Interval observers and fault tolerant control

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Outline

1. Introduction

2. Interval observers - LTI systems
   - Continuous-time systems
   - Discrete-time systems

3. Joint state and unknown input estimation
   - State estimation
   - Upper and lower bounds of the unknown input

4. Fault Tolerant Control
   - Problem statement
   - Interval observer design
   - Control design
   - Numerical simulations
Most of the material of this presentation is detailed in the overview:

1 Introduction

2 Interval observers - LTI systems

3 Joint state and unknown input estimation

4 Fault Tolerant Control
Context

**Observation/Control**

- **✓** Linear systems: several constructive results ⇒ freq. approaches, output/state feedback, ...

- **✓** Nonlinear systems: the solutions depend on the nonlinearity structure ⇒
  - Lipschitzian systems: $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$
  ⇒ linear approaches can be used to build observers/controllers.

- **✓** LPV systems (Linear Parameter-Varying): intermediate class between **Linear** and **Nonlinear** systems
  - Several techniques allow one to transform/approximate NL into LPV systems
    - $\dot{x} = f(x, u) \Rightarrow \dot{x} = A(\theta(t))x + B(\theta(t))u$
  - The nonlinear trajectory belongs into the LPV ones
  - NL ≡ Linear + parameter uncertainties ($\theta(t)$)
Several cases may be met

- Models **without** uncertainties
- Models with uncertain parameters (**constant** or **varying** uncertain parameters)
- Uncertain parameters & unknown inputs

Observers structures

- \( \dot{x} = Ax + Bu; y = Cx \)
  - \( \Rightarrow \) Luenberger Obs. \( \dot{z} = Az + Bu + L(y - Cz) \).
- \( \dot{x} = A(\theta)x + B(\theta)u; y = C(\theta)x \) \( \Rightarrow \) \( \theta \) is known or unknown?

Possible solutions

- Adaptive approaches \( \Rightarrow \) joint estimation of \( x \) and \( \theta \).
- Robust approaches \( \dot{z} = A_\alpha z + B_\alpha u + L(y - C_\alpha z) \) (for some average values \( A_\alpha, B_\alpha \) and \( C_\alpha \)).
- **Set-membership estimation / Interval observers.**
Estimation & Uncertainties

Several cases may be met

- Models \textbf{without} uncertainties
- Models with uncertain parameters (\textbf{constant} or \textbf{varying} uncertain parameters)
- Uncertain parameters & unknown inputs

Observers structures

- \[ \dot{x} = Ax + Bu; \ y = Cx \]
  \[ \Rightarrow \text{ Luenberger Obs.} \quad \dot{z} = Az + Bu + L(y - Cz). \]
- \[ \dot{x} = A(\theta)x + B(\theta)u; \ y = C(\theta)x \Rightarrow \theta \text{ is known or unknown?} \]

Possible solutions

- Adaptive approaches \Rightarrow \text{ joint estimation of } x \text{ and } \theta.
- Robust approaches \[ \dot{z} = A_a z + B_a u + L(y - C_a z) \text{ (for some average values } A_a, B_a \text{ and } C_a). \]
- \textbf{Set-membership estimation / Interval observers.}
Estimation & Uncertainties

Several cases may be met

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Observers structures

- \( \dot{x} = Ax + Bu; y = Cx \)  
  \( \Rightarrow \) Luenberger Obs.  
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- \( \dot{x} = A(\theta)x + B(\theta)u; y = C(\theta)x \)  
  \( \Rightarrow \theta \) is known or unknown?

Possible solutions

- Adaptive approaches  \( \Rightarrow \) joint estimation of \( x \) and \( \theta \).
- Robust approaches  
  \( \dot{z} = A_a z + B_a u + L(y - C_a z) \) (for some average values \( A_a, B_a \) and \( C_a \)).
- **Set-membership estimation / Interval observers.**

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Several cases may be met

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- Models with uncertain parameters (\textbf{constant} or \textbf{varying} uncertain parameters)
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Observers structures

- $\dot{x} = Ax + Bu; y = Cx$  
  $\Rightarrow$ Luenberger Obs. $\dot{z} = Az + Bu + L(y - Cz)$.
- $\dot{x} = A(\theta)x + B(\theta)u; y = C(\theta)x \Rightarrow \theta$ is known or unknown?

Possible solutions

- Adaptive approaches $\Rightarrow$ joint estimation of $x$ and $\theta$.
- Robust approaches $\dot{z} = A_a z + B_a u + L(y - C_a z)$ (for some average values $A_a, B_a$ and $C_a$).
- \textbf{Set-membership estimation / Interval observers.}
Set-membership estimation 1/5

- Without uncertainties ⇒ point estimation.
- Systems subject to bounded uncertainties ⇒ estimation of a feasible solution set.

✓ Prediction/correction approach

✓ Interval Observers
Set-membership estimation 1/5

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Set-membership estimation 1/5

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- Systems subject to bounded uncertainties $\Rightarrow$ estimation of a feasible solution set.

✓ Prediction/correction approach

✓ Interval Observers
Set-membership estimation 2/5

Illustrative example
Let’s consider the following example:

\[
\dot{x}(t) = -x^3(t) + d(t), \quad t \geq 0, \tag{1}
\]

- \(d(t) \in [-1, 1]\) is uncertain input whose values belong into the interval \([-1, 1]\).
- Assume that the admissible values for initial conditions of this system is the interval \([-2, 2]\), i.e. \(x(0) \in [-2, 2]\).
- The system is nonlinear, the input and initial conditions are uncertain \(\Rightarrow\) it is hard to evaluate an exact value of the state \(x(t)\) at each time \(t\).
- **However**, it is possible to evaluate the admissible values for \(x(t)\) with initial conditions \(x(0) \in [-2, 2]\) and \(d(t) \in [-1, 1]\).
**Set-membership estimation 3/5**

**Illustrative example**
The set of admissible values is described by:

\[
\begin{align*}
\dot{x}(t) &= -x^3(t) - 1, \quad x(0) = -2 \\
\ddot{x}(t) &= -\dot{x}^3(t) + 1, \quad \dot{x}(0) = 2
\end{align*}
\]

(2)

*Figure: Admissible set of the solutions of (1) computed through (2)*
Illustrative example

- As we can conclude, the interval \([\underline{x}(t), \overline{x}(t)]\) represents a worst case estimate of admissible values of \(x(t)\) for given uncertainties.

- The main problem considered in this survey is how to design an interval estimator like (2) using all available information, including the output measurements (not taken into account in this example) and minimizing the width of the interval \([\underline{x}(t), \overline{x}(t)]\), i.e. \(\overline{x}(t) - \underline{x}(t)\).

- Linear Time-Invariant, Linear Parameter-Varying, Continuous-time, discrete-time systems are considered.
Set-membership estimation 5/5

Given a system described by

\[
\begin{cases}
\dot{x} = f(x,u) \\
y = h(x)
\end{cases}
\]  

(3)

Definition 1

The dynamical system

\[
\begin{cases}
\dot{z} = \alpha(z,y,u) \\
[x^T, \bar{x}^T]^T = \beta(z,y,u)
\end{cases}
\]

is an interval observer for (3) if:

\[\underline{x}(0) \leq x(0) \leq \overline{x}(0) \Rightarrow -\infty < \underline{x}(t) \leq x(t) \leq \overline{x}(t) < \infty, \quad \forall t \geq 0.\]  

(5)

Roughly speaking, an interval observer should verify two conditions:

- Inclusion: \(\underline{x}(t) \leq x(t) \leq \overline{x}(t)\), \(\forall t \geq t_0\)
- Stability of \(e = x - \underline{x}\) and \(\bar{e} = \overline{x} - x\)
1 Introduction

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4 Fault Tolerant Control
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Cooperative systems - nonnegative systems

- Given two vectors \( x_1, x_2 \) and two matrices \( A_1, A_2 \), the relations \( x_1 \leq x_2, \ x_1 \geq x_2, \ A_1 \leq A_2, \ A_1 \geq A_2 \) should be understood elementwise.

- A positive semi-definite matrix is denoted by \( P : P = P^T \succeq 0 \).

- A square matrix \( S \in \mathbb{R}^{n \times n} \) is called Metzler if \( S_{i,j} \geq 0, \forall \ 1 \leq i \neq j \leq n \). The set of all Metzler matrices is denoted by \( \mathcal{M} \).

**Theorem 1**

*Given a Metzler matrix \( S \ (S \in \mathcal{M}) \), the system*

\[
\dot{z} = Sz + r(t); \quad z \in \mathbb{R}^n; \quad r : \mathbb{R}_+ \to \mathbb{R}_+^n
\]

*is called cooperative or nonnegative. Its trajectories verify:*

\[
z(0) \geq 0 \Rightarrow z(t) \geq 0, \forall t \geq 0.
\]
Metzler matrices properties

A matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz stable ($A \in \mathcal{H}$) if all its eigenvalues have negative real parts.

For a Metzler matrix $A \in \mathcal{M} \subseteq \mathbb{R}^{n \times n}$, the following properties are similar:

- $A \in \mathcal{H}$;
- $A^{-1} \leq 0$;
- there exists $P \in \mathbb{R}^{n \times n}$, $P = P^T \succ 0$ such that
  
  $$A^T P + PA \prec 0;$$

- there exists a diagonal matrix $D \in \mathbb{R}^{n \times n}$, $D \succ 0$ such that
  
  $$A^T D + DA \prec 0;$$

- there exists a vector $\rho \in \mathbb{R}^n$, $\rho > 0$ such that
  
  $$A^T \rho < 0$$
Somme additional properties

Define $A^+ = \max(0, A)$, $A^- = A^+ - A \Rightarrow$ the matrix of absolute values of all elements by $|A| = A^+ + A^-$.  

**Lemma 1**

Given vectors $\underline{x}, x, \overline{x} \in \mathbb{R}^n$, $\underline{x} \leq x \leq \overline{x}$ and a matrix $A \in \mathbb{R}^{m \times n}$, then

$$A^+ \underline{x} - A^- \overline{x} \leq Ax \leq A^+ \overline{x} - A^- \underline{x}. \quad (6)$$

For $A \preceq A \leq \overline{A}$, $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$, then

$$A^+ \underline{x}^+ - \overline{A}^+ \underline{x}^+ - \overline{A}^- \overline{x}^- + \overline{A}^- \underline{x}^- \leq Ax \leq \overline{A}^+ \underline{x}^+ - A^+ \overline{x}^- - \overline{A}^- \overline{x}^+ + A^- \underline{x}^- \quad (7)$$
Somme additional properties

Define \( A^+ = \max(0, A), \ A^- = A^+ - A \Rightarrow \) the matrix of absolute values of all elements by \(|A| = A^+ + A^-\).

**Lemma 1**

Given vectors \( \underline{x}, \bar{x}, \underline{x} \in \mathbb{R}^n, \underline{x} \leq x \leq \bar{x} \) and a matrix \( A \in \mathbb{R}^{m \times n} \), then

\[
A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}. \tag{6}
\]

For \( \underline{A} \leq A \leq \bar{A}, \ \underline{A}, \bar{A} \in \mathbb{R}^{m \times n} \), then

\[
A^+ \underline{x}^+ - \bar{A}^+ \underline{x}^- - A^- \bar{x}^+ + \bar{A}^- \underline{x}^- \leq Ax \leq \bar{A}^+ \underline{x}^+ - A^+ \bar{x}^- - \bar{A}^- \underline{x}^+ + A^- \bar{x}^- \tag{7}
\]
LTI continuous-time systems

Given a system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bd(t), \quad d : \mathbb{R}_+ \rightarrow \mathbb{R}_+^q \\
y(t) &= Cx(t) + Dd(t)
\end{align*}
\]

where \( A \) is a Metzler matrix, then the solution \( x(t) \) is elementwise nonnegative for all \( t \geq 0 \) provided that \( x(0) \geq 0 \) and \( B \in \mathbb{R}_+^{n \times q} \).

The stability of the nonnegative system (8) can be checked by verifying a Linear Programming (LP) problem

\[
A^T \lambda < 0
\]

for some nonnegative \( \lambda \in \mathbb{R}_+^n \), or equivalently a Lyapunov matrix equation

\[
A^T P + PA < 0
\]

with \( P \) a diagonal matrix.
Lemma 2\textsuperscript{a}

\textsuperscript{a}Refer for instance to

The nonnegative system (8) (i.e. $A$ is Metzler, $B \geq 0$, $C \geq 0$ and $D \geq 0$) is asymptotically stable if and only if there exist a nonnegative $\lambda \in \mathbb{R}_+^n$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:

$$
\begin{pmatrix}
A^T \lambda + C^T E_p \\
B^T \lambda - \gamma E_p + D^T E_p
\end{pmatrix} < 0.
$$

Moreover, in this case, the $L_1$ gain of the transfer $d \to y$ is lower than $\gamma$. 
LTI continuous-time systems

Lemma 3\textsuperscript{a}

\textsuperscript{a} Refer for instance to


The nonnegative system (8) (i.e. $A$ is Metzler, $B \geq 0$, $C \geq 0$ and $D \geq 0$) is asymptotically stable \textit{if and only if} there exist a nonnegative $\lambda \in \mathbb{R}^n_+$ and a scalar $\gamma > 0$ such that the following \textbf{LP problem} is feasible:

\[
\begin{pmatrix}
A\lambda + BE_q \\
C\lambda - \gamma E_p + DE_q
\end{pmatrix} < 0.
\]

(10)

Moreover, in this case, the $L_\infty$ gain of the transfer $d \rightarrow y$ is lower than $\gamma$. 
LTI Continuous-time systems

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + d(t) \\
y(t) &= Cx(t) + v(t)
\end{align*}
\]  
(11)

Assumption 1

Assume that \( \underline{x}_0 \leq x(0) \leq \overline{x}_0, \underline{d}(t) \leq d(t) \leq \overline{d}(t), -V \leq v(t) \leq V, \forall t \geq 0. \)

Interval observer structure:

\[
\begin{align*}
\dot{x}(t) &= Ax + L [y(t) - C\overline{x}] - |L|E_pV + \overline{d}(t) \\
\dot{\overline{x}}(t) &= A\overline{x} + L [y(t) - C\overline{x}] + |L|E_pV + \overline{d}(t)
\end{align*}
\]  
(12)

How to ensure

\[ \underline{x}(t) \leq x(t) \leq \overline{x}(t) \]
LTI Continuous-time systems

Theorem 2

Let Assumption 1 hold and $x \in \mathcal{L}_\infty^n$, then, the solutions of the systems (11) and (12) satisfy

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

provided that the matrix $A - LC$ is Metzler. In addition, if $A - LC$ is Hurwitz, then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$.

Proof:

Define the estimation errors $e(t) = x(t) - \underline{x}(t)$, $\overline{e}(t) = \overline{x}(t) - x(t)$

$$\dot{e}(t) = (A - LC)e(t) + Lv(t) + |L|E_pV + d(t) - \overline{d}(t),$$

$$\dot{\overline{e}}(t) = (A - LC)\overline{e}(t) - Lv(t) + |L|E_pV + \overline{d}(t) - d(t).$$

By assumption 1, $|L|E_pV \pm Lv(t) \geq 0$, $d(t) - \overline{d}(t) \geq 0$, $\overline{d}(t) - d(t) \geq 0 \Rightarrow$

The inputs of $e(t), \overline{e}(t)$ are nonnegative and $e(0) \geq 0, \overline{e}(0) \geq 0$.

If $A - LC$ is Metzler, then $e(t) \geq 0, \overline{e}(t) \geq 0, \forall t \geq 0$

If $A - LC$ is Hurwitz, then $e(t), \overline{e}(t) \in \mathcal{L}_\infty^n \Rightarrow \underline{x}(t), \overline{x}(t)$ are also bounded

(inputs of $e(t), \overline{e}(t)$ and $x$ are bounded).
LTI Continuous-time systems

Theorem 2\(^a\)


Let Assumption 1 hold and \(x \in L_\infty^n\), then, the solutions of the systems (11) and (12) satisfy

\[ \underline{x}(t) \leq x(t) \leq \overline{x}(t) \]

provided that the matrix \(A - LC\) is Metzler. In addition, if \(A - LC\) is Hurwitz, then \(\underline{x}, \overline{x} \in L_\infty^n\).

Proof:

Define the estimation errors \(\underline{e}(t) = x(t) - \underline{x}(t), \quad \overline{e}(t) = \overline{x}(t) - x(t)\)

\[ \dot{\underline{e}}(t) = (A - LC)\underline{e}(t) +Lv(t) + |L|E_pV + d(t) - \overline{d}(t), \]

\[ \dot{\overline{e}}(t) = (A - LC)\overline{e}(t) - Lv(t) + |L|E_pV + \underline{d}(t) - d(t). \]

By assumption 1, \(|L|E_pV \pm Lv(t) \geq 0\), \(d(t) - \underline{d}(t) \geq 0\), \(\overline{d}(t) - d(t) \geq 0\) \(\Rightarrow\)

The inputs of \(\underline{e}(t), \overline{e}(t)\) are nonnegative and \(\underline{e}(0) \geq 0, \overline{e}(0) \geq 0\).

If \(A - LC\) is Metzler, then \(\underline{e}(t) \geq 0, \overline{e}(t) \geq 0, \forall t \geq 0\)

If \(A - LC\) is Hurwitz, then \(\underline{e}(t), \overline{e}(t) \in L_\infty^n \Rightarrow x(t), \overline{x}(t)\) are also bounded (inputs of \(\underline{e}(t), \overline{e}(t)\) and \(x\) are bounded).
Theorem 2


Let Assumption 1 hold and \( x \in L^\infty_n \), then, the solutions of the systems (11) and (12) satisfy

\[ \underline{x}(t) \leq x(t) \leq \bar{x}(t) \]

provided that the matrix \( A - LC \) is Metzler. In addition, if \( A - LC \) is Hurwitz, then \( x, \bar{x} \in L^\infty_n \).

Proof:

Define the estimation errors \( e(t) = x(t) - \underline{x}(t), \bar{e}(t) = \bar{x}(t) - x(t) \)

\[ \dot{e}(t) = (A - LC)e(t) + Lv(t) + |L|E_pV + \bar{d}(t) - d(t), \]

\[ \dot{\bar{e}}(t) = (A - LC)\bar{e}(t) - Lv(t) + |L|E_pV + \bar{d}(t) - d(t). \]

By assumption 1, \( |L|E_pV \pm L \bar{v}(t) \geq 0 \), \( d(t) - \bar{d}(t) \geq 0 \), \( \bar{d}(t) - d(t) \geq 0 \) \( \Rightarrow \) The inputs of \( e(t), \bar{e}(t) \) are nonnegative and \( e(0) \geq 0, \bar{e}(0) \geq 0 \).

If \( A - LC \) is Metzler, then \( e(t) \geq 0, \bar{e}(t) \geq 0, \forall t \geq 0 \)

If \( A - LC \) is Hurwitz, then \( e(t), \bar{e}(t) \in L^\infty_n \) \( \Rightarrow \) \( x(t), \bar{x}(t) \) are also bounded (inputs of \( e(t), \bar{e}(t) \) and \( x \) are bounded).
LTI Continuous-time systems

**Theorem 2**

Let Assumption 1 hold and \( x \in \mathcal{L}^n_{\infty} \), then, the solutions of the systems (11) and (12) satisfy

\[
\underline{x}(t) \leq x(t) \leq \overline{x}(t)
\]

provided that the matrix \( A - LC \) is Metzler. In addition, if \( A - LC \) is Hurwitz, then \( \underline{x}, \overline{x} \in \mathcal{L}^n_{\infty} \).

**Proof:**

Define the estimation errors \( \underline{e}(t) = x(t) - \underline{x}(t) \), \( \overline{e}(t) = \overline{x}(t) - x(t) \)

\[
\dot{\underline{e}}(t) = (A - LC)\underline{e}(t) + Lv(t) + |L|E_p V + d(t) - \overline{d}(t), \\
\dot{\overline{e}}(t) = (A - LC)\overline{e}(t) - Lv(t) + |L|E_p V + \underline{d}(t) - \overline{d}(t).
\]

By assumption 1, \( |L|E_p V \pm Lv(t) \geq 0 \), \( d(t) - \overline{d}(t) \geq 0 \), \( \overline{d}(t) - \underline{d}(t) \geq 0 \) \( \Rightarrow \)

The inputs of \( \underline{e}(t), \overline{e}(t) \) are nonnegative and \( \underline{e}(0) \geq 0, \overline{e}(0) \geq 0 \).

If \( A - LC \) is Metzler, then \( \underline{e}(t) \geq 0, \overline{e}(t) \geq 0, \forall t \geq 0 \)

If \( A - LC \) is Hurwitz, then \( \underline{e}(t), \overline{e}(t) \in \mathcal{L}_{\infty}^n \) \( \Rightarrow \) \( \underline{x}(t), \overline{x}(t) \) are also bounded

(inputs of \( \underline{e}(t), \overline{e}(t) \) and \( x \) are bounded).
LTI Continuous-time systems

**Theorem 2**

Let Assumption 1 hold and $x \in \mathcal{L}_\infty^n$, then, the solutions of the systems (11) and (12) satisfy

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

provided that the matrix $A - LC$ is Metzler. In addition, if $A - LC$ is Hurwitz, then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$.

**Proof:**

Define the estimation errors $\underline{e}(t) = x(t) - \underline{x}(t), \quad \overline{e}(t) = \overline{x}(t) - x(t)$

\[
\dot{\underline{e}}(t) = (A - LC)\underline{e}(t) + Lv(t) + |L|E_PV + d(t) - \overline{d}(t), \\
\dot{\overline{e}}(t) = (A - LC)\overline{e}(t) - Lv(t) + |L|E_PV + \overline{d}(t) - d(t).
\]

By assumption 1, $|L|E_PV \pmLv(t) \geq 0, \quad d(t) - \overline{d}(t) \geq 0, \quad \overline{d}(t) - d(t) \geq 0 \Rightarrow$

The inputs of $\underline{e}(t), \overline{e}(t)$ are nonnegative and $\underline{e}(0) \geq 0, \overline{e}(0) \geq 0$.

If $A - LC$ is Metzler, then $\underline{e}(t) \geq 0, \overline{e}(t) \geq 0, \forall t \geq 0$

If $A - LC$ is Hurwitz, then $\underline{e}(t), \overline{e}(t) \in \mathcal{L}_\infty^n \Rightarrow \underline{x}(t), \overline{x}(t)$ are also bounded (inputs of $\underline{e}(t), \overline{e}(t)$ and $x$ are bounded).
LTI Continuous-time systems

**Optimal interval observers**

- If the observer gain $L$ is designed such that $A - LC$ is Metzler \( \Rightarrow x(t) \leq \underline{x}(t) \leq \overline{x}(t), \forall t \geq 0. \)
- If the observer gain $L$ is designed such that $A - LC$ is Hurwitz stable \( \Rightarrow \) the error $\overline{x}(t) - \underline{x}(t)$ is bounded.
- What about the width of $w([\underline{x}(t) - \overline{x}(t)]) = \overline{x}(t) - \underline{x}(t)$?

The $[\underline{x}(t), \overline{x}(t)]$ width optimization in the $L_1$ framework can be formulated as a Linear Programming problem:

**Theorem 3**

Consider the interval observer (12) for (11). If there exist a nonnegative vector $\lambda \in \mathbb{R}^n$, $W \in \mathbb{R}^n$ and a diagonal matrix $M \in \mathbb{R}^{n \times n}$ such that

\[
\begin{pmatrix}
A^T \lambda - C^T W + E_n \\
\lambda - \gamma E_n \\
A^T \lambda - C^T W + M \lambda
\end{pmatrix} \leq 0
\]

(13)

Then, $W = L^T \lambda$ and $d \rightarrow w(\overline{x} - \underline{x})$ has a $L_1$ gain lower than $\gamma$. 

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Optimal interval observers

- If the observer gain $L$ is designed such that $A - LC$ is Metzler $\Rightarrow \underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \geq 0$.
- If the observer gain $L$ is designed such that $A - LC$ is Hurwitz stable $\Rightarrow$ the error $\overline{x}(t) - \underline{x}(t)$ is bounded.
- What about the width of $w([\underline{x}(t) - \overline{x}(t)]) = \overline{x}(t) - \underline{x}(t)$?

The $[\underline{x}(t), \overline{x}(t)]$ width optimization in the $L_1$ framework can be formulated as a Linear Programming problem:

**Theorem 3**

Consider the interval observer (12) for (11). If there exist a nonnegative vector $\lambda \in \mathbb{R}^n$, $W \in \mathbb{R}^n$ and a diagonal matrix $M \in \mathbb{R}^{n \times n}$ such that

$$
\begin{bmatrix}
A^T \lambda - C^T W + E_n \\
\lambda - \gamma E_n \\
A^T \lambda - C^T W + M \lambda
\end{bmatrix} < 0
$$

$$
A^T \lambda - C^T W + M \lambda \geq 0
$$

$\lambda > 0, M \geq 0$

Then, $W = L^T \lambda$ and $d \rightarrow w(\overline{x} - x)$ has a $L_1$ gain lower than $\gamma$. 
Optimal interval observers

- If the observer gain $L$ is designed such that $A - LC$ is Metzler $\Rightarrow x(t) \leq \bar{x}(t) \leq \underline{x}(t), \forall t \geq 0$.
- If the observer gain $L$ is designed such that $A - LC$ is Hurwitz stable $\Rightarrow$ the error $\bar{x}(t) - \underline{x}(t)$ is bounded.
- What about the width of $w([\underline{x}(t), \bar{x}(t)]) = \bar{x}(t) - \underline{x}(t)$?

The $[\underline{x}(t), \bar{x}(t)]$ width optimization in the $L_1$ framework can be formulated as a Linear Programming problem:

**Theorem 3**

Consider the interval observer (12) for (11). If there exist a nonnegative vector $\lambda \in \mathbb{R}^n$, $W \in \mathbb{R}^n$ and a diagonal matrix $M \in \mathbb{R}^{n \times n}$ such that

$$
\begin{pmatrix}
A^T \lambda - C^T W + E_n \\
\lambda - \gamma E_n \\
A^T \lambda - C^T W + M \lambda
\end{pmatrix} < 0
\Rightarrow
\begin{align*}
A^T \lambda & - C^T W + E_n \\
\lambda & - \gamma E_n \\
A^T \lambda & - C^T W + M \lambda \\
\lambda & > 0, M \geq 0
\end{align*}
$$

Then, $W = L^T \lambda$ and $d \rightarrow w(\bar{x} - \underline{x})$ has a $L_1$ gain lower than $\gamma$. 
Optimal interval observers

- If the observer gain $L$ is designed such that $A - LC$ is Metzler $\Rightarrow$ $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \geq 0$.
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$$\begin{bmatrix} A^T \lambda - C^T W + E_n \\ \lambda - \gamma E_n \\ A^T \lambda - C^T W + M \lambda \end{bmatrix} \leq 0$$

Then, $W = L^T \lambda$ and $d \rightarrow w(\overline{x} - x)$ has a $L_1$ gain lower than $\gamma$. 

Tarek RAISSI
LTI Continuous-time systems

Optimal interval observers

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  \( \underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \geq 0 \).
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\]

Then, \( W = L^T \lambda \) and \( d \rightarrow w(\overline{x} - \underline{x}) \) has a \( L_1 \) gain lower than \( \gamma \).
LTI Continuous-time systems

- Theorem 3 provides an effective and simple computational tool to design interval observers for LTI systems.
- It gives only sufficient conditions \( \Rightarrow \) in some cases this LP problem may have no solution.
- The LP problem has no solution if it is not possible to find \( L \) such that \( A - LC \) is simultaneously Metzler and Hurwitz.

A counterexample: Given a system described by:

\[
\dot{x} = Ax + Bu, \quad y = Cx,
\]

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},
\]

This system is observable, whereas the matrix

\[
A - LC = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix}
\]

cannot be Hurwitz and Metzler!
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cannot be Hurwitz and Metzler!
The detectability of $(A, C)$ implies the existence of $L$ such that $A - LC$ is Hurwitz stable.

The Hurwitz property of matrices is preserved under similarity transformations of coordinates $\Rightarrow$

- To overcome this issue, it is possible to design the gain $L$ such that the matrix $A - LC$ is Hurwitz.
- Find a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ such that in the new coordinates $z = Sx$ the state matrix $D = S(A - LC)S^{-1}$ is Metzler.
The detectability of \((A, C)\) implies the existence of \(L\) such that \(A - LC\) is Hurwitz stable.

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1. To overcome this issue, it is possible to design the gain \(L\) such that the matrix \(A - LC\) is Hurwitz.
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1. To overcome this issue, it is possible to design the gain \(L\) such that the matrix \(A - LC\) is Hurwitz

2. Find a nonsingular matrix \(S \in \mathbb{R}^{n \times n}\) such that in the new coordinates \(z = Sx\) the state matrix \(D = S(A - LC)S^{-1}\) is Metzler.
Lemma 4

Given the matrices $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times n}$, and $C \in \mathbb{R}^{p \times n}$. If there exists a matrix $L \in \mathbb{R}^{n \times p}$ such that the matrices $A - LC$ and $D$ have the same eigenvalues, then there exists a matrix $S \in \mathbb{R}^{n \times n}$ such that $D = S(A - LC)S^{-1}$ provided that the pairs $(A - LC, e_1)$ and $(D, e_2)$ are observable for some $e_1 \in \mathbb{R}^{1 \times n}$, $e_2 \in \mathbb{R}^{1 \times n}$.
By adding and subtracting $Ly(t)$ to (11), the dynamics of this LTI system is described by:

$$\dot{x}(t) = (A - LC)x(t) + Ly(t) - Lv(t) + d(t).$$

(14)

Under conditions of Lemma 4 (slide 25), in the new coordinates $z = Sx$, the system (14) takes the form:

$$\dot{z}(t) = Dz(t) + SLy(t) + \delta(t), \quad \delta(t) = S[d(t) - Lv(t)].$$

(15)

Let

$$\left\{ \begin{array}{l}
\delta(t) = S^+d(t) - S^-\bar{d}(t) - |SL|E_pV \\
\bar{\delta}(t) = S^+\bar{d}(t) - S^-d(t) + |SL|E_pV.
\end{array} \right.$$ 

By using Lemma 1 (slide 16), we get

$$\underline{\delta}(t) \leq \delta(t) \leq \bar{\delta}(t).$$
LTI Continuous-time systems

Under the detectability property for the pair \((A, C)\), all the conditions of Theorem 2 in slide 21 are satisfied in the coordinates \(z = Sx\). An interval observer candidate is given by:

\[
\begin{align*}
\dot{z}(t) &= Dz(t) + SLy(t) + \delta(t) \\
\dot{\bar{z}}(t) &= D\bar{z}(t) + SLy(t) + \bar{\delta}(t)
\end{align*}
\]  

(16)

**Initial conditions:**

\[
\begin{align*}
\bar{z}(0) &= S^+x_0 - S^-\bar{x}_0 \\
\bar{\bar{z}}(0) &= S^+\bar{x}_0 - S^-\bar{x}_0
\end{align*}
\]

**Estimation in the original coordinates:** Let \(R = S^{-1}\)

\[
\begin{align*}
x(t) &= R^+z(t) - R^-\bar{z}(t) \\
\bar{x}(t) &= R^+\bar{z}(t) - R^-z(t)
\end{align*}
\]  

(17)
LTI Continuous-time systems

Remark:
If the eigenvalues of \((A - LC)\) are complex-valued, the change of coordinates \(z = Sx\) could be time-varying.

Lemma\textsuperscript{a}

\textsuperscript{a}F. Mazenc and O. Bernard, Interval observers for linear time-invariant systems with disturbances, Automatica, vol. 47, no. 1, pp. 140–147, 2011.

Let \(A - LC\) be Hurwitz, then there exists an invertible matrix function \(S : \mathbb{R} \to \mathbb{R}^{n \times n}\) of class \(C^\infty\) elementwise, \(\|S(t)\|_2 < +\infty\) for all \(t \in \mathbb{R}\), such that for all \(t \in \mathbb{R}\),

\[
\dot{S}(t) = DS(t) - S(t)(A - LC)
\]

where \(D \in \mathbb{R}^{n \times n}\) is a Hurwitz and Metzler matrix.

\(D\) can for instance be chosen as the Jordan canonical form of \(A - LC\).
LTI Continuous-time systems

Remark:

If the eigenvalues of \((A - LC')\) are complex-valued, the change of coordinates \(z = Sx\) could be time-varying.

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Let \(A - LC\) be Hurwitz, then there exists an invertible matrix function \(S : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}\) of class \(C^\infty\) elementwise, \(\|S(t)\|_2 < +\infty\) for all \(t \in \mathbb{R}\), such that for all \(t \in \mathbb{R}\),

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LTI Continuous-time systems

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Numerical example 1/2

\[ \dot{x} = Ax + B(p_1, p_2) f(x) u(t), \quad y = Cx, \]

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
1 & -4 & \sqrt{3} \\
-1 & -\sqrt{3} & -4
\end{bmatrix}, \quad B(p_1, p_2) = \begin{bmatrix}
-2p_1 \\
0 \\
p_2
\end{bmatrix},
\]

\[ C = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}, \]

\[ f(x) = x_1 x_2, \quad p_1 = 4.48, \quad \bar{p}_1 = 6.12, \quad p_2 = 3.2, \quad \bar{p}_2 = 3.6. \]

- The pair \((A, C)\) is not observable and there is no observer gain \(L\) such that the matrix \(A - LC\) is Metzler.
- Only one eigenvalue can be assigned with the gain \(L\).
- The matrix

\[
D = \begin{bmatrix}
-a & b & 0 \\
0 & -a & b \\
b & 0 & -a
\end{bmatrix}
\]

has the following eigenvalues \(b - a, -a - 0.5b \pm 0.5b\sqrt{3}i\) (we take here \(b = 2\) and \(a = 3\)).
Numerical example 2/2

- For $L = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T$, the matrix $A - LC \in \mathcal{H}$ and its eigenvalues are $-1, -4 \pm \sqrt{3}i$.

- The pairs $(A - LC, e_1)$ and $(R, e_2)$ are observable for $e_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, then

$$S = O_2^{-1}O_1 = \begin{bmatrix} 0.158 & 0.866 & 0.5 \\ 0.842 & -0.866 & 0.5 \\ 0.658 & 0 & -1 \end{bmatrix}.$$
1 Introduction

2 Interval observers - LTI systems
   - Continuous-time systems
   - Discrete-time systems

3 Joint state and unknown input estimation
   - State estimation
   - Upper and lower bounds of the unknown input

4 Fault Tolerant Control
   - Problem statement
   - Interval observer design
   - Control design
   - Numerical simulations
Notations and definitions

- A matrix $A \in \mathbb{R}^{n \times n}$ is called Schur stable if its spectral radius is less than one.
- A matrix $A \in \mathbb{R}^{n \times n}$ is called nonnegative if all its elements are nonnegative.

**Definition: nonnegative systems**

Consider the linear system

$$x(k+1) = Ax(k) + \omega(k) \quad (18)$$

where $\omega \in \mathbb{R}^n$ and $A$ is a nonnegative matrix.

- $\forall k \geq 0$, $x(k) \geq 0$ provided that $x(0) \geq 0$.
- Such dynamical systems are called nonnegative (or cooperative).
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**Definition: nonnegative systems**

Consider the linear system

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(18)

where $\omega \in \mathbb{R}^n_+$ and $A$ is a nonnegative matrix.

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Such dynamical systems are called nonnegative (or cooperative).
LTI discrete-time systems: observer design

- Consider the following system:

\[
\begin{align*}
    x(t + 1) &= Ax(t) + d(t), \\
    y(t) &= Cx(t) + v(t),
\end{align*}
\tag{19}
\]

- **Assumption:**
  Let \(x(0) \in [x_0, \overline{x}_0]\), two functions \(\underline{d}, \overline{d} \in \mathcal{L}_\infty^n\) and a constant \(V > 0\) such that

\[
\underline{d}(t) \leq d(t) \leq \overline{d}(t), \quad |v(t)| \leq V, \forall t \in \mathbb{Z}_+.
\]

- **Interval observer structure:**

\[
\begin{align*}
    x(t + 1) &= Ax(t) + L(y(t) - Cx(t)) - |L|E_p V + \underline{d}(t) \\
    \overline{x}(t + 1) &= A\overline{x}(t) + L(y(t) - C\overline{x}(t)) + |L|E_p V + \overline{d}(t) \\
    \underline{x}(0) &= x_0, \quad \overline{x}(0) = \overline{x}_0.
\end{align*}
\tag{20}
LTI discrete-time systems: observer design

**Theorem 4**


Let the assumption given above hold and \( x \in \mathcal{L}_\infty^n \), the solutions of (19) and (20) satisfy

\[
\underline{x}(t) \leq x(t) \leq \bar{x}(t), \forall t \in \mathbb{Z}_+
\]

provided that the matrix \( A - LC \) is nonnegative. In addition, \( \underline{x} \) and \( \bar{x} \in \mathcal{L}_\infty^n \) if \( A - LC \) is Schur stable.

**Proof sketch:**

The estimation errors dynamics \( e(t) = x(t) - \underline{x}(t) \) and \( \bar{e}(t) = \bar{x}(t) - x(t) \) follow the dynamics:

\[
\begin{cases}
    e(t+1) = (A - LC)e(t) + d(t) - \bar{d}(t) + |L|E_p V - Lv(t) \\
    \bar{e}(t+1) = (A - LC)\bar{e}(t) + \bar{d}(t) - d(t) + |L|E_p V + Lv(t)
\end{cases}
\]  

(21)

- The relation \( x(t) \leq x(t) \leq \bar{x}(t) \), \( \forall t \in \mathbb{Z}_+ \) is ensured based on the assumption of slide 33 and nonnegativity of \( (A - LC) \).
- The stability is ensured if \( (A - LC) \) is Schur stable.
LTI discrete-time systems: observer design

Theorem 4

Let the assumption given above hold and $x \in \mathcal{L}_\infty^n$, the solutions of (19) and (20) satisfy

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \in \mathbb{Z}_+$$

provided that the matrix $A - LC$ is nonnegative. In addition, $\underline{x}$ and $\overline{x} \in \mathcal{L}_\infty^n$ if $A - LC$ is Schur stable.

Proof sketch:
The estimation errors dynamics $\underline{e}(t) = x(t) - \underline{x}(t)$ and $\overline{e}(t) = \overline{x}(t) - x(t)$ follow the dynamics:

$$\begin{cases} 
\underline{e}(t+1) = (A - LC)\underline{e}(t) + d(t) - \underline{d}(t) + |L|E_p V - Lv(t) \\
\overline{e}(t+1) = (A - LC)\overline{e}(t) + \overline{d}(t) - d(t) + |L|E_p V + Lv(t)
\end{cases}$$

The relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \in \mathbb{Z}_+$ is ensured based on the assumption of slide 33 and nonnegativity of $(A - LC)$.

The stability is ensured if $(A - LC)$ is Schur stable.
LTI discrete-time systems: observer design

The observer gain $L$ can be computed as a solution of the following Linear Matrix Inequality (LMI):

\[
\begin{align*}
\begin{bmatrix}
P & PA - WC \\
A^T P - C^T W^T & P
\end{bmatrix} & \succeq 0 \\
PA - WC & \succeq 0 \\
P & = P^T \succ 0
\end{align*}
\] (22)

- The diagonal matrix $P \in \mathbb{R}^{n \times n}$ and $W \in \mathbb{R}^{n \times p}$ are the variables to determine $\Rightarrow$ then $L = P^{-1}W$.

- A gain optimization problem (similar to the continuous-time case) can also be formulated to find $L$ providing a minimal interval width $\bar{x}(t) - \underline{x}(t)$ with respect to a chosen norm.
LTI discrete-time systems: observer design

**Restrictive condition:** existence of $L$ such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

Let $\text{assumption of slide 33 be verified and } A - LC$ is Schur stable for a gain $L$. Given a matrix $R \in \mathbb{R}^{n \times n}$ and $e_1 \in \mathbb{R}^{1 \times n}$, $e_2 \in \mathbb{R}^{1 \times n}$ such that $\lambda(A - LC) = \lambda(R)$ and the pairs $(A - LC, e_1)$, $(R, e_2)$ are observable. Then, an interval estimation for (19) is given by:

\[
\begin{align*}
    z(t+1) &= Rz(t) + Fy(t) - |F|EpV + S^+d(t) - S^-d(t) \\
    z(t) &= Rz(t) + Fy(t) + |F|EpV + S^+d(t) - S^-d(t)
\end{align*}
\]

where $S = O_{A - LC}^1$ and $O_{A - LC}$ and $O_{R}$ are the observability matrices of the pairs $(A - LC, e_1)$, $(R, e_2)$ and $P = PA$. 

---

Theorem 5 by D. Efimov, W. Perruquetti, T. Raissi, and A. Zolghadri. On interval observer design for time-invariant discrete-time systems, European Control Conference (Zurich), 2013.
LTI discrete-time systems: observer design

Restrictive condition: existence of $L$ such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

Theorem 5\(^a\)

\(^a\)D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, On interval observer design for time-invariant discrete-time systems, European Control Conference (Zurich), 2013.

Let the assumption of slide 33 be verified and $A - LC$ is Schur stable for a gain $L$. Given a matrix $R \in \mathbb{R}^{n \times n}$ and $e_1 \in \mathbb{R}^{1 \times n}$, $e_2 \in \mathbb{R}^{1 \times n}$ such that $\lambda (A - LC) = \lambda (R)$ and the pairs $(A - LC, e_1)$, $(R, e_2)$ are observable. Then, an interval estimation for (19) is given by:

\[
\begin{align*}
\begin{cases}
\dot{z}(t+1) &= Rz(t) + Fy(t) - |F|E_p V + S^+ e(t) - S^- e(t) \\
\dot{z}(t+1) &= Rz(t) + Fy(t) + |F|E_p V + S^+ e(t) - S^- e(t)
\end{cases}
\end{align*}
\]

where $S = O_{A - LC} O^{-1}$ ($O_{A - LC}$ and $O_R$ are the observability matrices of the pairs $(A - LC, e_1)$, $(R, e_2)$) and $F = SL$.\[z_0 = S^+ x_0 - S^- x_0$

$\bar{x}_0 = S^+ x_0 - S^- x_0$
LTI discrete-time systems: observer design

**Restrictive condition:** existence of $L$ such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

---

**Theorem 5$^a$**

---

$^a$D. Efimov, W. Perruquet, T. Raïssi, and A. Zolghadri, On interval observer design for time-invariant discrete-time systems, European Control Conference (Zurich), 2013.

Let the assumption of slide 33 be verified and $A - LC$ is Schur stable for a gain $L$. Given a matrix $R \in \mathbb{R}^{n \times n}$ and $e_1 \in \mathbb{R}^{1 \times n}$, $e_2 \in \mathbb{R}^{1 \times n}$ such that $\lambda(A - LC) = \lambda(R)$ and the pairs $(A - LC, e_1), (R, e_2)$ are observable. Then, an interval estimation for (19) is given by:

\[
\begin{align*}
    z(t + 1) &= Rz(t) + Fy(t) - |F|EpV + S^+d(t) - S^-\bar{d}(t) \\
    \bar{z}(t + 1) &= R\bar{z}(t) + Fy(t) + |F|EpV + S^+\bar{d}(t) - S^-d(t)
\end{align*}
\]

\[
\begin{align*}
    \begin{cases}
        z_0 = S^+x_0 - S^-\bar{x}_0 \\
        \bar{z}_0 = S^+\bar{x}_0 - S^-x_0
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    \begin{cases}
        \bar{x}(t) = (S^{-1})^+\bar{z}(t) - (S^{-1})^-\bar{z}(t), \\
        \bar{x}(t) = (S^{-1})^+\bar{z}(t) - (S^{-1})^-\bar{z}(t)
    \end{cases}
\end{align*}
\]

where $S = O_{A-LC}O^{-1}$ ($O_{A-LC}$ and $O_R$ are the observability matrices of the pairs $(A - LC, e_1)$, $(R, e_2)$) and $F = SL$. 
LTI discrete-time systems: observer design

Restrictive condition: existence of $L$ such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

Theorem 5


Let the assumption of slide 33 be verified and $A - LC$ is Schur stable for a gain $L$. Given a matrix $R \in \mathbb{R}^{n \times n}$ and $e_{1} \in \mathbb{R}^{1 \times n}$, $e_{2} \in \mathbb{R}^{1 \times n}$ such that $\lambda(A - LC) = \lambda(R)$ and the pairs $(A - LC, e_{1})$, $(R, e_{2})$ are observable. Then, an interval estimation for (19) is given by:

\[
\begin{align*}
\underline{z}(t+1) &= R\underline{z}(t) + Fy(t) - |F|E_{p}V + S^{+}\underline{d}(t) - S^{-}\overline{d}(t) \\
\overline{z}(t+1) &= R\overline{z}(t) + Fy(t) + |F|E_{p}V + S^{+}\overline{d}(t) - S^{-}\overline{d}(t)
\end{align*}
\]

\[
\begin{align*}
\underline{x}_{0} &= S^{+}\underline{x}_{0} - S^{-}\overline{x}_{0} \\
\overline{x}_{0} &= S^{+}\overline{x}_{0} - S^{-}\underline{x}_{0} \\
\underline{x}(t) &= (S^{-1})^{+}\underline{z}(t) - (S^{-1})^{-}\overline{z}(t), \\
\overline{x}(t) &= (S^{-1})^{+}\overline{z}(t) - (S^{-1})^{-}\underline{z}(t)
\end{align*}
\]

where $S = O_{A - LC}O_{A - LC}^{-1}$ ($O_{A - LC}$ and $O_{R}$ are the observability matrices of the pairs $(A - LC, e_{1})$, $(R, e_{2})$) and $F = SL$. 

---

Tarek RAISII
LTI discrete-time systems: observer design

**Restrictive condition:** existence of $L$ such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

**Theorem 5**

$^a$D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, On interval observer design for time-invariant discrete-time systems, European Control Conference (Zurich), 2013.

Let the assumption of slide 33 be verified and $A - LC$ is Schur stable for a gain $L$. Given a matrix $R \in \mathbb{R}^{n \times n}$ and $e_1 \in \mathbb{R}^{1 \times n}$, $e_2 \in \mathbb{R}^{1 \times n}$ such that $\lambda(A - LC) = \lambda(R)$ and the pairs $(A - LC, e_1)$, $(R, e_2)$ are observable. Then, an interval estimation for (19) is given by:

$$
\begin{align*}
\zeta(t+1) &= R\zeta(t) + F y(t) - |F| E_p V + S^+d(t) - S^-\bar{d}(t) \\
\bar{z}(t+1) &= R\bar{z}(t) + F y(t) + |F| E_p V + S^+\bar{d}(t) - S^-d(t)
\end{align*}
$$

$$
\begin{align*}
\left\{ \begin{array}{l}
\zeta_0 = S^+ x_0 - S^- x_0 \\
\bar{z}_0 = S^+ \bar{x}_0 - S^- \bar{x}_0
\end{array} \right.
\end{align*}
$$

$$
\begin{align*}
\begin{array}{l}
\bar{x}(t) = (S^{-1})^+\zeta(t) - (S^{-1})^-\bar{z}(t), \\
\bar{x}(t) = (S^{-1})^+\bar{z}(t) - (S^{-1})^-\zeta(t)
\end{array}
\end{align*}
$$

where $S = O_{A-LC}O^{-1}$ ($O_{A-LC}$ and $O_R$ are the observability matrices of the pairs $(A - LC, e_1)$, $(R, e_2)$) and $F = SL$. 

Tarek RAIISSI
LTI discrete-time systems: observer design

**Restrictive condition:** existence of $L$ such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

**Theorem 5**


Let the assumption of slide 33 be verified and $A - LC$ is Schur stable for a gain $L$. Given a matrix $R \in \mathbb{R}^{n \times n}$ and $e_1 \in \mathbb{R}^{1 \times n}$, $e_2 \in \mathbb{R}^{1 \times n}$ such that $\lambda(A - LC) = \lambda(R)$ and the pairs $(A - LC, e_1)$, $(R, e_2)$ are observable. Then, an interval estimation for (19) is given by:

\[
\begin{align*}
\underline{z}(t+1) &= R\underline{z}(t) + Fy(t) - |F|E_p V + S^+ \underline{d}(t) - S^- \overline{d}(t) \\
\overline{z}(t+1) &= R\overline{z}(t) + Fy(t) + |F|E_p V + S^+ \overline{d}(t) - S^- \underline{d}(t)
\end{align*}
\]

\[
\begin{align*}
\underline{x}_0 &= S^+ \underline{x}_0 - S^- \overline{x}_0 \\
\overline{x}_0 &= S^+ \overline{x}_0 - S^- \underline{x}_0
\end{align*}
\]

\[
\begin{align*}
\underline{x}(t) &= (S^{-1})^+ \underline{z}(t) - (S^{-1})^- \overline{z}(t), \\
\overline{x}(t) &= (S^{-1})^+ \overline{z}(t) - (S^{-1})^- \underline{z}(t)
\end{align*}
\]

where $S = O_{A - LC} O^{-1}$ ($O_{A - LC}$ and $O_R$ are the observability matrices of the pairs $(A - LC, e_1)$, $(R, e_2)$) and $F = SL$. 

Tarek RAÏSSI
Numerical example:

\[
\begin{align*}
  x(t + 1) &= \begin{pmatrix} 0.3 & -0.7 \\ 0.6 & -0.5 \end{pmatrix} x(t) + \begin{pmatrix} \sin(0.1t) \\ \cos(0.2t) \end{pmatrix} + 0.5 \begin{pmatrix} \sin(0.5tx_2(t)) \\ \sin(0.3t) \end{pmatrix} \\
  y(t) &= (1 \quad 0) x(t) + 0.1 \sin(t)
\end{align*}
\]

Let \( L = (-0.8000 \quad -0.7000)^T \)

and \( D = A - LC = \begin{pmatrix} 0.3 & 0.1 \\ 0.6 & 0.2 \end{pmatrix} \).

\( D \) is a nonnegative matrix \( \Rightarrow \) no need of a change of coordinates.

The nonlinear term is bounded \( \Rightarrow \) it can be considered as a disturbance.
Numerical example:
The simulations are performed using the interval observer given in the slide 36.
1 Introduction

2 Interval observers - LTI systems

3 Joint state and unknown input estimation
   - State estimation
   - Upper and lower bounds of the unknown input

4 Fault Tolerant Control
Interval observers and linear systems with unknown inputs

- LTI discrete-time system with unknown inputs:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\
    y(k) &= Cx(k) + \delta(k)
\end{align*}
\]  

(23)

Is it possible to estimate $x$ and $d$? ⇒ Such estimation can be useful for instance for diagnosis and Fault Tolerant Control.

For more details, refer to:


Interval observers and linear systems with unknown inputs

- LTI discrete-time system with unknown inputs:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\
    y(k) &= Cx(k) + \delta(k)
\end{align*}
\]

(23)

Is it possible to estimate \(x\) and \(d\)? \(\Rightarrow\) Such estimation can be useful for instance for diagnosis and Fault Tolerant Control.

For more details, refer to:


Methodology to jointly estimate the bounds of $x$ and $d$

1. Change of coordinates to divide the system (23) into two subsytems:
   - one affected by the unknown input
   - the second one is unknown input-free

2. Change of coordinates to ensure the nonnegativity property of the observation error in the new coordinates

3. Design of an interval observer in the new basis to compute $\underline{x}$ and $\overline{x}$.

4. Compute $\underline{d}$ and $\overline{d}$. 
1 Introduction

2 Interval observers - LTI systems
   - Continuous-time systems
   - Discrete-time systems

3 Joint state and unknown input estimation
   - State estimation
   - Upper and lower bounds of the unknown input

4 Fault Tolerant Control
   - Problem statement
   - Interval observer design
   - Control design
   - Numerical simulations
Step 1: State and unknown input decoupling

LTI discrete-time system with unknown inputs:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\
y(k) &= Cx(k) + \delta(k)
\end{align*}
\]  

(24)

Assumption 2

\( C \) is a full row rank matrix and \( D \) is a full column rank matrix.

- There exist matrices \( H \in \mathbb{R}^{n \times n} \), \( R_0 \in \mathbb{R}^{q \times q} \) and \( K \in \mathbb{R}^{q \times q} \) such that:

\[
    D = H \begin{bmatrix} R_0 \\ 0 \end{bmatrix} K^T
\]  

(25)
Step 1: State and unknown input decoupling

LTI discrete-time system with unknown inputs:

\[
\begin{align*}
    x(k + 1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\
    y(k) &= Cx(k) + \delta(k)
\end{align*}
\]  

(24)

Assumption 2

\(C\) is a full row rank matrix and \(D\) is a full column rank matrix.

There exist matrices \(H \in \mathbb{R}^{n \times n}\), \(R_0 \in \mathbb{R}^{q \times q}\) and \(K \in \mathbb{R}^{q \times q}\) such that:

\[
D = H \begin{bmatrix} R_0 \\ 0 \end{bmatrix} K^T
\]  

(25)
Step 1: State and unknown input decoupling

LTI discrete-time system with unknown inputs:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\
    y(k) &= Cx(k) + \delta(k)
\end{align*}
\]  

(24)

Assumption 2

\( C \) is a full row rank matrix and \( D \) is a full column rank matrix.

- There exist matrices \( H \in \mathbb{R}^{n \times n} \), \( R_0 \in \mathbb{R}^{q \times q} \) and \( K \in \mathbb{R}^{q \times q} \) such that:

\[
D = H \begin{bmatrix} R_0 \\ 0 \end{bmatrix} K^T
\]  

(25)
Step 1: obtain an unknown input-free system

Transformation of the initial system into an equivalent one: Let

\[ z(k) = H^T x(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \]

\[
\begin{cases}
  z(k+1) = \tilde{A}z(k) + \tilde{B}u(k) + \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tilde{d}(k) + \tilde{\omega}(k) \\
  y(k) = \tilde{C}z(k) + \delta(k)
\end{cases}
\]  

(26)

where:

\[ H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \tilde{A} = H^T A H = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \]

\[ \tilde{B} = H^T B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = CH = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \tilde{d}(k) = K^T d(k) \tilde{\omega}(k) = H^T \omega = \begin{bmatrix} \tilde{\omega}_1(k) \\ \tilde{\omega}_2(k) \end{bmatrix} \]

\[ H^T \] is supposed to be bounded, therefore \(|\tilde{\omega}| \leq \bar{\omega}\).
Step 1: obtain an unknown input-free system

- Transformation of the initial system into an equivalent one: Let
\[ z(k) = H^T x(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \]

\[
\begin{align*}
  z(k + 1) &= \tilde{A}z(k) + \tilde{B}u(k) + \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tilde{d}(k) + \tilde{\omega}(k) \\
  y(k) &= \tilde{C}z(k) + \delta(k)
\end{align*}
\]

(26)

where:
\[
H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \tilde{A} = H^T A H = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}
\]
\[
\tilde{B} = H^T B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = C H = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}
\]
\[
\tilde{d}(k) = K^T d(k) \tilde{\omega}(k) = H^T \omega = \begin{bmatrix} \tilde{\omega}_1(k) \\ \tilde{\omega}_2(k) \end{bmatrix}
\]

- $H^T$ is supposed to be bounded, therefore $|\tilde{\omega}| \leq \overline{\omega}$. 

Tarek RAISSI
Step 1: obtain an unknown input-free system

\[
\begin{aligned}
    z(k+1) &= \tilde{A}z(k) + \tilde{B}u(k) + \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tilde{d}(k) + \tilde{\omega}(k) \\
y(k) &= \tilde{C}z(k) + \delta(k)
\end{aligned}
\]  

(27)

The system (27) is decomposed into an unknown input depending subsystem and an unknown input-free subsystem:

\[
\begin{aligned}
    z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\
z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\
y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k)
\end{aligned}
\]  

(28)
Step 1: obtain an unknown input-free system

\[
\begin{align*}
    z_1(k + 1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\
    z_2(k + 1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\
    y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k)
\end{align*}
\]

(29)

Transformation of (29) into a conventional linear system:

- \(\tilde{C}_1\) is supposed to be a full column rank matrix and can be decomposed as:
  \[
  \tilde{C}_1 = H_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix} K_1^T
  \]  
  \text{(30)}
  
  with \(H_1 = \begin{bmatrix} H_{011} & H_{012} \end{bmatrix} (H_{011} \in \mathbb{R}^{p \times q} \text{ and } H_{012} \in \mathbb{R}^{p \times (p-q)})\).

- Measurements equation can be decomposed as \(\tilde{y}(k) = H_1^T y(k)\)

\[
\begin{align*}
    \tilde{y}_1(k) &= R_1K_1^T z_1(k) + H_{011}^T \tilde{C}_2z_2(k) + H_{011}^T \delta(k) \\
    \tilde{y}_2(k) &= H_{012}^T \tilde{C}_2z_2(k) + H_{012}^T \delta(k) = C_2z_2(k) + H_{012}^T \delta(k)
\end{align*}
\]

(31)
Step 1: obtain an unknown input-free system

\[
\begin{align*}
    z_1(k+1) &= \tilde{A}_{11} z_1(k) + \tilde{A}_{12} z_2(k) + \tilde{B}_1 u(k) + R_0 \tilde{d}(k) + \tilde{\omega}_1(k) \\
    z_2(k+1) &= \tilde{A}_{21} z_1(k) + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \\
    y(k) &= \tilde{C}_1 z_1(k) + \tilde{C}_2 z_2(k) + \delta(k)
\end{align*}
\]

(29)

Transformation of (29) into a conventional linear system:

- \( \tilde{C}_1 \) is supposed to be a full column rank matrix and can be decomposed as:
  \[
  \tilde{C}_1 = H_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix} K_1^T
  \]  
  (30)

  with \( H_1 = \begin{bmatrix} H_{011} & H_{012} \end{bmatrix} \) (\( H_{011} \in \mathbb{R}^{p \times q} \) and \( H_{012} \in \mathbb{R}^{p \times (p-q)} \)).

- Measurements equation can be decomposed as \( \tilde{y}(k) = H_1^T y(k) \)

\[
\begin{align*}
    \tilde{y}_1(k) &= R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\
    \tilde{y}_2(k) &= H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k)
\end{align*}
\]  

(31)
Step 1: obtain an unknown input-free system

\[
\begin{align*}
\begin{cases}
    z_1(k+1) &= \tilde{A}_{11} z_1(k) + \tilde{A}_{12} z_2(k) + \tilde{B}_1 u(k) + R_0 \tilde{d}(k) + \tilde{\omega}_1(k) \\
    z_2(k+1) &= \tilde{A}_{21} z_1(k) + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \\
    y(k) &= \tilde{C}_1 z_1(k) + \tilde{C}_2 z_2(k) + \delta(k)
\end{cases}
\end{align*}
\]  

(29)

Transformation of (29) into a conventional linear system:

- \( \tilde{C}_1 \) is supposed to be a full column rank matrix and can be decomposed as:

\[
\tilde{C}_1 = H_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix} K_1^T
\]

(30)

with \( H_1 = \begin{bmatrix} H_{011} & H_{012} \end{bmatrix} \) (\( H_{011} \in \mathbb{R}^{p \times q} \) and \( H_{012} \in \mathbb{R}^{p \times (p-q)} \)).

- Measurements equation can be decomposed as \( \tilde{y}(k) = H_1^T y(k) \)

\[
\begin{align*}
\begin{cases}
    \tilde{y}_1(k) &= R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\
    \tilde{y}_2(k) &= H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k)
\end{cases}
\end{align*}
\]

(31)
Step 1: obtain an unknown input-free system

- Measurements equation can be decomposed as

\[
\begin{align*}
\tilde{y}_1(k) &= R_1 K_{11}^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\
\tilde{y}_2(k) &= H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k)
\end{align*}
\] (32)

- As \( \tilde{y}_1(k) = G_s^T \tilde{y}(k) \) with \( G_s^T = [I_q \quad O_{q \times (p-q)}] \), the expression of \( z_1 \) is extracted from the first equation of (32):

\[
z_1(k) = E (y(k) - \tilde{C}_2 z_2(k) - \delta(k))
\] (33)

with \( E = K_1 R_1^{-1} G_s^T H_1^T \).

- By replacing (33) in the dynamics of \( z_2 \) in (29) we obtain:

\[
z_2(k+1) = \tilde{A}_2 E [y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_2 z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k)
\] (34)
Step 1: obtain an unknown input-free system

• Measurements equation can be decomposed as

\[
\begin{aligned}
\tilde{y}_1(k) &= R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\
\tilde{y}_2(k) &= H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k)
\end{aligned}
\]  

(32)

• As \( \tilde{y}_1(k) = G_s^T \tilde{y}(k) \) with \( G_s^T = [I_q \ O_{q \times (p-q)}] \), the expression of \( z_1 \) is extracted from the first equation of (32):

\[
z_1(k) = E(y(k) - \tilde{C}_2 z_2(k) - \delta(k))
\]

(33)

with \( E = K_1 R_1^{-1} G_s^T H_1^T \).

• By replacing (33) in the dynamics of \( z_2 \) in (29) we obtain:

\[
z_2(k+1) = \tilde{A}_21 E[y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_22 z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k)
\]

(34)
Step 1: obtain an unknown input-free system

- Measurements equation can be decomposed as

\[
\begin{align*}
\tilde{y}_1(k) &= R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\
\tilde{y}_2(k) &= H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k)
\end{align*}
\]  

(32)

- As \( \tilde{y}_1(k) = G_s^T \tilde{y}(k) \) with \( G_s^T = \begin{bmatrix} I_q & O_{q \times (p-q)} \end{bmatrix} \), the expression of \( z_1 \) is extracted from the first equation of (32):

\[
z_1(k) = E(y(k) - \tilde{C}_2 z_2(k) - \delta(k))
\]  

(33)

with \( E = K_1 R_1^{-1} G_s^T H_1^T \).

- By replacing (33) in the dynamics of \( z_2 \) in (29) we obtain:

\[
z_2(k+1) = \tilde{A}_{21} E[y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k)
\]  

(34)
Step 1: obtain an unknown input-free system

Finally we obtain the following unknown input-free LTI discrete-time system:

\[
\begin{align*}
    z_2(k+1) &= A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\
    \tilde{y}_2(k) &= C_2 z_2(k) + H_{012}^T \delta(k)
\end{align*}
\]  \hspace{1cm} (35)

where \( A_2 = \tilde{A}_{22} - \tilde{A}_{21} E \tilde{C}_2 \), \( B_2 = \tilde{B}_2 \), \( C_2 = H_{012}^T \tilde{C}_2 \) and \( D_2 = \tilde{A}_{21} E \).
Step 2: Nonnegativity of the observation error in the new coordinates

- **Unknown input-free LTI discrete-time system:**

\[
\begin{align*}
\dot{z}_2(k+1) &= A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{w}_2(k) \\
\tilde{y}_2(k) &= C_2 z_2(k) + H_{012}^T \delta(k)
\end{align*}
\]

(36)

**Assumption**

- The pair \((A_2, C_2)\) is detectable.
- There exists a gain \(L \in \mathbb{R}^{(n-q) \times (p-q)}\) and a matrix \(P\) such that \((A_2 - LC_2)\) is Schur stable and \(R = P(A_2 - LC_2)P^{-1}\) is nonnegative.

After the change of coordinates \(r_2 = P z_2\), the system (36) is described in the new coordinates by:

\[
\begin{align*}
\dot{r}_2(k+1) &= R r_2(k) + PB_2 u(k) + My(k) - M \delta(k) + P \tilde{w}_2(k) \\
\tilde{y}_2(k) &= C_2 P^{-1} r_2(k) + H_{012}^T \delta(k)
\end{align*}
\]
Step 2: Nonnegativity of the observation error in the new coordinates

- Unknown input-free LTI discrete-time system:

\[
\begin{align*}
    z_2(k + 1) &= A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\
    \tilde{y}_2(k) &= C_2 z_2(k) + H_{012}^T \delta(k)
\end{align*}
\]

(36)

Assumption

- The pair \((A_2, C_2)\) is detectable.
- There exists a gain \(L \in \mathbb{R}^{(n-q) \times (p-q)}\) and a matrix \(P\) such that \((A_2 - LC_2)\) is Schur stable and \(R = P(A_2 - LC_2)P^{-1}\) is nonnegative.

After the change of coordinates \(r_2 = Pz_2\), the system (36) is described in the new coordinates by:

\[
\begin{align*}
    r_2(k + 1) &= R r_2(k) + PB_2 u(k) + M y(k) - M \delta(k) + P \tilde{\omega}_2(k) \\
    \tilde{y}_2(k) &= C_2 P^{-1} r_2(k) + H_{012}^T \delta(k)
\end{align*}
\]
Step 2: Nonnegativity of the observation error in the new coordinates

- Unknown input-free LTI discrete-time system:

\[
\begin{align*}
\dot{z}_2(k + 1) &= A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{w}_2(k) \\
\tilde{y}_2(k) &= C_2 z_2(k) + H^T_{012} \delta(k)
\end{align*}
\]

(36)

Assumption

- The pair \((A_2, C_2)\) is detectable.
- There exists a gain \(L \in \mathbb{R}^{(n-q) \times (p-q)}\) and a matrix \(P\) such that \((A_2 - LC_2)\) is Schur stable and \(R = P(A_2 - LC_2)P^{-1}\) is nonnegative.

After the change of coordinates \(r_2 = Pz_2\), the system (36) is described in the new coordinates by:

\[
\begin{align*}
\dot{r}_2(k + 1) &= Rr_2(k) + PB_2 u(k) + My(k) - M\delta(k) + P\tilde{w}_2(k) \\
\tilde{y}_2(k) &= C_2 P^{-1}r_2(k) + H^T_{012} \delta(k)
\end{align*}
\]

(37)
To sum up the previous steps

- First change of coordinates $z = H^T x$ to obtain an unknown input-free system:

  \[
  \begin{align*}
  z_2(k+1) &= A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\
  \tilde{y}_2(k) &= C_2 z_2(k) + H_{012}^T \delta(k)
  \end{align*}
  \]
  (38)

- Second change of coordinates $r_2 = Pz_2$ to ensure the cooperativeness property of the observation error in the new coordinates:

  \[
  \begin{align*}
  r_2(k+1) &= Rr_2(k) + PB_2 u(k) + My(k) - M \delta(k) + P\tilde{\omega}_2(k) \\
  \tilde{y}_2(k) &= C_2 P^{-1} r_2(k) + H_{012}^T \delta(k)
  \end{align*}
  \]
  (39)

  where $(A_2 - LC_2)$ is Schur stable and $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

  ⇒ The interval observer given in slide 36 can be used to estimate the state $r_2$ and ... $z_2$. 
To sum up the previous steps

1. First change of coordinates $z = H^\top x$ to obtain an unknown input-free system:

$$
\begin{align*}
    z_2(k + 1) &= A_2z_2(k) + B_2u(k) + D_2y(k) - D_2\delta(k) + \tilde{\omega}_2(k) \\
    \tilde{y}_2(k) &= C_2z_2(k) + H_{012}^T\delta(k)
\end{align*}
$$  \hfill (38)

2. Second change of coordinates $r_2 = Pz_2$ to ensure the cooperativity property of the observation error in the new coordinates:

$$
\begin{align*}
    r_2(k + 1) &= Rr_2(k) + PB_2u(k) + My(k) - M\delta(k) + P\tilde{\omega}_2(k) \\
    \tilde{y}_2(k) &= C_2P^{-1}r_2(k) + H_{012}^T\delta(k)
\end{align*}
$$  \hfill (39)

where $(A_2 - LC_2)$ is Schur stable and $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

⇒ The interval observer given in slide 36 can be used to estimate the state $r_2$ and ... $z_2$. 

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To sum up the previous steps

First change of coordinates \( z = H^\top x \) to obtain an unknown input-free system:

\[
\begin{align*}
    z_2(k + 1) &= A_2z_2(k) + B_2u(k) + D_2y(k) - D_2\delta(k) + \tilde{w}_2(k) \\
    \tilde{y}_2(k) &= C_2z_2(k) + H_{012}^T\delta(k)
\end{align*}
\]  
(38)

Second change of coordinates \( r_2 = Pz_2 \) to ensure the cooperativity property of the observation error in the new coordinates:

\[
\begin{align*}
    r_2(k + 1) &= Rr_2(k) + PB_2u(k) + My(k) - M\delta(k) + P\tilde{w}_2(k) \\
    \tilde{y}_2(k) &= C_2P^{-1}r_2(k) + H_{012}^T\delta(k)
\end{align*}
\]  
(39)

where \((A_2 - LC_2)\) is Schur stable and \( R = P(A_2 - LC_2)P^{-1} \) is nonnegative.

⇒ The interval observer given in slide 36 can be used to estimate the state \( r_2 \) and \( \ldots \ z_2 \).
Step 3: Compute $\underline{x}$ and $\overline{x}$

Step 3a: Compute $\underline{r}$ and $\overline{r}$

- State estimation is first performed in the coordinates $r_2$.
- We define $\Delta^T = [\delta \ -\delta]$, $\Delta^T = [-\delta \ \delta]$ and $\Omega^T = [\omega \ -\omega]$, $\Omega^T = [-\omega \ \omega]$.

Theorem 5

Assume that $\underline{r}_2(0) \leq r_2(0) \leq \overline{r}_2(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{r}_2(k)$ and $\overline{r}_2(k)$ given by

$$
\begin{align*}
\underline{r}_2(k+1) &= R\underline{r}_2(k) + P B_2 u(k) + M y(k) + (-M)^* \Delta + P^* \overline{\Omega}_2 \\
\overline{r}_2(k+1) &= R\overline{r}_2(k) + P B_2 u(k) + M y(k) + (-M)^* \Delta + P^* \overline{\Omega}_2
\end{align*}
$$

are bounded and verify

$$
\underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k)
$$
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Step 3: Compute $\underline{x}$ and $\overline{x}$

Step 3a: Compute $\underline{r}$ and $\overline{r}$

- State estimation is first performed in the coordinates $r_2$.
- We define $\Delta^T = [\delta \ -\delta]$, $\Delta = [-\delta \ \delta]$ and $\Omega^T = [\omega \ -\omega]$, $\Omega = [-\omega \ \omega]$.

Theorem 5

Assume that $r_2(0) \leq r_2(0) \leq \overline{r}_2(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{r}_2(k)$ and $\overline{r}_2(k)$ given by

$$
\begin{align*}
\underline{r}_2(k+1) &= R\underline{r}_2(k) + PB_2u(k) + My(k) + (-M)^*\Delta + P^*\Omega_2 \\
\overline{r}_2(k+1) &= R\overline{r}_2(k) + PB_2u(k) + My(k) + (-M)^*\Delta + P^*\Omega_2 \\
\end{align*}
$$

are bounded and verify

$$
\underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k)
$$
Step 3: Compute \( \underline{x} \) and \( \overline{x} \)

Step 3b: Compute \( \underline{z}_2 \) and \( \overline{z}_2 \)

Since \( r_2 = Pz_2 \), the bounds of \( z_2(k) \) are given by:

Corollary

The bounds of the state \( z_2 \) is given by:

\[
\begin{align*}
\underline{z}_2(k) &= (P^{-1})^+ r_2(k) + (P^{-1})^- r_2(k) \\
\overline{z}_2(k) &= (P^{-1})^+ \overline{r}_2(k) + (P^{-1})^- \overline{r}_2(k)
\end{align*}
\]

(42)

with \( \underline{z}_2(k) \leq z_2(k) \leq \overline{z}_2(k) \)
Step 3: Compute $\underline{x}$ and $\overline{x}$

Step 3c: Return into the initial coordinates $\underline{x}$ and $\overline{x}$

Using $x = Hz$ the following theorem ensures the interval estimation of the state vector in the original coordinates.

**Theorem 7**

Assume $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. Then, for all $k \in \mathbb{Z}^+$ the estimates $\underline{x}(k)$ and $\overline{x}(k)$ given by

$$
\begin{align*}
\underline{x}_1(k) &= H_{11}Ey + (H_{12})^\ast \overline{Z}_2(k) + (E_1)^\ast \overline{Z}_2(k) + (\overline{H}_{11}E)^\ast \overline{\Delta} \\
\overline{x}_1(k) &= H_{11}Ey + (H_{12})^\ast \overline{Z}_2(k) + (E_1)^\ast \overline{Z}_2(k) + (\overline{H}_{11}E)^\ast \overline{\Delta} \\
\underline{x}_2(k) &= H_{21}Ey + (H_{22})^\ast \overline{Z}_2(k) + (E_2)^\ast \overline{Z}_2(k) + (\overline{H}_{21}E)^\ast \overline{\Delta} \\
\overline{x}_2(k) &= H_{21}Ey + (H_{22})^\ast \overline{Z}_2(k) + (E_2)^\ast \overline{Z}_2(k) + (\overline{H}_{21}E)^\ast \overline{\Delta}
\end{align*}
$$

are bounded and verify

$$
\underline{x}(k) \leq x(k) \leq \overline{x}(k)
$$

with $E_1 = H_{11}E\tilde{C}_2$ and $E_2 = H_{21}E\tilde{C}_2$. 

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Step 4: Compute $d$ and $\bar{d}$

\[
\begin{align*}
  z_1(k + 1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\
  z_2(k + 1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\
  y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k)
\end{align*}
\] (45)

- Expression of $d$ is obtained from (45) with $\tilde{d} = K^\top d$:

\[
d(k) = KR_0^{-1}\left[z_1(k + 1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)\right]
\]

- Then using

\[
z_1(k) = E(y(k) - \tilde{C}_2z_2(k) - \delta(k))
\] (46)

- The expression of the unknown input $d$ is

\[
d(k) = KR_0^{-1}\left[Ey(k + 1) - E\tilde{C}_2z_2(k + 1) - E\delta(k + 1) - \tilde{A}_{11}(Ey(k) - E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)\right]
\] (47)
Step 4: Compute $d$ and $\bar{d}$

\[
\begin{align*}
    z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\
    z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\
    y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k)
\end{align*}
\]

(45)

- **Expression of $d$** is obtained from (45) with $\tilde{d} = K^\top d$:

\[
d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]
\]

- Then using

\[
z_1(k) = E(y(k) - \tilde{C}_2z_2(k) - \delta(k))
\]

(46)

- The expression of the unknown input $d$ is

\[
d(k) = KR_0^{-1}E[y(k+1) - E\tilde{C}_2z_2(k+1) - E\delta(k+1) - \tilde{A}_{11}(Ey(k) - E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]
\]

(47)
Step 4: Compute \( \underline{d} \) and \( \overline{d} \)

\[
\begin{align*}
    z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\
    z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\
    y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k)
\end{align*}
\]

(45)

- Expression of \( d \) is obtained from (45) with \( \tilde{d} = K^\top d \):

\[
d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]
\]

- Then using

\[
z_1(k) = E(y(k) - \tilde{C}_2z_2(k) - \delta(k))
\]

(46)

- The expression of the unknown input \( d \) is

\[
d(k) = KR_0^{-1}[Ey(k+1) - E\tilde{C}_2z_2(k+1) - E\delta(k+1) - \tilde{A}_{11}(Ey(k) - E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]
\]

(47)
Step 4: Compute $\underline{d}$ and $\overline{d}$

\[
\begin{align*}
    z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\
    z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\
    y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k)
\end{align*}
\]

(45)

- Expression of $\underline{d}$ is obtained from (45) with $\tilde{d} = K^\top d$:

\[
d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]
\]

- Then using

\[
z_1(k) = E(y(k) - \tilde{C}_2z_2(k) - \delta(k))
\]

(46)

- The expression of the unknown input $\underline{d}$ is

\[
d(k) = KR_0^{-1}[Ey(k+1) - E\tilde{C}_2z_2(k+1) - E\delta(k+1) - \tilde{A}_{11}(Ey(k) - E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]
\]

(47)
Step 4: Compute $\underline{d}$ and $\overline{d}$

- The following theorem ensures the interval estimation of the unknown input:

**Theorem**

For all $k \in \mathbb{Z}^+$ the estimates $\underline{d}(k)$ and $\overline{d}(k)$ given by

$$
\begin{cases}
\underline{d}(k) = Qy(k + 1) - Q\tilde{A}_{11}Ey(k) - Q\tilde{B}_1u(k) + G_1^*\overline{Z}_2(k + 1) + G_2^*\overline{Z}_2(k) \\
+ G_3^*\Delta + G_4^*\Delta + G_5^*\tilde{\Omega}_1 \\
\overline{d}(k) = QEy(k + 1) - Q\tilde{A}_{11}Ey(k) - Q\tilde{B}_1u(k) + G_1^*\underline{Z}_2(k + 1) + G_2^*\underline{Z}_2(k) \\
+ G_3^*\Delta + G_4^*\Delta + G_5^*\tilde{\Omega}_1
\end{cases}
$$

(48)

are bounded and verify

$$\underline{d}(k) \leq d(k) \leq \overline{d}(k)$$

(49)

With $Q = KR_0^{-1}$, $G_1 = -QE\tilde{C}_2$, $G_2 = Q(\tilde{A}_{11}E\tilde{C}_2 - \tilde{A}_{12})$, $G_3 = -QE$, $G_4 = Q\tilde{A}_{11}E$ and $G_5 = -Q$. 

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Dynamical systems can be subject to several kinds of faults such as:

- Actuators faults,
- System faults,
- Sensors faults.

Figure: Faulty system.
Dynamical systems can be subject to several kinds of faults such as:
- Actuators faults,
- System faults,
- Sensors faults.

Figure: Faulty system.
Dynamical systems can be subject to several kinds of faults such as:

- Actuators faults,
- System faults,
- Sensors faults.

Figure: Faulty system.
Problem statement

Dynamical systems can be subject to several kinds of faults such as:
- Actuators faults,
- System faults,
- Sensors faults.

Figure: Faulty system.
Problem statement

Fault Tolerant Control is required to maintain stability and additional performances in presence of faults.

Figure: Active FTC system.
Problem statement

Figure: Passive FTC system.
Problem statement

Consider a discrete-time LTI system:

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + w_k \\
    y_k &= Cx_k + v_k
\end{align*}
\]  

(50)

The system dynamics with actuator additive faults can be modeled by:

\[
\begin{align*}
    x_{k+1} &= Ax_k + (B + B_{f,k})u_k + w_k \\
    y_k &= Cx_k + v_k
\end{align*}
\]  

(51)

where \(B_{f,k} \in \mathbb{R}^{n \times q}\) is a time-varying fault parameter. \(w\) and \(v\) are respectively disturbance and noise sequences.
Consider a discrete-time LTI system:

\[
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + w_k \\
y_k &= Cx_k + v_k
\end{aligned}
\]  

(50)

The system dynamics with actuator additive faults can be modeled by:

\[
\begin{aligned}
x_{k+1} &= Ax_k + (B + B_{f,k})u_k + w_k \\
y_k &= Cx_k + v_k
\end{aligned}
\]  

(51)

where $B_{f,k} \in \mathbb{R}^{n \times q}$ is a time-varying fault parameter. $w$ and $v$ are respectively disturbance and noise sequences.
Assumptions and Goal

Assumption

- $\Delta B \leq B_{f,k} \leq \overline{\Delta B}$ $\forall B_{f,k} \in \mathbb{R}^{n \times q}$.
- $\underline{w}_k \leq w_k \leq \overline{w}_k$ are satisfied $\forall k \in \mathbb{N}$. $\|v\| < V < +\infty$.

Goal

The goal of this section is to stabilize the system (50) with a robust feedback control keeping the required performances despite the appearance of actuator faults and external disturbances.
Assumptions and Goal

Assumption

- $\Delta B \leq B_{f,k} \leq \overline{\Delta B}$ $\forall$ $B_{f,k} \in \mathbb{R}^{n \times q}$.
- $w_k \leq w_k \leq \overline{w_k}$ are satisfied $\forall$ $k \in \mathbb{N}$. $\|v\| < V < +\infty$.

Goal

The goal of this section is to stabilize the system (50) with a robust feedback control keeping the required performances despite the appearance of actuator faults and external disturbances.
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Interval observers design for (51) requires the following assumption.

**Assumption**

The pair $(A, C)$ is detectable and there exists a matrix $L \in \mathbb{R}^{n \times p}$ such that $A - LC$ is Schur stable.

The proposed interval observer structure for (51) is:

\[
\begin{align*}
\overline{x}_{k+1} &= (A - LC)\overline{x}_k + B u_k + \overline{B}u^+_k - \overline{B}u^-_k + \overline{w}_k \\
& \quad + Ly_k + |L| V E_p \\
\underline{x}_{k+1} &= (A - LC)\underline{x}_k + B u_k + \underline{B}u^+_k - \underline{B}u^-_k + \underline{w}_k \\
& \quad + Ly_k - |L| V E_p
\end{align*}
\]

(52)

with $u^+_k = \max(u_k, 0)$ and $u^-_k = u^+_k - u_k$. 
Interval observers design for (51) requires the following assumption.

**Assumption**

*The pair (A, C) is detectable and there exists a matrix \( L \in \mathbb{R}^{n \times p} \) such that \( A - LC \) is Schur stable.*

The proposed interval observer structure for (51) is:

\[
\begin{cases}
\bar{x}_{k+1} = (A - LC)\bar{x}_k + B u_k + \Delta B u^+_k - \Delta B u^-_k + w_k \\
\quad + Ly_k + |L| V V E_p \\
\under{x}_{k+1} = (A - LC)\bar{x}_k + B u_k + \Delta B u^+_k - \Delta B u^-_k + w_k \\
\quad + Ly_k - |L| V V E_p
\end{cases}
\]

(52)

with \( u^+_k = max(u_k, 0) \) and \( u^-_k = u^+_k - u_k \).
Interval observers design for (51) requires the following assumption.

**Assumption**

*The pair \((A, C)\) is detectable and there exists a matrix \(L \in \mathbb{R}^{n \times p}\) such that \(A - LC\) is Schur stable.*

The proposed interval observer structure for (51) is:

\[
\begin{align*}
\overline{x}_{k+1} &= (A - LC)\overline{x}_k + Bu_k + \Delta B u^+_k - \Delta B u^-_k + w_k \\
& \quad + Ly_k + |L|VE_p \\
\underline{x}_{k+1} &= (A - LC)\underline{x}_k + Bu_k + \Delta B u^+_k - \Delta B u^-_k + w_k \\
& \quad + Ly_k - |L|VE_p
\end{align*}
\]

(52)

with \(u^+_k = \max(u_k, 0)\) and \(u^-_k = u^+_k - u_k\).
Theorem 8

Let the assumptions of this section be satisfied, $A - LC$ is nonnegative and the initial state $x_0$ verifies $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, then the state $x_k$ solution of (52) satisfies:

$$\underline{x}_k \leq x_k \leq \overline{x}_k, \quad \forall k \in \mathbb{N} \tag{53}$$

In addition if $A - LC$ is Schur stable, it follows that $\overline{e}_k, \underline{e}_k \in L_\infty^\infty$ with

$$\overline{e}_k = \overline{x}_k - x_k \quad \text{and} \quad \underline{e}_k = x_k - \underline{x}_k.$$
If \( \not\exists L \) such that \( A - LC \) is Schur stable and nonnegative \( \Rightarrow \) A change of coordinates \( z_k = Rx_k \) with a nonsingular matrix \( R \) such that \( E = R(A - LC)R^{-1} \) is Schur stable and nonnegative. In the new coordinates \( z \), the interval observer can be written as:

\[
\begin{align*}
\bar{z}_{k+1} &= E\bar{z}_k + RBu_k + \varphi(u_k^-, u_k^+) + \rho(w_k, w_k) + RLY_k + |F|VE_p \\
z_{k+1} &= Ez_k + RBu_k + \varphi(u_k^-, u_k^+) + \rho(w_k, w_k) + RLY_k - |F|VE_p
\end{align*}
\]

(54)

with \( \varphi(u_k^-, u_k^+) = (R^+ \Delta B - R^- \Delta B)u_k^+ - (R^+ \Delta B - R^- \Delta B)u_k^- \),

\( \varphi(u_k^-, u_k^+) = (R^+ \Delta B - R^- \Delta B)u_k^+ - (R^+ \Delta B - R^- \Delta B)u_k^- \),

\( \rho(w_k, w_k) = R^+ w_k - R^- w_k, \rho(w_k, w_k) = R^+ w_k - R^- w_k, F = RL \).
If \( \not \exists L \) such that \( A - LC \) is Schur stable and nonnegative \( \Rightarrow \) A change of coordinates \( z_k = Rx_k \) with a nonsingular matrix \( R \) such that \( E = R(A - LC)R^{-1} \) is Schur stable and nonnegative. In the new coordinates \( z \), the interval observer can be written as:

\[
\begin{align*}
\overline{z}_{k+1} &= E\overline{z}_k + RBu_k + \overline{\varphi}(u_k^-, u_k^+) + \overline{\rho}(\overline{w}_k, \overline{w}_k) \\
&\quad + RLy_k + |F|V\overline{E}_p \\
\underline{z}_{k+1} &= E\underline{z}_k + RBu_k + \underline{\varphi}(u_k^-, u_k^+) + \underline{\rho}(\underline{w}_k, \underline{w}_k) \\
&\quad + RLy_k - |F|V\overline{E}_p
\end{align*}
\]

with \( \overline{\varphi}(u_k^-, u_k^+) = (R^+\Delta B - R^-\Delta B)u_k^+ - (R^+\Delta B - R^-\Delta B)u_k^- \),
\( \underline{\varphi}(u_k^-, u_k^+) = (R^+\Delta B - R^-\Delta B)u_k^+ - (R^+\Delta B - R^-\Delta B)u_k^- \),
\( \overline{\rho}(\overline{w}_k, \overline{w}_k) = R^+\overline{w}_k - R^-\overline{w}_k \),
\( \underline{\rho}(\underline{w}_k, \underline{w}_k) = R^+\underline{w}_k - R^-\underline{w}_k \), \( F = RL \).
Theorem 9

Given a nonsingular matrix $R$ such that $E = R(A - LC)S$ is Schur stable and nonnegative. Then, the solutions of (51) and (54) satisfy (in the coordinates $z$):

$$
\underline{z}_k \leq z_k \leq \bar{z}_k, \quad \forall k \in \mathbb{N}
$$

(55)

provided that $\underline{z}_0 \leq z_0 \leq \bar{z}_0$. In addition, if $A - LC$, then the interval observer errors $\underline{e}_k$ and $\bar{e}_k \in \mathcal{L}^n_\infty$ with $\bar{e}_k = \bar{z}_k - z_k$ and $\underline{e}_k = z_k - \underline{z}_k$. 
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\[
\begin{cases}
x_{k+1} = Ax_k + (B + B_{f,k})u_k + w_k \\
y_k = Cx_k + v_k
\end{cases}
\]

Figure: Faulty system.
Faulty system

Interval observer

$u_k = -K \bar{x}_k \quad (56)$

**Figure:** System with feedback control.
Figure: System with feedback control.

\[ u_k = -K\bar{x}_k \]  \hspace{1cm} (56)
Substituting the control (56) into the interval observer (52), we get:

\[
\begin{align*}
\overline{x}_{k+1} &= (A - LC)\overline{x}_k - BK\overline{x}_k + \Delta B \max\{0, -K\overline{x}_k\} \\
&\quad -\Delta B \max\{0, K\overline{x}_k\} + \overline{w}_k + Ly_k + |L| V E_p \\
\underline{x}_{k+1} &= (A - LC)\underline{x}_k - BK\underline{x}_k + \Delta B \max\{0, -K\underline{x}_k\} \\
&\quad -\Delta B \max\{0, K\underline{x}_k\} + \underline{w}_k + Ly_k - |L| V E_p
\end{align*}
\] (57)
Theorem 10

Let the assumptions of this section be satisfied and $A - LC \succeq 0$. If there exist $P \in \mathbb{R}^{2n \times 2n}$, $P = P^T \succ 0$ and $\gamma > 0$, such that the dynamic state feedback $K$ satisfies the following constraint:

$$|K|^2 \leq \frac{1}{8\gamma} \frac{1}{10|\Delta B|^2}$$

(58)

with $D^T PD - P \leq -I$, $D = A - LC$ and $\gamma = \frac{3}{2} |D^T P|^2 + |P|$, then, the system (57) is asymptotically stable.
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4 Fault Tolerant Control
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Consider the discrete-time LTI system:

\[
\begin{cases}
x_{k+1} = Ax_k + (B + B_{f,k})u_k + w_k, \\
y_k = Cx_{1,k} + v_k,
\end{cases}
\] (59)

\[A = \begin{bmatrix} 1.1 & -0.1 & 0.35 \\ 0.9 & 0.2 & -0.2 \\ 0.85 & -0.2 & 0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad B_{f,k} = \begin{bmatrix} 0 \\ \sin(k) \\ 0 \end{bmatrix}, \quad \Delta B = -\Delta \bar{B} = [0 \ 1 \ 0]^T, \quad w_k = [0 \ 0.1 \sin(0.1k) \ 0]^T, \quad \overline{w}_k = -
\overline{w}_k = [0 \ 0.1 \ 0]^T, \quad v_k = 0.01 \cos(k) \text{ and } V = 0.01.
\]
Consider the discrete-time LTI system:

\[
\begin{align*}
\begin{cases}
x_{k+1} = A x_k + (B + B_{f,k}) u_k + w_k, \\
y_k = C x_{1,k} + v_k,
\end{cases}
\end{align*}
\]

(59)

\[
A = \begin{bmatrix} 1.1 & -0.1 & 0.35 \\ 0.9 & 0.2 & -0.2 \\ 0.85 & -0.2 & 0.25 \end{bmatrix},
B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},
B_{f,k} = \begin{bmatrix} 0 \\ sin(k) \\ 0 \end{bmatrix}
\]

\[
\Delta B = -\Delta B = [0 \ 1 \ 0]^T, \ w_k = [0 \ 0.1 \ sin(0.1 \ k) \ 0]^T,
\]

\[
\overline{w}_k = -\underline{w}_k = [0 \ 0.1 \ 0]^T, \ v_k = 0.01 \ cos(k) \ \text{and} \ V = 0.01.
\]
Numerical simulations

For $L = [1 \ 1 \ 0.5]^T$, the matrix $A - LC$ is not nonnegative. Thus a transformation of coordinates,

$$S = \begin{bmatrix} -0.058 & 0.997 & -0.052 \\ 0.134 & -0.044 & -0.99 \\ 0.989 & 0.064 & 0.131 \end{bmatrix}$$

is used such that

$$E = R(A - LC)S, \text{ with } R = S^{-1},$$

is nonnegative.

\[
\begin{align*}
\overline{z}_{k+1} &= Ez_k + RBu_k + \varphi(u_k^-, u_k^+) + \rho(w_k, \overline{w}_k) \\
&\quad + RLy_k + |F| VE_p
\\
z_{k+1} &= Ez_k + RBu_k + \varphi(u_k^-, u_k^+) + \rho(w_k, \overline{w}_k) \\
&\quad + RLy_k - |F| VE_p
\end{align*}
\]

(60)

with $u_k = -K \overline{x}_k$ and $K = [0.9140 \ 0.9128 \ 0.3999]$. 
Numerical simulations

For $L = [1 \ 1 \ 0.5]^{T}$, the matrix $A - LC$ is not nonnegative. Thus a transformation of coordinates,

$$S = \begin{bmatrix}
-0.058 & 0.997 & -0.052 \\
0.134 & -0.044 & -0.99 \\
0.989 & 0.064 & 0.131
\end{bmatrix}$$

is used such that $E = R(A - LC)S$, with $R = S^{-1}$, is nonnegative.

\[
\begin{cases}
\overline{z}_{k+1} = E\overline{z}_k + RBu_k + \overline{\varphi}(u^-_k, u^+_k) + \overline{\rho}(\overline{w}_k, \overline{w}_k) \\
\quad + RLy_k + |F|VE_p \\
\underline{z}_{k+1} = E\underline{z}_k + RBu_k + \underline{\varphi}(u^-_k, u^+_k) + \underline{\rho}(\underline{w}_k, \underline{w}_k) \\
\quad + RLy_k - |F|VE_p
\end{cases}
\]

(60)

with $u_k = -K \overline{x}_k$ and $K = [0.9140 \ 0.9128 \ 0.3999]$. 

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Numerical simulations

For $L = [1 \ 1 \ 0.5]^T$, the matrix $A - LC$ is not nonnegative. Thus a transformation of coordinates,

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-0.058 & 0.997 & -0.052 \\
0.134 & -0.044 & -0.99 \\
0.989 & 0.064 & 0.131 
\end{bmatrix}$$

is used such that

$$E = R(A - LC)S$$

with $R = S^{-1}$, is nonnegative.

$$
\begin{align*}
\bar{z}_{k+1} &= E\bar{z}_k + RBu_k + \varphi(u^-_k, u^+_k) + \rho(w_k, w_k) \\
& \quad + RLy_k + |F|VE_p \\
\underline{z}_{k+1} &= E\underline{z}_k + RBu_k + \varphi(u^-_k, u^+_k) + \rho(w_k, w_k) \\
& \quad + RLy_k - |F|VE_p 
\end{align*}
$$

(60)

with $u_k = -K \bar{x}_k$ and $K = [0.9140 \ 0.9128 \ 0.3999]$. 
Numerical simulations

Figure: Simulations results for the case of LTI system without fault.
**Figure:** Simulations results for the case of LTI system with fault.
More results about Interval estimation at the **Open Invited Track on Interval estimation applied to diagnosis and control of uncertain systems - IFAC-WC’2017 - July 11th.**