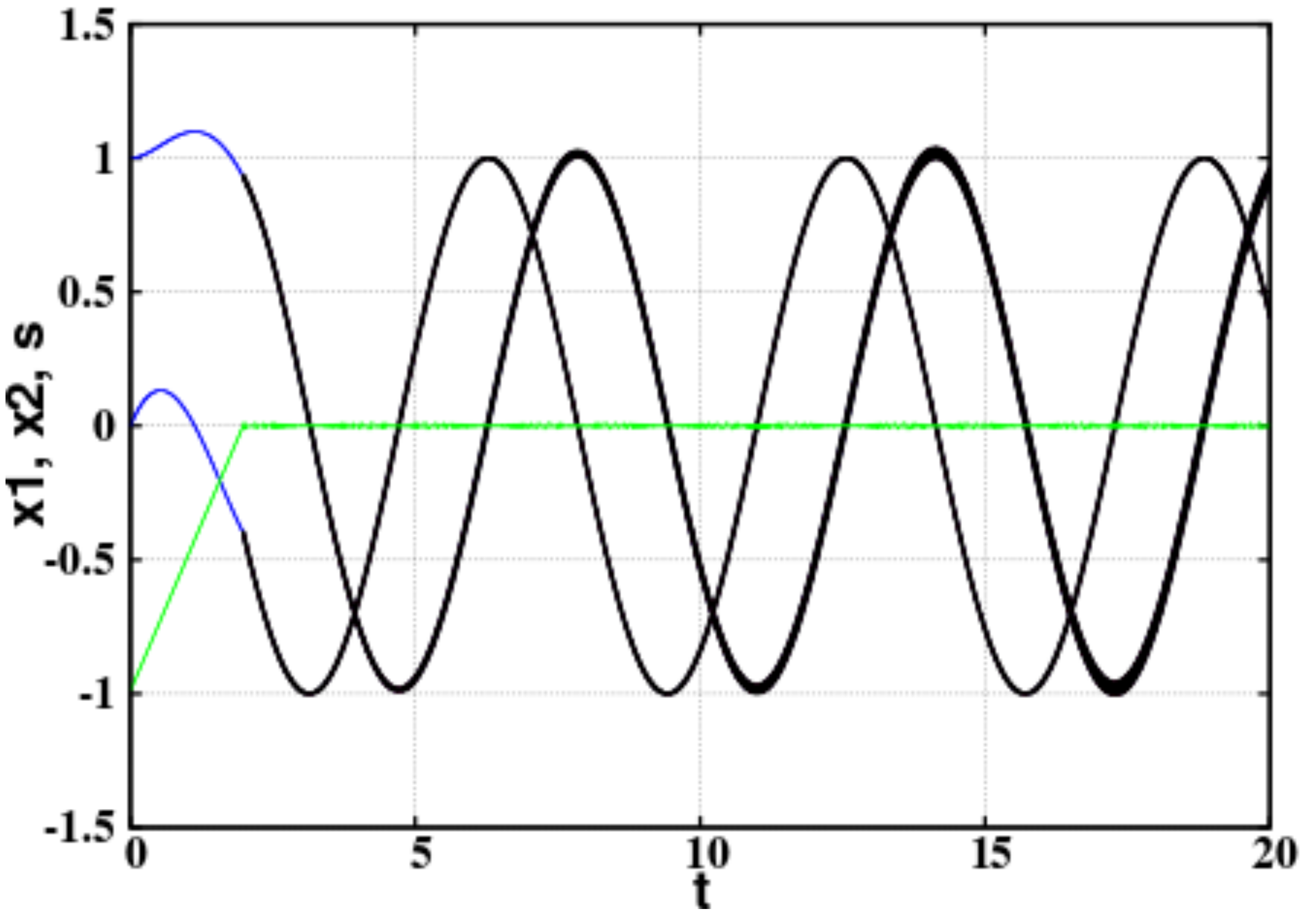

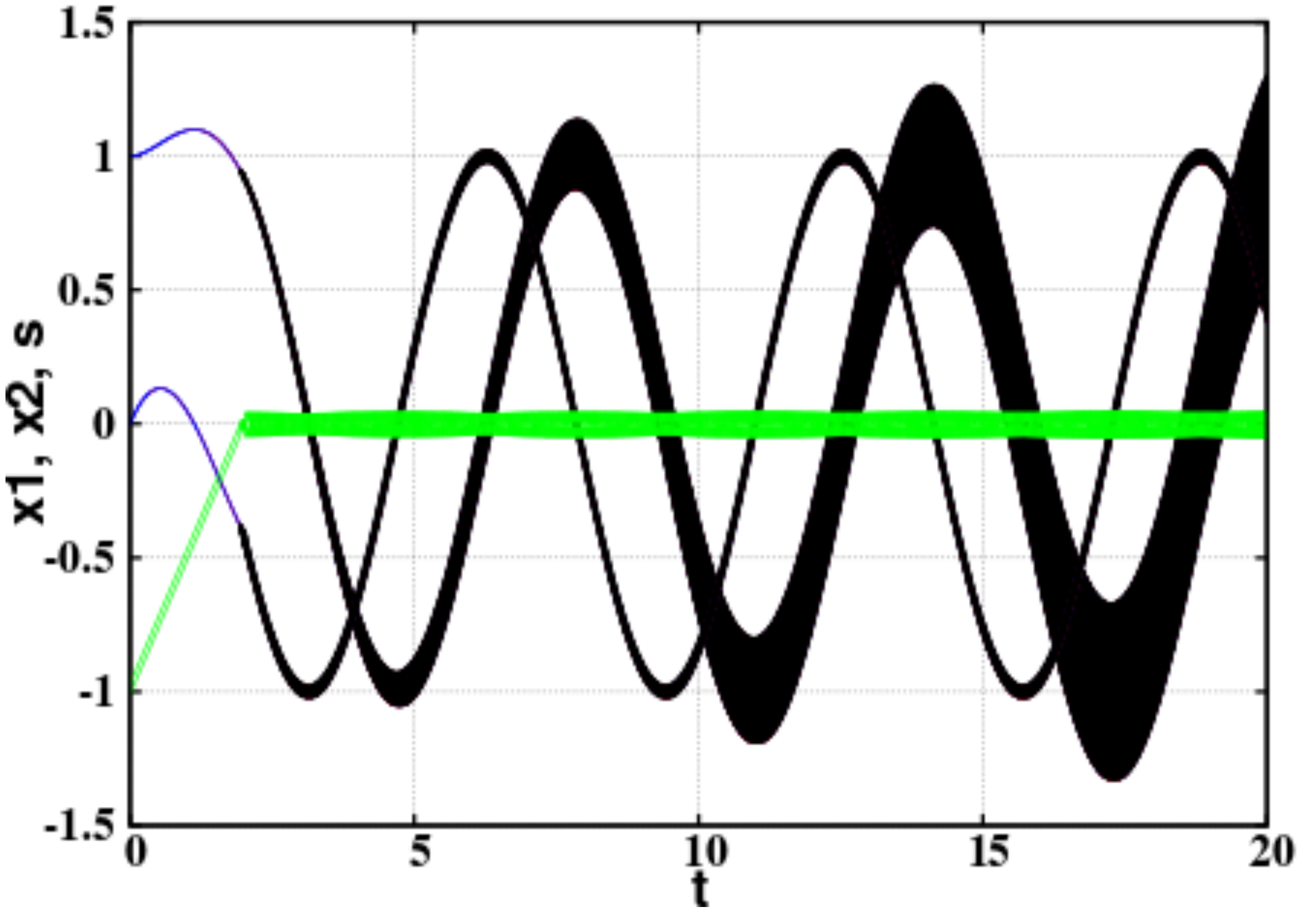

UNIVERSITE D'ORLEANS

- **Set-membership Parameter Estimation with Hybrid Systems**

Set Membership Estimation

- **Parameter estimation with hybrid systems**
  - Branch-&-bound, branch-&-prune, interval contractors …
    (Eggers, Ramdani et al., 2012), (Maïga, Ramdani et al., 2015)

$$
\begin{aligned}
\mathbb{S} = \{\mathbf{p} \in \mathbb{P}_0 | \quad & (\forall t \in [t_0, T_{end}], \\
& flow(q) \wedge Inv(q) \wedge guard(e)) \\
& \wedge \forall t_j \in \{t_1, t_2, ..., T_n\}, g_q(x, \mathbf{p}, t) \in \mathbb{Y}_j \quad \}
\end{aligned}
$$

Set Membership Estimation

Parameter estimation with hybrid systems

- Branch-&-bound, branch-&-prune, interval contractors …
  (Eggers, Ramdani et al., 2012), (Maïga, Ramdani et al., 2015)

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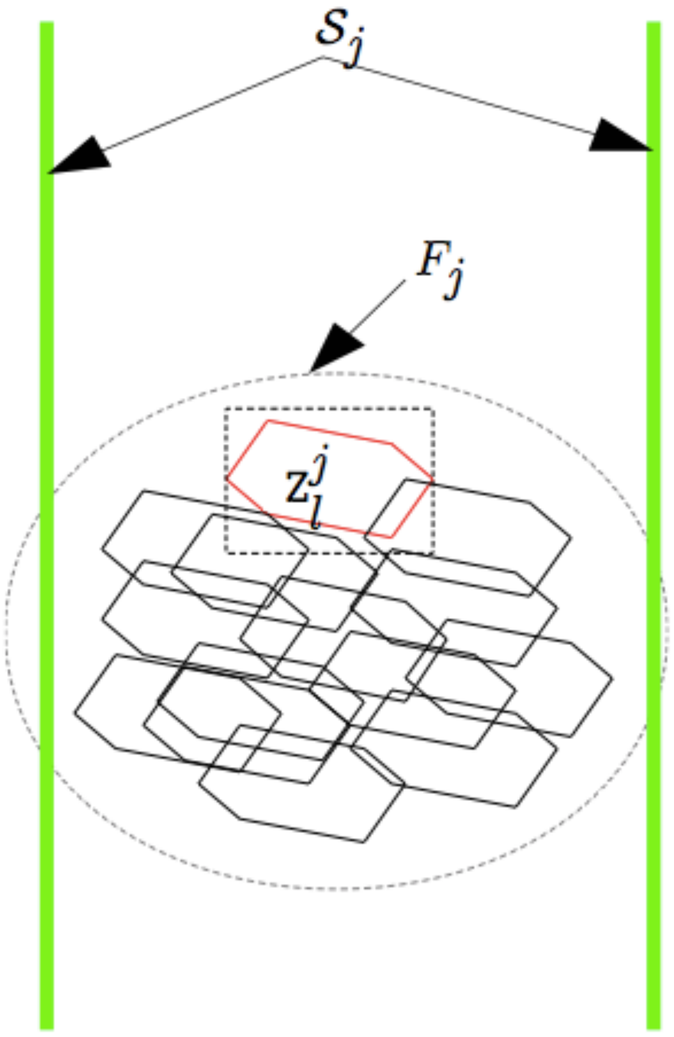
$$\underline{\mathbb{S}} \underline{\subset} \mathbb{S} \underline{\subset} \underline{\mathbb{S}} \cup \Delta\mathbb{S} \equiv \overline{\mathbb{S}}$$

Need an inclusion test!

Inclusion test

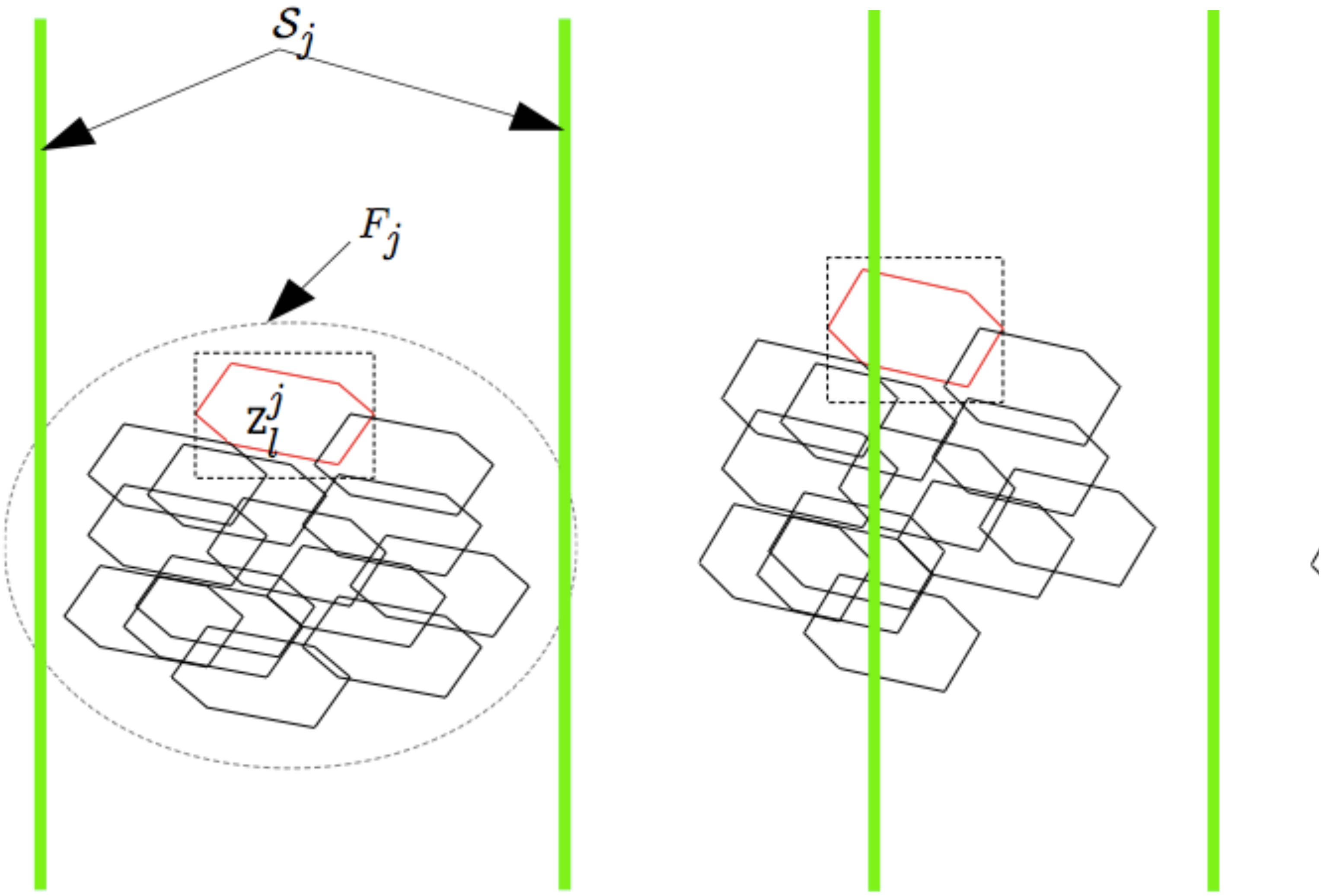
Frontier of the reachable set = union of zonotopes



(a) Test: is true

Inclusion test

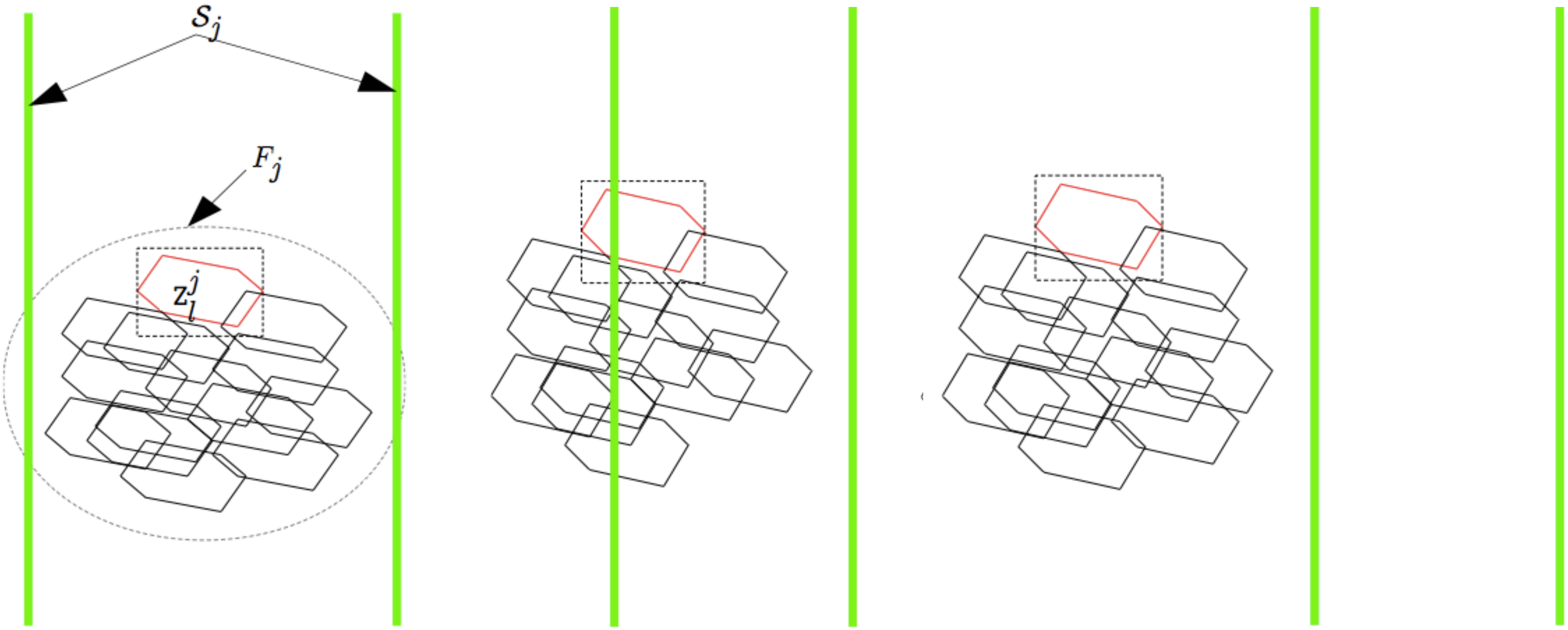
Frontier of the reachable set = union of zonotopes



(a) Test: is true (b) is ambiguous

Inclusion test

Frontier of the reachable set = union of zonotopes



(a) Test: is true (b) is ambiguous (c) is false

Inclusion test

$$\text{Zonotope} \quad \mathsf{Z} = c \oplus R\boldsymbol{B}^p$$

$$\text{Strip} \quad \mathcal{S}_j = \{x \in \mathbb{R}^n |\ |\eta^\top x - d_j| \leq \sigma_j\} \equiv [y_j]$$

$$\text{Zonotope support strip} \quad \mathcal{S}_{\mathsf{Z}} = \{x \in \mathbb{R}^n |\ q_d \leq \eta^\top x \leq q_u\}$$

$$q_u = \min_{x \in \mathsf{Z}} \eta^\top x = \eta^\top c - \|R^\top \eta\|_1$$

$$q_d = \max_{x \in \mathsf{Z}} \eta^\top x = \eta^\top c + \|R^\top \eta\|_1$$

Theorem [(Vicino and Zappa (1996))]

$$\mathsf{Z} \cap \mathcal{S}_j = \emptyset \iff (q_d \geq d_j - \sigma_j) \wedge (q_u \leq d_j + \sigma_j)$$

$$\mathsf{Z} \subseteq \mathcal{S}_j \iff (q_u < d_j - \sigma_j) \vee (q_d > d_j + \sigma_j)$$

Inclusion test

$$\text{Zonotope} \quad \mathsf{Z} = c \oplus R\boldsymbol{B}^p$$

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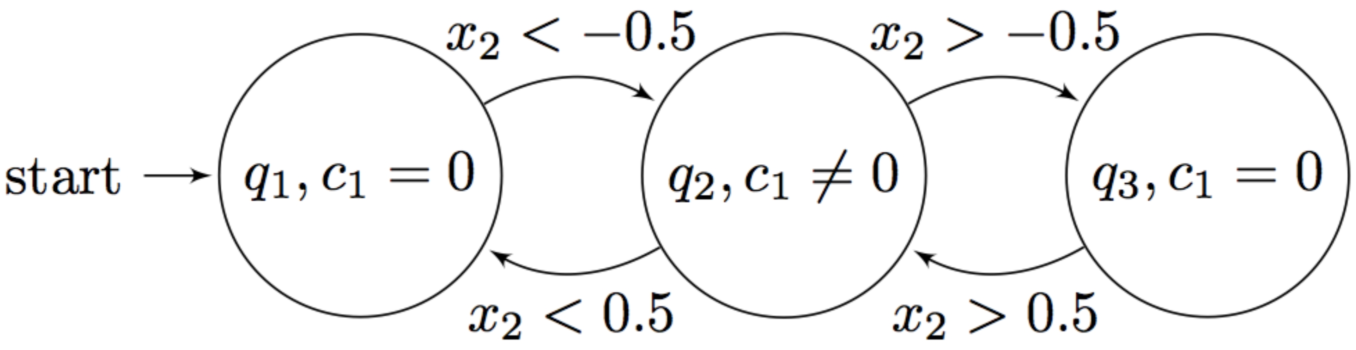
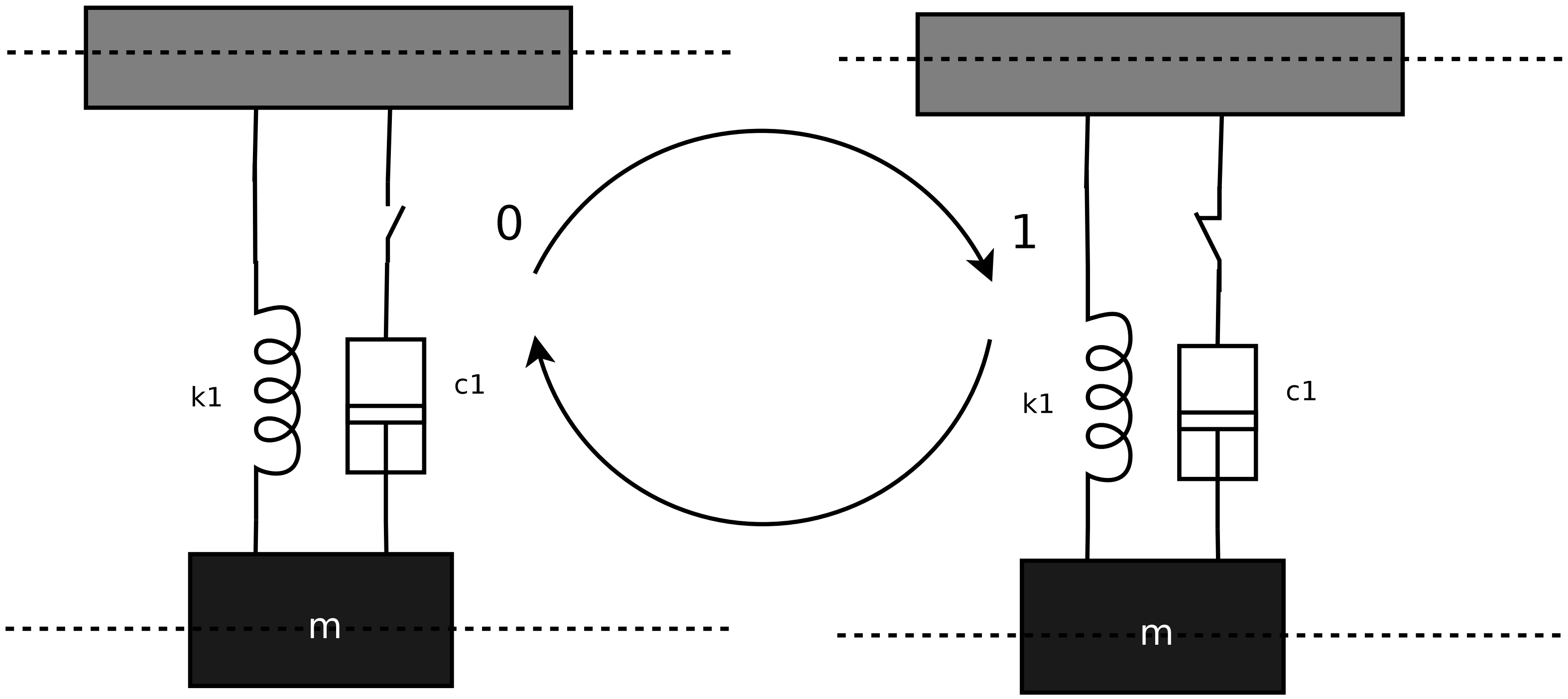
Parameter identification

Hybrid Mass-Spring

- Velocity-dependent damping. Mode switching driven by

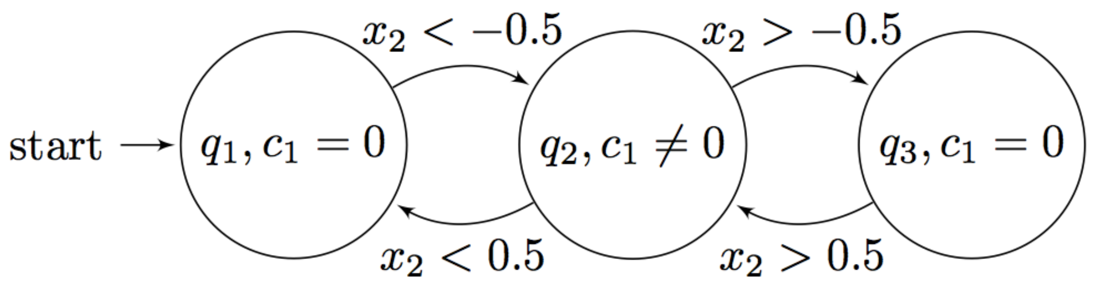


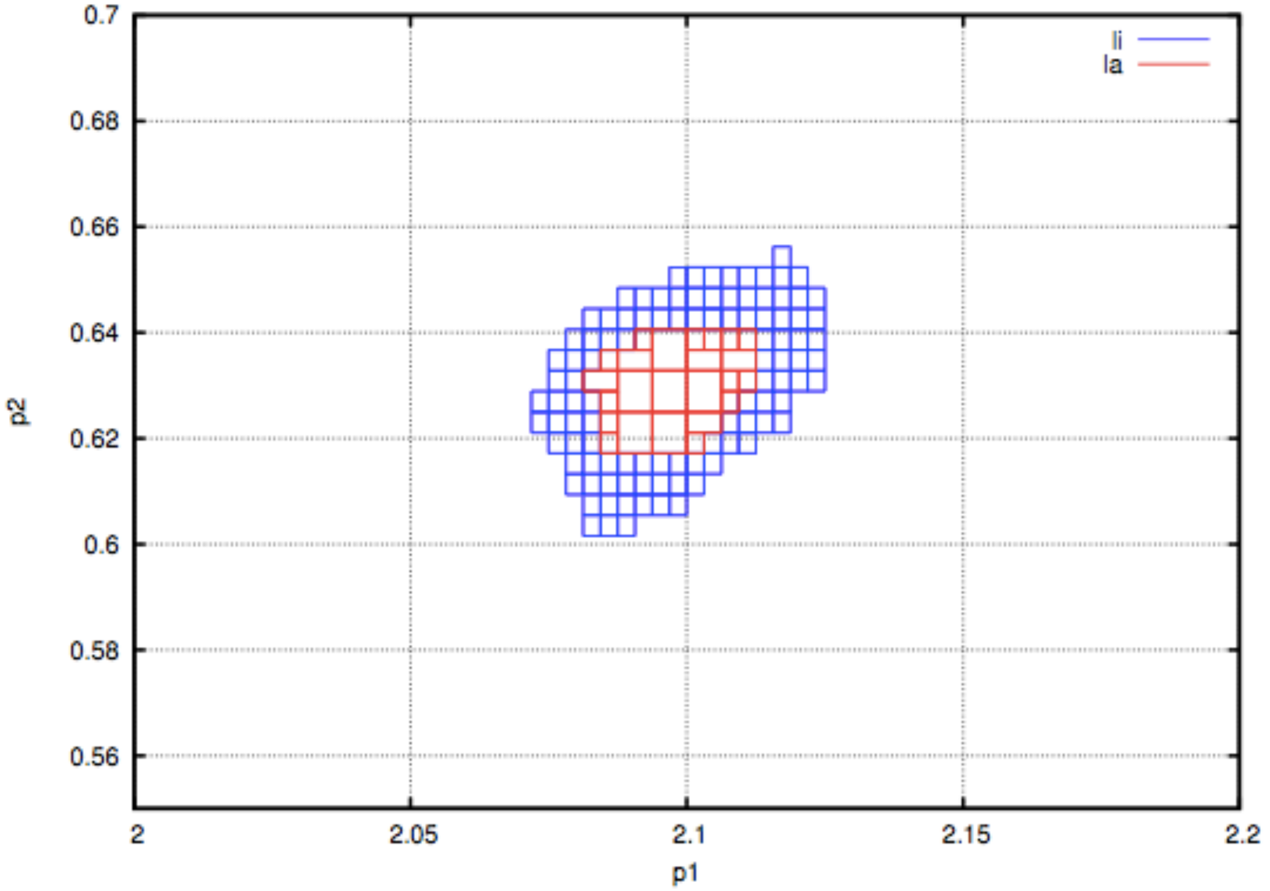


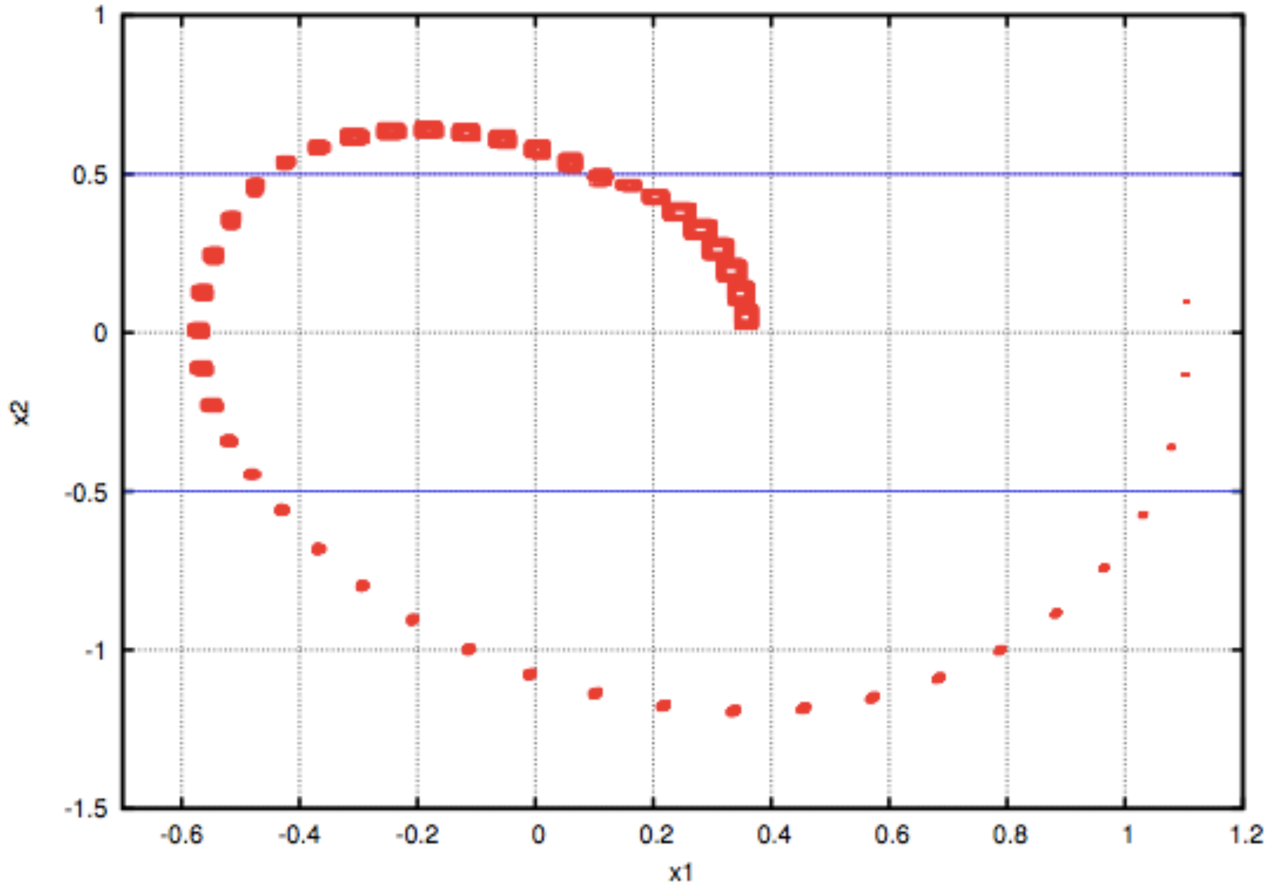
Parameter identification

Hybrid Mass-Spring

- case 1 : Parameters acting on continuous dynamics.
  - *CPU time approx. 140 mn!*



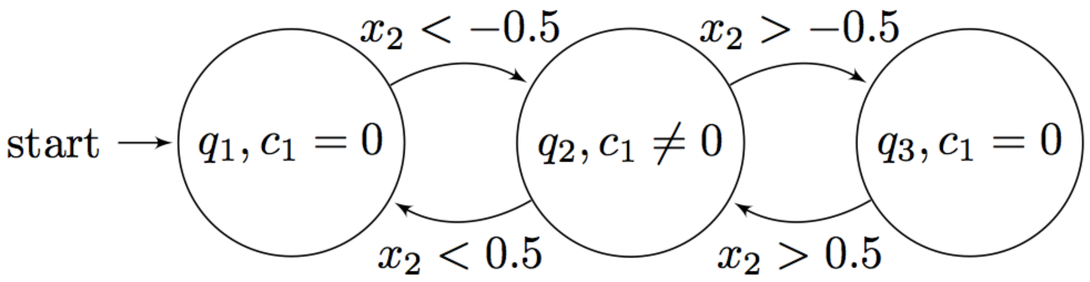


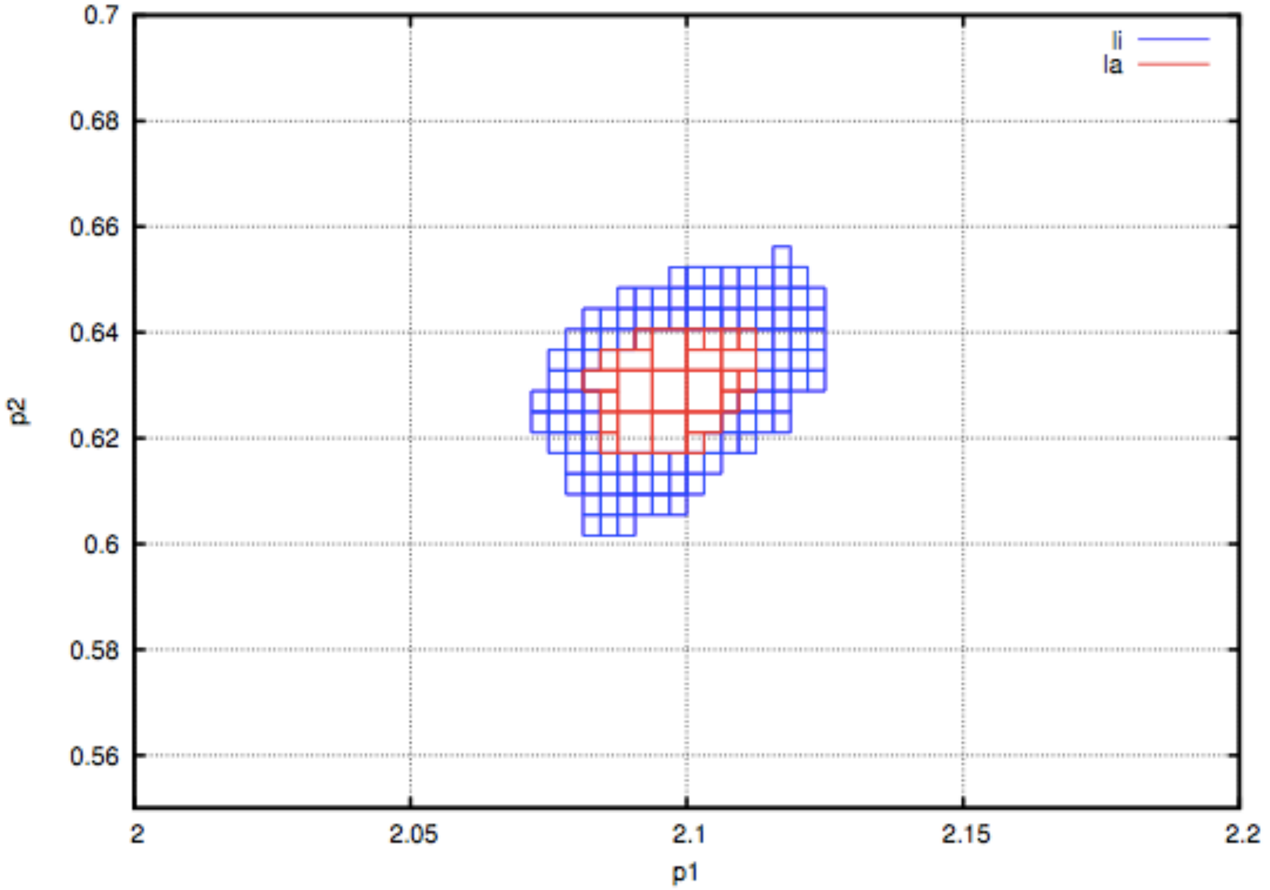


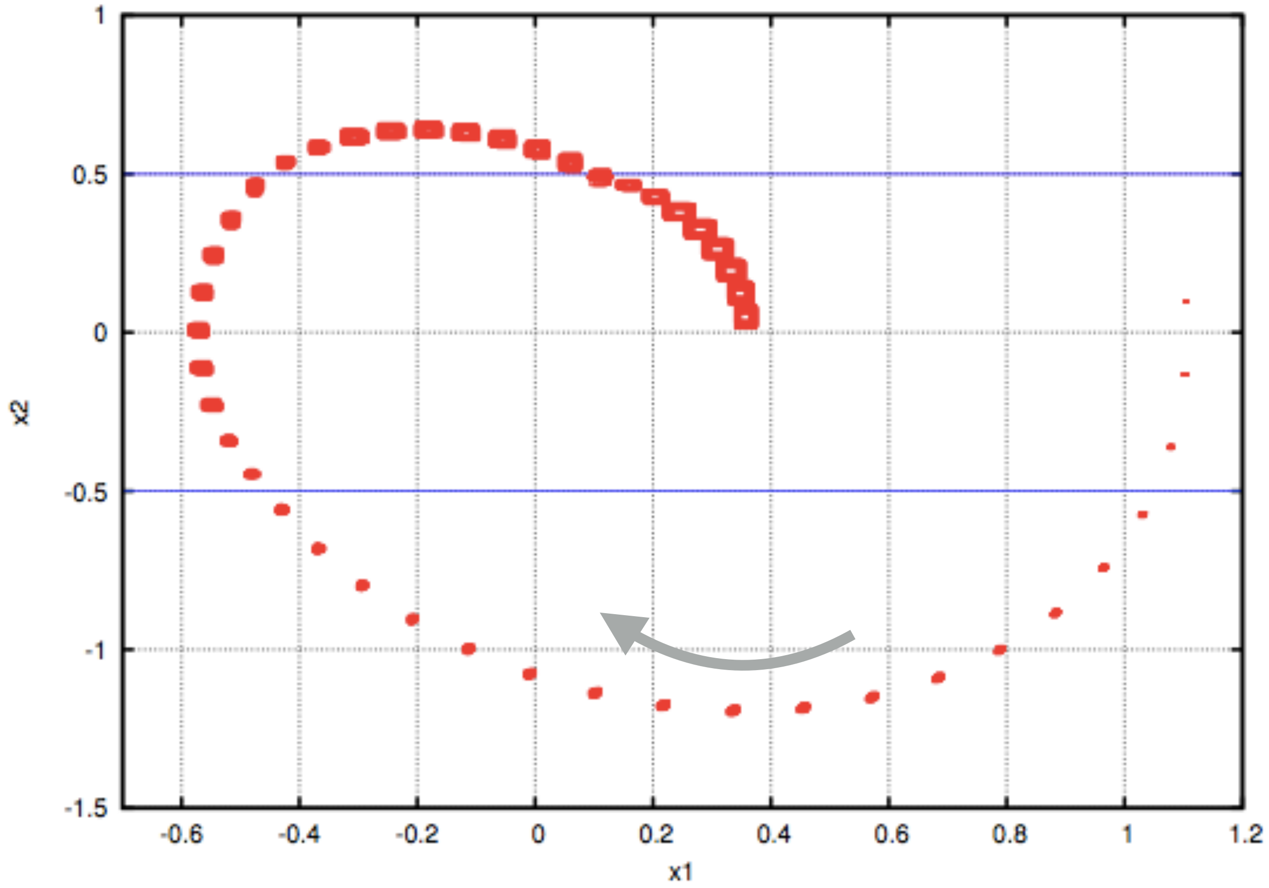
Parameter identification

Hybrid Mass-Spring

- case 1 : Parameters acting on continuous dynamics.
  - *CPU time approx. 140 mn!*



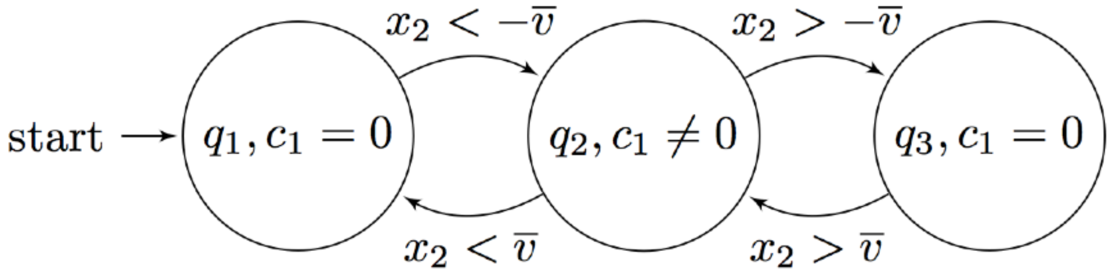


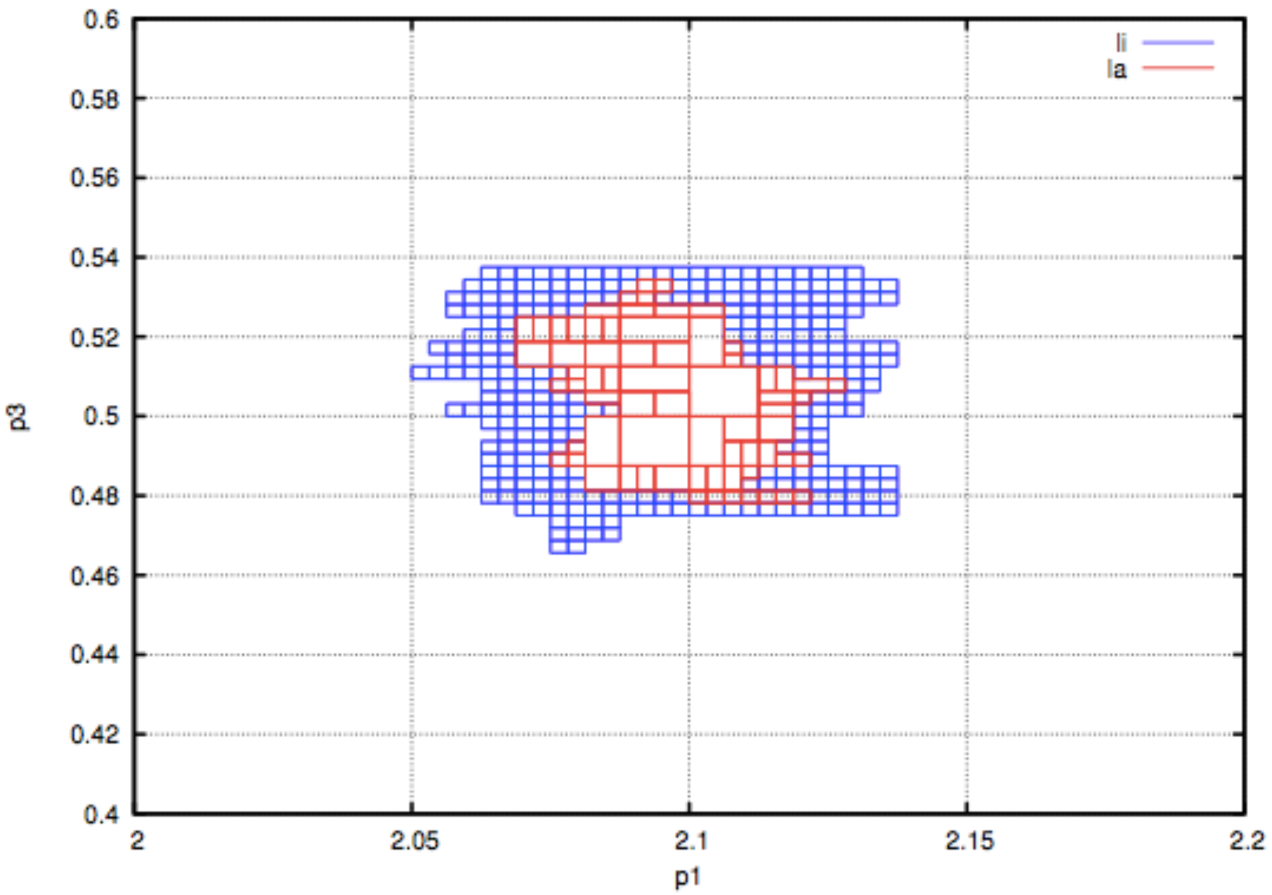


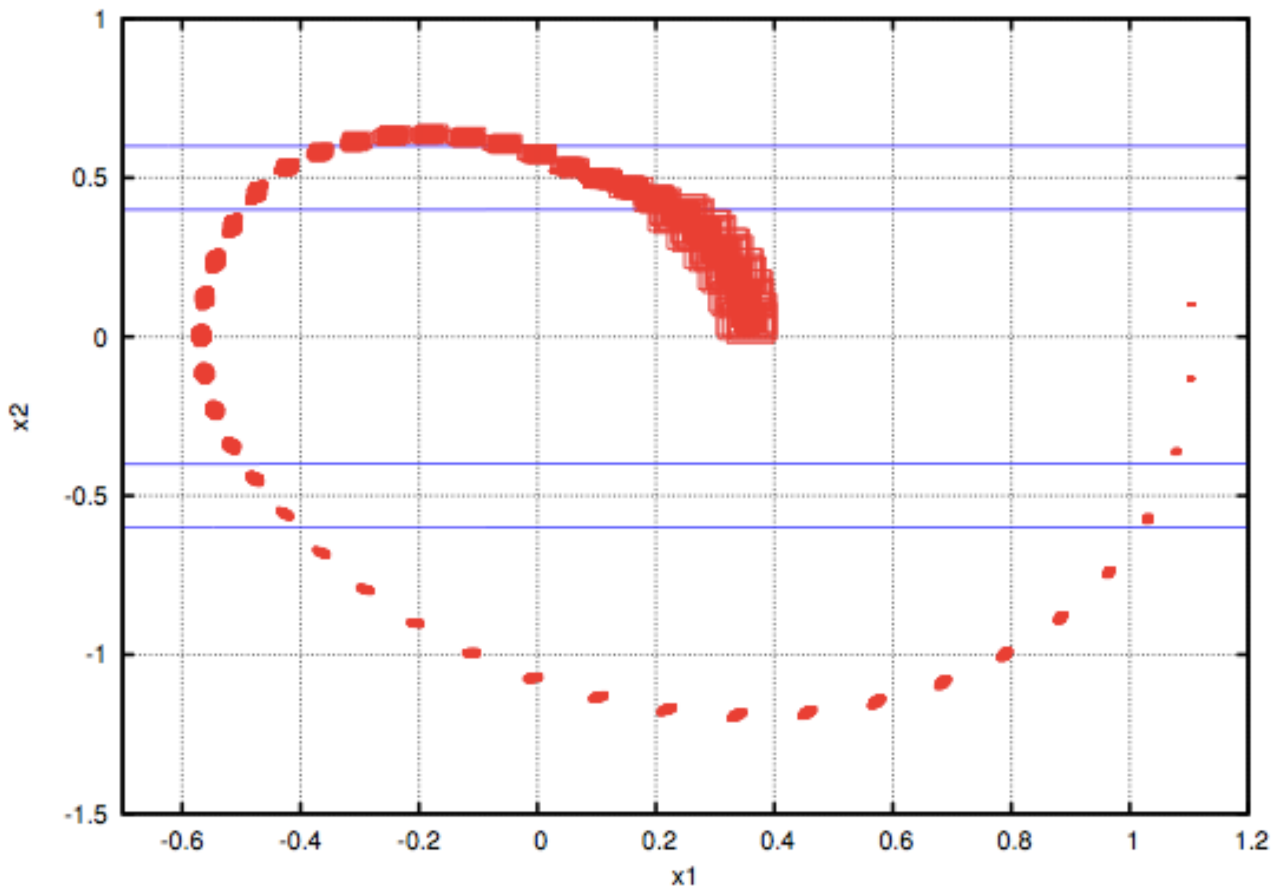
Parameter identification

Hybrid Mass-Spring

- case 2 : parameters acting on discrete transition.
  - *CPU time approx. 40 mn*



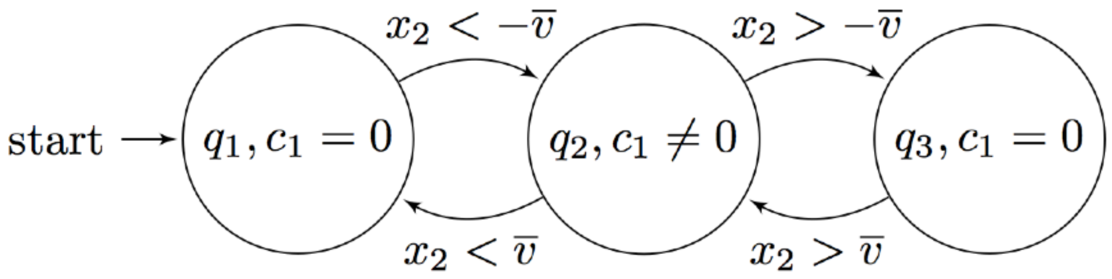


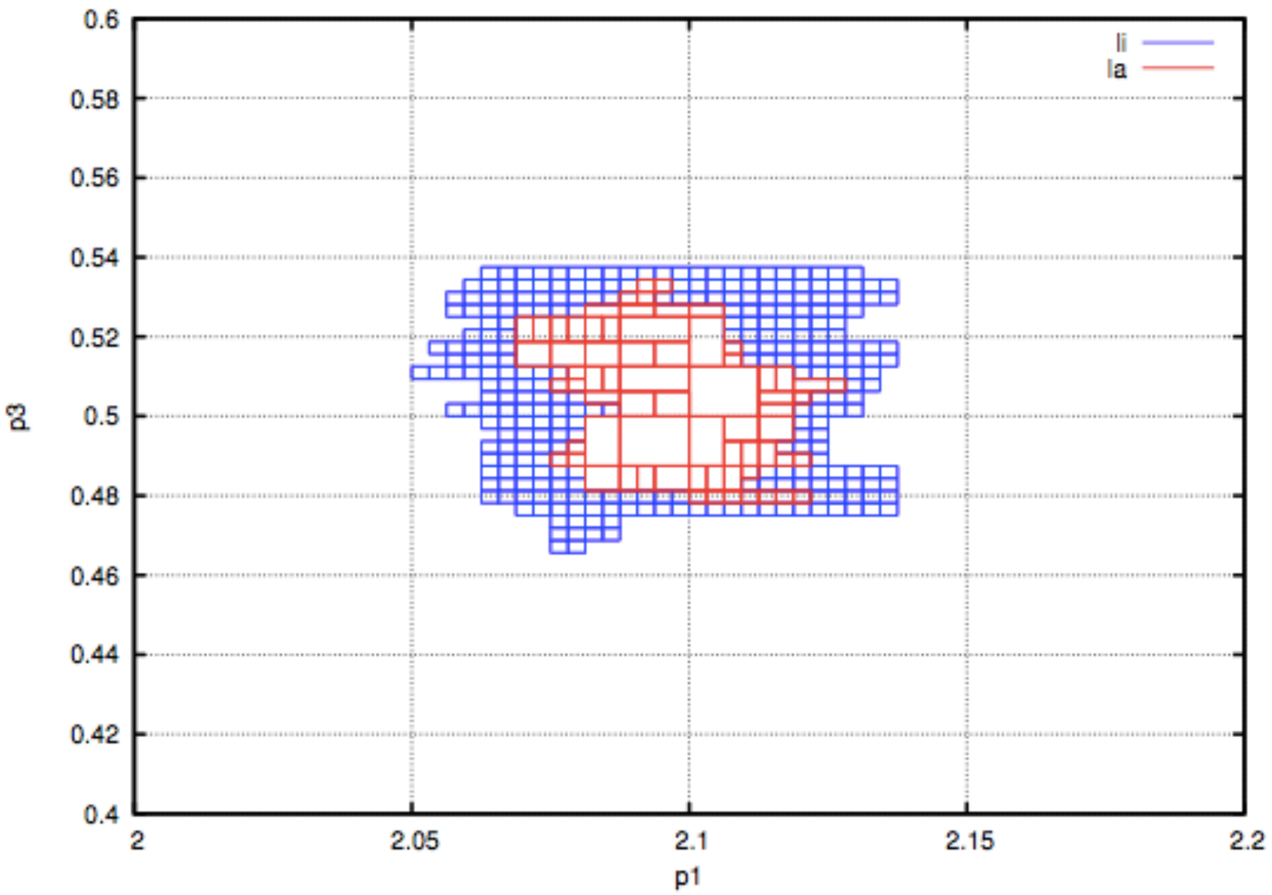


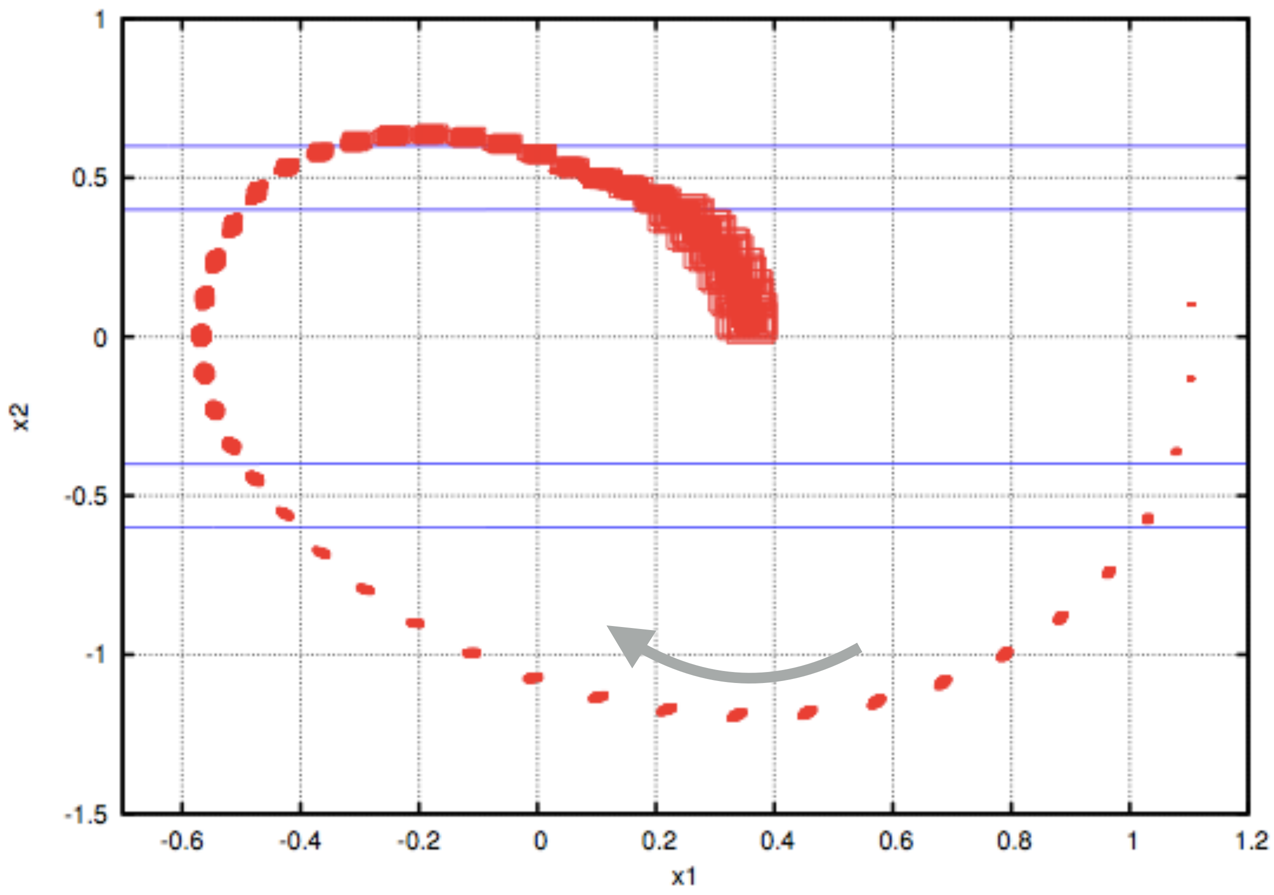
Parameter identification

Hybrid Mass-Spring

- case 2 : parameters acting on discrete transition.
  - *CPU time approx. 40 mn*







Selected References

- N. Meslem, N. Ramdani, Reliable stabilising controller based on set-value parameter synthesis, **IMA Journal of Mathematical Control and Information,** vol. 34, issue 1, 2017, pp. 159-178.
- M. Maïga, N. Ramdani, L. Travé-Massuyès, C. Combastel, A comprehensive method for reachability analysis of uncertain nonlinear hybrid systems, **IEEE Transactions on Automatic Control**, vol. 61, n.9, 2016, pp. 2341-2356.
- A. Eggers, N. Ramdani, N.S. Nedialkov & M. Fränzle. Improving the SAT Modulo ODE Approach to Hybrid Systems Analysis by Combining Different Enclosure Methods, **Software & Systems Modeling**, February 2015, Volume 14, Issue 1, pp 121-148.
- N.Meslem & N.Ramdani, Interval observer design based on nonlinear hybridization and practical stability analysis, **International Journal of Adaptive Control and Signal Processing**, 25(3): 189–294, 2011.
- N.Ramdani & N.S.Nedialkov, Computing reachable sets for uncertain nonlinear hybrid systems using interval constraint propagation techniques, **Nonlinear Analysis: Hybrid Systems**, 5(2): 149-162, 2011.
- N.Meslem, N.Ramdani & Y.Candau, Using hybrid automata for set-membership state estimation with uncertain nonlinear continuous-time systems, **Journal Process Control,** 20(4): 481-489, 2010.
- N.Ramdani, N.Meslem & Y.Candau, Computing reachable sets for uncertain nonlinear monotone systems, **Nonlinear Analysis: Hybrid Systems** 4(2): 263-278, 2010.
- N.Ramdani, N.Meslem & Y.Candau, A hybrid bounding method for computing an over-approximation for the reachable space of uncertain nonlinear systems, **IEEE Transactions on Automatic Control** 54(10): 2352–2364, 2009.
- T.Raïssi, N.Ramdani & Y.Candau, Set membership state and parameter estimation for systems described by nonlinear differential equations, **Automatica** 40:1771–1777, 2004.