Interval observers and fault tolerant control

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Outline

1. Introduction
2. Interval observers - LTI systems
   - Continuous-time systems
   - Discrete-time systems
3. Joint state and unknown input estimation
   - State estimation
   - Upper and lower bounds of the unknown input
4. Fault Tolerant Control
   - Problem statement
   - Interval observer design
   - Control design
   - Numerical simulations

Most of the material of this presentation is detailed in the overview:

Denis Efimov & **Tarek Raïssi**, Design of interval observers for uncertain dynamical systems, Automation and Remote Control, Volume 77, Issue 2, pp 191-225, 2016.

1. Introduction
2. Interval observers - LTI systems
3. Joint state and unknown input estimation
4. Fault Tolerant Control

Context

Observation/Control

- ✓ Linear systems: several constructive results ⇒ freq. approaches, output/state feedback,...
- ✓ Nonlinear systems: the solutions depend on the nonlinearity structure ⇒
  Lipschitzian systems: $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$
  ⇒ linear approaches can be used to build observers/controllers.
- ✓ LPV systems (Linear Parameter-Varying): intermediate class between **Linear** and **Nonlinear** systems
  - Several techniques allow one to transform/approximate NL into LPV systems
    $\dot{x} = f(x, u) \Rightarrow \dot{x} = A(\theta(t))x + B(\theta(t))u$
  - The nonlinear trajectory belongs into the LPV ones
  - NL $\equiv$ Linear $+$ parameter uncertainties $(\theta(t))$

Estimation & Uncertainties

Several cases may be met

- Models **without** uncertainties
- Models with uncertain parameters (**constant** or **varying** uncertain parameters)
- Uncertain parameters & unknown inputs

Observers structures

- $\dot{x} = Ax + Bu; y = Cx$
  $\Rightarrow$ Luenberger Obs. $\dot{z} = Az + Bu + L(y - Cz)$.
- $\dot{x} = A(\theta)x + B(\theta)u; y = C(\theta)x \Rightarrow \theta$ is known or unknown?

Possible solutions

- Adaptive approaches $\Rightarrow$ joint estimation of x and θ.
- Robust approaches $\dot{z} = A_a z + B_a u + L(y - C_a z)$ (for some average values A_a, B_a and C_a).
- **Set-membership estimation / Interval observers**.

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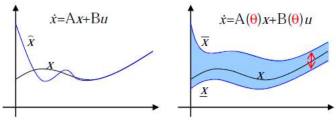
Set-membership estimation 1/5

- Without uncertainties ⇒ point estimation.
- Systems subject to bounded uncertainties ⇒ estimation of a feasible solution set.



✓ Prediction/correction approach

Jaulin, 2002 ; Kieffer & Walter, 2004 ; Raïssi, Ramdani, Candau, 2004 . . .

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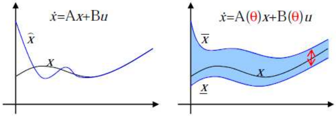

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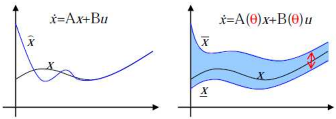

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Set-membership estimation 2/5

Illustrative example

Let's consider the following example:

$$\dot{x}(t) = -x^3(t) + d(t), \quad t \geq 0, \tag{1}$$

- $d(t) \in [-1, 1]$ is uncertain input whose values belong into the interval $[-1, 1]$.
- Assume that the admissible values for initial conditions of this system is the interval $[-2, 2]$, i.e. $x(0) \in [-2, 2]$.
- The system is nonlinear, the input and initial conditions are uncertain $\Rightarrow$ it is hard to evaluate an exact value of the state $x(t)$ at each time t.
- However, it is possible to evaluate the admissible values for $x(t)$ with initial conditions $x(0) \in [-2, 2]$ and $d(t) \in [-1, 1]$.

Set-membership estimation 3/5

Illustrative example

The set of admissible values is described by:

$$\begin{cases} \dot{\underline{x}}(t) = -\underline{x}^3(t) - 1, & \underline{x}(0) = -2 \\ \dot{\overline{x}}(t) = -\overline{x}^3(t) + 1, & \overline{x}(0) = 2 \end{cases} \tag{2}$$

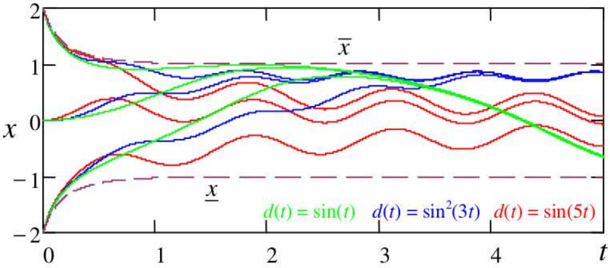

Figure: Admissible set of the solutions of (1) computed through (2)

Set-membership estimation 4/5

Illustrative example

- As we can conclude, the interval $[\underline{x}(t), \overline{x}(t)]$ represents a worst case estimate of admissible values of $x(t)$ for given uncertainties.
- The main problem considered in this survey is how to design an interval estimator like (2) using all available information, including the output measurements (not taken into account in this example) and minimizing the width of the interval $[\underline{x}(t), \overline{x}(t)]$, i.e. $\overline{x}(t) - \underline{x}(t)$.
- Linear Time-Invariant, Linear Parameter-Varying, Continuous-time, discrete-time systems are considered.

Set-membership estimation 5/5

Given a system described by

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{3}$$

Definition 1

The dynamical system

$$\begin{cases} \dot{z} & = & \alpha(z,y,u) \\ \left[\underline{x}^T, \overline{x}^T\right]^T & = & \beta(z,y,u) \end{cases} \tag{4}$$

is an interval observer for (3) if:

$$\underline{x}(0) \leq x(0) \leq \overline{x}(0) \quad \Rightarrow \quad -\infty < \underline{x}(t) \leq x(t) \leq \overline{x}(t) < \infty, \quad \forall t \geq 0. \tag{5}$$

Roughly speaking, an interval observer should verify two conditions:

- Inclusion: $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, $\forall t \geq t_0$
- Stability of $\underline{e} = x - \underline{x}$ and $\overline{e} = \overline{x} - x$

Cooperative systems - nonnegative systems

- Given two vectors x_1, x_2 and two matrices A_1, A_2, the relations $x_1 \leq x_2$, $x_1 \geq x_2$, $A_1 \leq A_2$, $A_1 \geq A_2$ should be understood elementwise.
- A positive semi-definite matrix is denoted by $P : P = P^T \succeq 0$.
- A square matrix $S \in R^{n \times n}$ is called Metzler if $S_{i,j} \geq 0, \forall \quad 1 \leq i \neq j \leq n$. The set of all Metzler matrices is denoted by $\mathcal{M}$.

Theorem 1

Given a Metzler matrix S ($S \in \mathcal{M}$), the system

$$\dot{z} = Sz + r(t); \quad z \in \mathbb{R}^n; \quad r : \mathbb{R}_+ \to \mathbb{R}^n_+$$

is called cooperative or nonnegative. Its trajectories verify:

$$z(0) \geq 0 \Rightarrow z(t) \geq 0, \forall t \geq 0.$$

Metzler matrices properties

A matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz stable ($A \in \mathcal{H}$) if all its eigenvalues have negative real parts.

For a Metzler matrix $A \in \mathcal{M} \subseteq \mathbb{R}^{n\times n}$, the following properties are similar:

- $A \in \mathcal{H}$;
- $A^{-1} \leq 0$;
- there exists $P \in \mathbb{R}^{n\times n}$, $P = P^T \succ 0$ such that
$$A^T P + PA \prec 0;$$
- there exists a diagonal matrix $D \in \mathbb{R}^{n\times n}$, $D \succ 0$ such that
$$A^T D + DA \prec 0;$$
- there exists a vector $\rho \in \mathbb{R}^n$, $\rho > 0$ such that
$$A^T \rho < 0$$

Somme additional properties

Define $A^+ = \max(0, A)$, $A^- = A^+ - A \Rightarrow$ the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1

Given vectors $x, \underline{x}, \overline{x} \in \mathbb{R}^n$, $\underline{x} \leq x \leq \overline{x}$ and a matrix $A \in \mathbb{R}^{m\times n}$, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \qquad (6)$$

For $\underline{A} \leq A \leq \overline{A}$, $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then

$$\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax \leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^- \qquad (7)$$

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LTI continuous-time systems

Given a system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bd(t), \quad d : \mathbb{R}_+ \to \mathbb{R}_+^q \\ y(t) = Cx(t) + Dd(t) \end{cases} \tag{8}$$

where A is a Metzler matrix, then the solution $x(t)$ is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}_+^{n \times q}$.

The stability of the nonnegative system (8) can be checked by verifying a Linear Programming (LP) problem

$$A^T \lambda < 0$$

for some nonnegative $\lambda \in \mathbb{R}_+^n$, or equivalently a Lyapunov matrix equation

$$A^T P + PA \prec 0$$

with P a diagonal matrix.

LTI continuous-time systems

Lemma 2[a]

[a] Refer for instance to

- C. Briat, Robust stability analysis of uncertain linear positive systems via integral linear constraints: L_1- and linfty-gain characterizations, 50th IEEE CDC and ECC, (Orlando), pp. 6337-6342, 2011.
- Y. Ebihara, D. Peaucelle, and D. Arzelier, L_1 gain analysis of linear positive systems and its application, 50th IEEE CDC and ECC, (Orlando), pp. 4029-4035, 2011.

The nonnegative system (8) (i.e. A is Metzler, $B \geq 0$, $C \geq 0$ and $D \geq 0$) is asymptotically stable if and only if there exist a nonnegative $\lambda \in \mathbb{R}^n_+$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:

$$\begin{pmatrix} A^T\lambda + C^T E_p \\ B^T\lambda - \gamma E_p + D^T E_p \end{pmatrix} < 0. \tag{9}$$

Moreover, in this case, the L_1 gain of the transfer $d \to y$ is lower than γ.

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$$\begin{pmatrix} A\lambda + BE_q \\ C\lambda - \gamma E_p + DE_q \end{pmatrix} < 0. \tag{10}$$

Moreover, in this case, the L_∞ gain of the transfer $d \to y$ is lower than γ.

LTI Continuous-time systems

$$\begin{cases} \dot{x}(t) = Ax(t) + d(t) \\ y(t) = Cx(t) + v(t) \end{cases} \quad (11)$$

Assumption 1

Assmume that $\underline{x}_0 \leq x(0) \leq \overline{x}_0$, $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$, $-V \leq v(t) \leq V$, $\forall t \geq 0$.

Interval observer structure:

$$\begin{cases} \dot{\underline{x}}(t) = A\underline{x} + L\left[y(t) - C\underline{x}\right] - |L|E_pV + \underline{d}(t) \\ \dot{\overline{x}}(t) = A\overline{x} + L\left[y(t) - C\overline{x}\right] + |L|E_pV + \overline{d}(t) \end{cases} \quad (12)$$

How to ensure

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

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Theorem 2[a]

[a] J.L. Gouzé, A. Rapaport, and M. Hadj-Sadok, Interval observers for uncertain biological systems, Ecological Modelling, vol. 133, pp. 46-56, 2000.

Let Assumption 1 hold and $x \in \mathcal{L}_\infty^n$, then, the solutions of the systems (11) and (12) satisfy

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

provided that the matrix $A - LC$ is Metzler. In addition, if $A - LC$ is Hurwitz, then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$.

Proof:

Define the estimation errors $\underline{e}(t) = x(t) - \underline{x}(t), \quad \overline{e}(t) = \overline{x}(t) - x(t)$

$$\dot{\underline{e}}(t) = (A - LC)\underline{e}(t) + Lv(t) + |L|E_pV + d(t) - \underline{d}(t),$$
$$\dot{\overline{e}}(t) = (A - LC)\overline{e}(t) - Lv(t) + |L|E_pV + \overline{d}(t) - d(t).$$

By assumption 1, $|L|E_pV \pm Lv(t) \geq 0, \quad d(t) - \underline{d}(t) \geq 0, \quad \overline{d}(t) - d(t) \geq 0 \Rightarrow$
The inputs of $\underline{e}(t), \overline{e}(t)$ are nonnegative and $\underline{e}(0) \geq 0$, $\overline{e}(0) \geq 0$.

If $A - LC$ is Metzler, then $\underline{e}(t) \geq 0, \underline{e}(t) \geq 0, \forall t \geq 0$

If $A - LC$ is Hurwitz, then $\underline{e}(t), \overline{e}(t) \in \mathcal{L}_\infty^n \Rightarrow \underline{x}(t), \overline{x}(t)$ are also bounded (inputs of $\underline{e}(t), \overline{e}(t)$ and x are bounded).

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If $A - LC$ is Hurwitz, then $\underline{e}(t), \overline{e}(t) \in \mathcal{L}_\infty^n \Rightarrow \underline{x}(t), \overline{x}(t)$ are also bounded (inputs of $\underline{e}(t), \overline{e}(t)$ and x are bounded).

LTI Continuous-time systems

Optimal interval observers

- If the observer gain L is designed such that $A - LC$ is Metzler $\Rightarrow$ $\underline{x}(t) \le x(t) \le \overline{x}(t), \forall t \ge 0$.
- If the observer gain L is designed such that $A - LC$ is Hurwitz stable $\Rightarrow$ the error $\overline{x}(t) - \underline{x}(t)$ is bounded.
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  The $[\underline{x}(t), \overline{x}(t)]$ width otimization in the L_1 framework can be formulated as a Linear Programming problem:

Theorem 3

Consider the interval observer (12) *for* (11). *If there exist a nonnegative vector $\lambda \in \mathbb{R}^n$, $W \in \mathbb{R}^n$ and a diagonal matrix $M \in \mathbb{R}^{n\times n}$ such that*

$$\begin{array}{l} \begin{pmatrix} A^T\lambda - C^T W + E_n \\ \lambda - \gamma E_n \end{pmatrix} < 0 \\ A^T\lambda - C^T W + M\lambda \ge 0 \\ \lambda > 0, M \ge 0 \end{array} \qquad (13)$$

Then, $W = L^T\lambda$ and $d \to w(\overline{x} - \underline{x})$ has a L_1 gain lower than γ.

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- Theorem 3 provides an effective and simple computational tool to design interval observers for LTI systems.
- It gives only sufficient conditions $\Rightarrow$ in some cases this LP problem may have no solution.
- The LP problem has no solution if it is not possible to find L such that $A - LC$ is simultaneously Metzler and Hurwitz.

A counterexample: Given a system described by:

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

This system is observable, whereas the matrix

$$A - LC = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix}$$

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  1. To overcome this issue, it is possible to design the gain L such that the matrix $A - LC$ is Hurwitz
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LTI Continuous-time systems

Lemma 4[a]

[a] T. Raïssi, D. Efimov, and A. Zolghadri, Interval state estimation for a class of nonlinear systems, IEEE Transactions on Automatic Control, vol. 57, no. 1, pp. 260-265, 2012.

Given the matrices $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$. If there exists a matrix $L \in \mathbb{R}^{n\times p}$ such that the matrices $A - LC$ and D have the same eigenvalues, then there exists a matrix $S \in \mathbb{R}^{n\times n}$ such that $D = S(A - LC)S^{-1}$ provided that the pairs $(A - LC, e_1)$ and (D, e_2) are observable for some $e_1 \in \mathbb{R}^{1\times n}$, $e_2 \in \mathbb{R}^{1\times n}$.

LTI Continuous-time systems

By adding and substracting $Ly(t)$ to (11), the dynamics of this LTI system is described by:

$$\dot{x}(t) = (A - LC)x(t) + Ly(t) - Lv(t) + d(t). \tag{14}$$

Under conditions of Lemma 4 (slide 25), in the new coordinates $z = Sx$, the system (14) takes the form:

$$\dot{z}(t) = Dz(t) + SLy(t) + \delta(t), \quad \delta(t) = S[d(t) - Lv(t)]. \tag{15}$$

Let

$$\begin{cases} \underline{\delta}(t) = S^+\underline{d}(t) - S^-\overline{d}(t) - |SL|E_pV \\ \overline{\delta}(t) = S^+\overline{d}(t) - S^-\underline{d}(t) + |SL|E_pV. \end{cases}$$

By using Lemma 1 (slide 16), we get

$$\underline{\delta}(t) \leq \delta(t) \leq \overline{\delta}(t).$$

LTI Continuous-time systems

Under the detectability property for the pair (A, C), all the conditions of Theorem 2 in slide 21 are satisfied in the coordinates $z = Sx$. An interval observer candidate is given by:

$$\begin{cases} \dot{\underline{z}}(t) = D\underline{z}(t) + SLy(t) + \underline{\delta}(t) \\ \dot{\overline{z}}(t) = D\overline{z}(t) + SLy(t) + \overline{\delta}(t) \end{cases} \tag{16}$$

Initital conditions:

$$\begin{cases} \underline{z}(0) = S^{+}\underline{x}_0 - S^{-}\overline{x}_0 \\ \overline{z}(0) = S^{+}\overline{x}_0 - S^{-}\underline{x}_0 \end{cases}$$

Estimation in the original coordinates: Let $R = S^{-1}$

$$\begin{cases} \underline{x}(t) = R^{+}\underline{z}(t) - R^{-}\overline{z}(t) \\ \overline{x}(t) = R^{+}\overline{z}(t) - R^{-}\underline{z}(t) \end{cases} \tag{17}$$

LTI Continuous-time systems

Remark:

If the eigenvalues of $(A - LC)$ are complex-valued, the change of coordinates $z = Sx$ could be time-varying.

Lemma[a]

[a] F. Mazenc and O. Bernard, Interval observers for linear time-invariant systems with disturbances. Automatica, vol. 47, no. 1, pp. 140-147, 2011.

Let $A - LC$ be Hurwitz, then there exists an invertible matrix function $S : \mathbb{R} \to \mathbb{R}^{n \times n}$ of class C^∞ elementwise, $||S(t)||_2 < +\infty$ for all $t \in \mathbb{R}$, such that for all $t \in \mathbb{R}$,

$$\dot{S}(t) = DS(t) - S(t)(A - LC)$$

where $D \in \mathbb{R}^{n \times n}$ is a Hurwitz and Metzler matrix.

D can for instance be chosen as the Jordan canonical form of $A - LC$.

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Numerical example 1/2

$$\dot{x} = Ax + B(p_1, p_2)f(x)u(t), \; y = Cx,$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & \sqrt{3} \\ -1 & -\sqrt{3} & -4 \end{bmatrix}, \quad B(p_1, p_2) = \begin{bmatrix} -2p_1 \\ 0 \\ p_2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$f(x) = x_1 x_2$, $\underline{p}_1 = 4.48$, $\bar{p}_1 = 6.12$, $\underline{p}_2 = 3.2$, $\bar{p}_2 = 3.6$.

- The pair (A, C) is not observable and there is no observer gain L such that the matrix $A - LC$ is Metzler.
- Only one eigenvalue can be assigned with the gain L.
- The matrix

$$D = \begin{bmatrix} -a & b & 0 \\ 0 & -a & b \\ b & 0 & -a \end{bmatrix}$$

has the following eigenvalues $b - a$, $-a - 0.5b \pm 0.5b\sqrt{3}i$ (we take here $b = 2$ and $a = 3$).

Numerical example 2/2

- For $L = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T$, the matrix $A - LC \in \mathcal{H}$ and its eigenvalues are -1, $-4 \pm \sqrt{3}i$.
- The pairs $(A - LC, e_1)$ and (R, e_2) are observable for

$$e_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},\ e_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

then

$$S = O_2^{-1} O_1 = \begin{bmatrix} 0.158 & 0.866 & 0.5 \\ 0.842 & -0.866 & 0.5 \\ 0.658 & 0 & -1 \end{bmatrix}.$$

Notations and definitions

- A matrix $A \in \mathbb{R}^{n\times n}$ is called Schur stable if its spectral radius is less than one.
- A matrix $A \in \mathbb{R}^{n\times n}$ is called nonnegative if all its elements are nonnegative.

Definition: nonnegative systems

- Consider the linear system

$$x(k+1) = Ax(k) + \omega(k) \qquad (18)$$

where $\omega \in \mathbb{R}^n_+$ and A is a nonnegative matrix.
- $\forall k \geq 0$, $x(k) \geq 0$ provided that $x(0) \geq 0$.
- Such dynamical systems are called nonnegative (or cooperative).

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LTI discrete-time systems: observer design

- Consider the following system:

$$\begin{cases} x(t+1) = Ax(t) + d(t) \\ y(t) = Cx(t) + v(t), \end{cases}, \quad t \in \mathbb{Z}_+ \tag{19}$$

- **Assumption:**
Let $x(0) \in [\underline{x}_0, \overline{x}_0]$, two functions $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$ and a constant $V > 0$ such that

$$\underline{d}(t) \leq d(t) \leq \overline{d}(t), \quad |v(t)| \leq V, \forall t \in \mathbb{Z}_+.$$

- **Interval observer structure:**

$$\begin{cases} \underline{x}(t+1) = A\underline{x}(t) + L(y(t) - C\underline{x}(t)) - |L|E_p V + \underline{d}(t) \\ \overline{x}(t+1) = A\overline{x}(t) + L(y(t) - C\overline{x}(t)) + |L|E_p V + \overline{d}(t) \\ \underline{x}(0) = \underline{x}_0, \quad \overline{x}(0) = \overline{x}_0. \end{cases} \tag{20}$$

LTI discrete-time systems: observer design

Theorem 4[a]

[a] D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, On interval observer design for time-invariant discrete-time systems, European Control Conference (Zurich), 2013.

Let the assumption given above hold and $x \in \mathcal{L}_\infty^n$, the solutions of (19) and (20) satisfy

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \in \mathbb{Z}_+$$

provided that the matrix $A - LC$ is nonnegative. In addition, $\underline{x}$ and $\overline{x} \in \mathcal{L}_\infty^n$ if $A - LC$ is Schur stable.

Proof sketch:

The estimation errors dynamics $\underline{e}(t) = x(t) - \underline{x}(t)$ and $\overline{e}(t) = \overline{x}(t) - x(t)$ follow the dynamics:

$$\begin{cases} \underline{e}(t+1) = (A - LC)\underline{e}(t) + d(t) - \underline{d}(t) + |L|E_pV - Lv(t) \\ \overline{e}(t+1) = (A - LC)\overline{e}(t) + \overline{d}(t) - d(t) + |L|E_pV + Lv(t) \end{cases} \quad (21)$$

- The relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \in \mathbb{Z}_+$ is ensured based on the assumption of slide 33 and nonnegativity of $(A - LC)$.
- The stability is ensured if $(A - LC)$ is Schur stable.

LTI discrete-time systems: observer design

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Proof sketch:

The estimation errors dynamics $\underline{e}(t) = x(t) - \underline{x}(t)$ and $\overline{e}(t) = \overline{x}(t) - x(t)$ follow the dynamics:

$$\begin{cases} \underline{e}(t+1) = (A - LC)\underline{e}(t) + d(t) - \underline{d}(t) + |L|E_pV - Lv(t) \\ \overline{e}(t+1) = (A - LC)\overline{e}(t) + \overline{d}(t) - d(t) + |L|E_pV + Lv(t) \end{cases} \tag{21}$$

- The relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \in \mathbb{Z}_+$ is ensured based on the assumption of slide 33 and nonnegativity of $(A - LC)$.
- The stability is ensured if $(A - LC)$ is Schur stable.

LTI discrete-time systems: observer design

The observer gain L can be computed as a solution of the following Linear Matrix Inequality (LMI):

$$\begin{cases} \begin{pmatrix} P & PA - WC \\ A^T P - C^T W^T & P \end{pmatrix} \succ 0 \\ PA - WC \geq 0 \\ P = P^T \succ 0 \end{cases} \tag{22}$$

- The diagonal matrix $P \in \mathbb{R}^{n \times n}$ and $W \in \mathbb{R}^{n \times p}$ are the variables to determine $\Rightarrow$ then $L = P^{-1} W$.
- A gain optimization problem (similar to the continuous-time case) can also be formulated to find L providing a minimal interval width $\overline{x}(t) - \underline{x}(t)$ with respect to a chosen norm.

LTI discrete-time systems: observer design

Restrictive condition: existence of L such that $(A - LC)$ is nonnegative $\Rightarrow$ Change of coordinates.

LTI discrete-time systems: observer design

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Theorem 5[a]

[a] D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, On interval observer design for time-invariant discrete-time systems, European Control Conference (Zurich), 2013.

Let the assumption of slide 33 be verified and $A - LC$ is Schur stable for a gain L. Given a matrix $R \in \mathbb{R}^{n\times n}$ and $e_1 \in \mathbb{R}^{1\times n}$, $e_2 \in \mathbb{R}^{1\times n}$ such that $\lambda(A - LC) = \lambda(R)$ and the pairs $(A - LC, e_1)$, (R, e_2) are observable. Then, an interval estimation for (19) is given by:

$$\begin{cases} \underline{z}(t+1) = R\underline{z}(t) + Fy(t) - |F|E_pV + S^+\underline{d}(t) - S^-\overline{d}(t) \\ \\ \overline{z}(t+1) = R\overline{z}(t) + Fy(t) + |F|E_pV + S^+\overline{d}(t) - S^-\underline{d}(t) \end{cases}$$

$$\begin{cases} \underline{z}_0 = S^+\underline{x}_0 - S^-\overline{x}_0 \\ \overline{z}_0 = S^+\overline{x}_0 - S^-\underline{x}_0 \end{cases}$$

$$\begin{cases} \underline{x}(t) = (S^{-1})^+\underline{z}(t) - (S^{-1})^-\overline{z}(t), \\ \overline{x}(t) = (S^{-1})^+\overline{z}(t) - (S^{-1})^-\underline{z}(t) \end{cases}$$

where $S = O_{A-LC}O^{-1}$ (O_{A-LC} and O_R are the observability matrices of the pairs $(A - LC, e_1)$, (R, e_2)) and $F = SL$.

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where $S = O_{A-LC}O^{-1}$ (O_{A-LC} and O_R are the observability matrices of the pairs $(A - LC, e_1)$, (R, e_2)) and $F = SL$.

LTI discrete-time systems: observer design

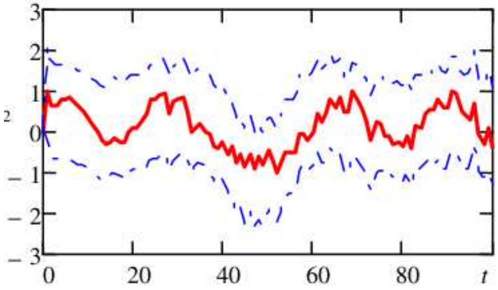
Numerical example:

$$\begin{cases} x(t+1) = \begin{pmatrix} 0.3 & -0.7 \\ 0.6 & -0.5 \end{pmatrix} x(t) + \begin{pmatrix} sin(0.1t) \\ cos(0.2t) \end{pmatrix} + 0.5 \begin{pmatrix} sin(0.5tx_2(t)) \\ sin(0.3t) \end{pmatrix} \\ y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + 0.1 sin(t) \end{cases}$$

Let $L = \begin{pmatrix} -0.8000 & -0.7000 \end{pmatrix}^T$
and $D = A - LC = \begin{pmatrix} 0.3 & 0.1 \\ 0.6 & 0.2 \end{pmatrix}$.
D is a nonnegative matrix $\Rightarrow$ no need of a change of coordinates.
The nonlinear term is bounded $\Rightarrow$ it can be considered as a disturbance.

LTI discrete-time systems: observer design

Numerical example:
The simulations are performed using the interval observer given in the slide 36.



1. Introduction
2. Interval observers - LTI systems
3. Joint state and unknown input estimation
   - State estimation
   - Upper and lower bounds of the unknown input
4. Fault Tolerant Control

Interval observers and linear systems with unknown inputs

- LTI discrete-time system with unknown inputs:

$$\begin{cases} x(k+1) & = Ax(k) + Bu(k) + Dd(k) + \omega(k) \\ y(k) & = Cx(k) + \delta(k) \end{cases} \tag{23}$$

Is it possible to estimate x and d? $\Rightarrow$ Such estimation can be useful for instance for diagnosis and Fault Tolerant Control.

For more details, refer to:

- Elinirina Irena Robinson, Julien Marzat, Tarek Raïssi, Interval Observer Design for Unknown Input Estimation of Linear Time-Invariant Discrete-Time Systems, IFAC World Congress, Toulouse, France. 9-14 July, 2017.
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Interval observers and linear systems with unknown inputs

Methodology to jointly estimate the bounds of x and d

1. Change of coordinates to divide the system (23) into two subsytems:
   $\rightarrow$ one affected by the unknown input
   $\rightarrow$ the second one is unknown input-free
2. Change of coordinates to ensure the nonnegativity property of the observation error in the new coordinates
3. Design of an interval observer in the new basis to compute $\underline{x}$ and $\overline{x}$.
4. Compute $\underline{d}$ and $\overline{d}$.

Step 1: State and unknown input decoupling

LTI discrete-time system with unknown inputs:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\ y(k) &= Cx(k) + \delta(k) \end{cases} \tag{24}$$

Assumption 2

C is a full row rank matrix and D is a full column rank matrix.

- There exist matrices $H \in \mathbb{R}^{n\times n}$, $R_0 \in \mathbb{R}^{q\times q}$ and $K \in \mathbb{R}^{q\times q}$ such that:

$$D = H \begin{bmatrix} R_0 \\ 0 \end{bmatrix} K^T \tag{25}$$

Step 1: State and unknown input decoupling

LTI discrete-time system with unknown inputs:

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Step 1: obtain an unknown input-free system

- Transformation of the initial system into an equivalent one: Let $z(k) = H^T x(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}$

$$\begin{cases} z(k+1) & = \tilde{A}z(k) + \tilde{B}u(k) + \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tilde{d}(k) + \tilde{\omega}(k) \\ y(k) & = \tilde{C}z(k) + \delta(k) \end{cases} \tag{26}$$

where:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \tilde{A} = H^T A H = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

$$\tilde{B} = H^T B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = CH = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \tilde{d}(k) = K^T d(k) \tilde{\omega}(k) \quad = H^T \omega = \begin{bmatrix} \tilde{\omega}_1(k) \\ \tilde{\omega}_2(k) \end{bmatrix}$$

- H^T is supposed to be bounded, therefore $|\tilde{\omega}| \leq \overline{\tilde{\omega}}$.

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$$\tilde{B} = H^T B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = CH = [\tilde{C}_1 \quad \tilde{C}_2]\, \tilde{d}(k) = K^T d(k) \tilde{\omega}(k) \quad = H^T \omega = \begin{bmatrix} \tilde{\omega}_1(k) \\ \tilde{\omega}_2(k) \end{bmatrix}$$

- H^T is supposed to be bounded, therefore $|\tilde{\omega}| \leq \overline{\tilde{\omega}}$.

Step 1: obtain an unknown input-free system

$$\begin{cases} z(k+1) & = \tilde{A}z(k) + \tilde{B}u(k) + \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tilde{d}(k) + \tilde{\omega}(k) \\ y(k) & = \tilde{C}z(k) + \delta(k) \end{cases} \quad (27)$$

- The system (27) is decomposed into an unknown input depending subsystem and an unknown input-free subsytem:

$$\begin{cases} z_1(k+1) & = \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) & = \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\ y(k) & = \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k) \end{cases} \quad (28)$$

Step 1: obtain an unknown input-free system

$$\begin{cases} z_1(k+1) & = \tilde{A}_{11} z_1(k) + \tilde{A}_{12} z_2(k) + \tilde{B}_1 u(k) + R_0 \tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) & = \tilde{A}_{21} z_1(k) + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \\ y(k) & = \tilde{C}_1 z_1(k) + \tilde{C}_2 z_2(k) + \delta(k) \end{cases} \tag{29}$$

Transformation of (29) into a conventional linear system:

- $\tilde{C}_1$ is supposed to be a full column rank matrix and can be decomposed as:

$$\tilde{C}_1 = H_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix} K_1^T \tag{30}$$

with $H_1 = \begin{bmatrix} H_{011} & H_{012} \end{bmatrix}$ ($H_{011} \in \mathbb{R}^{p \times q}$ and $H_{012} \in \mathbb{R}^{p \times (p-q)}$).

- Measurements equation can be decomposed as $\tilde{y}(k) = H_1^T y(k)$

$$\begin{cases} \tilde{y}_1(k) & = R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\ \tilde{y}_2(k) & = H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k) \end{cases} \tag{31}$$

Step 1: obtain an unknown input-free system

$$\begin{cases} z_1(k+1) &= \tilde{A}_{11} z_1(k) + \tilde{A}_{12} z_2(k) + \tilde{B}_1 u(k) + R_0 \tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) &= \tilde{A}_{21} z_1(k) + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \\ y(k) &= \tilde{C}_1 z_1(k) + \tilde{C}_2 z_2(k) + \delta(k) \end{cases} \tag{29}$$

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$$\begin{cases} z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\ y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k) \end{cases} \tag{29}$$

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- As $\tilde{y}_1(k) = G_s^T \tilde{y}(k)$ with $G_s^T = \begin{bmatrix} I_q & O_{q\times(p-q)} \end{bmatrix}$, the expression of z_1 is extracted from the first equation of (32):

$$z_1(k) = E(y(k) - \tilde{C}_2 z_2(k) - \delta(k)) \quad (33)$$

with $E = K_1 R_1^{-1} G_s^T H_1^T$.

- By replacing (33) in the dynamics of z_2 in (29) we obtain:

$$z_2(k+1) = \tilde{A}_{21} E[y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \quad (34)$$

Step 1: obtain an unknown input-free system

- Measurements equation can be decomposed as

$$\begin{cases} \tilde{y}_1(k) & = R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\ \tilde{y}_2(k) & = H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k) \end{cases} \tag{32}$$

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with $E = K_1 R_1^{-1} G_s^T H_1^T$.

- By replacing (33) in the dynamics of z_2 in (29) we obtain:

$$z_2(k+1) = \tilde{A}_{21} E[y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \tag{34}$$

Step 1: obtain an unknown input-free system

- Measurements equation can be decomposed as

$$\begin{cases} \tilde{y}_1(k) & = R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\ \tilde{y}_2(k) & = H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k) \end{cases} \tag{32}$$

- As $\tilde{y}_1(k) = G_s^T \tilde{y}(k)$ with $G_s^T = \begin{bmatrix} I_q & O_{q\times(p-q)} \end{bmatrix}$, the expression of z_1 is extracted from the first equation of (32):

$$z_1(k) = E(y(k) - \tilde{C}_2 z_2(k) - \delta(k)) \tag{33}$$

with $E = K_1 R_1^{-1} G_s^T H_1^T$.

- By replacing (33) in the dynamics of z_2 in (29) we obtain:

$$z_2(k+1) = \tilde{A}_{21} E[y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_{22} z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \tag{34}$$

Step 1: obtain an unknown input-free system

Finally we obtain the following unknown input-free LTI discrete-time system:

$$\begin{cases} z_2(k+1) & = A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 z_{2(k)} + H_{012}^T \delta(k) \end{cases} \quad (35)$$

where $A_2 = \tilde{A}_{22} - \tilde{A}_{21} E \tilde{C}_2$, $B_2 = \tilde{B}_2$, $C_2 = H_{012}^T \tilde{C}_2$ and $D_2 = \tilde{A}_{21} E$.

Step 2: Nonnegativity of the observation error in the new coordinates

- Unknown input-free LTI discrete-time system:

$$\begin{cases} z_2(k+1) & = A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 z_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{36}$$

Assumption

- The pair (A_2, C_2) is detectable.
- There exists a gain $L \in \mathbb{R}^{(n-q)\times(p-q)}$ and a matrix P such that $(A_2 - LC_2)$ is Schur stable and $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

- After the change of coordinates $r_2 = Pz_2$, the system (36) is described in the new coordinates by:

$$\begin{cases} r_2(k+1) & = Rr_2(k) + PB_2 u(k) + My(k) - M\delta(k) + P\tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 P^{-1} r_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{37}$$

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- After the change of coordinates $r_2 = P z_2$, the system (36) is described in the new coordinates by:

$$\begin{cases} r_2(k+1) & = R r_2(k) + P B_2 u(k) + M y(k) - M \delta(k) + P \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 P^{-1} r_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{37}$$

To sum up the previous steps

1. First change of coordinates $z = H^\top x$ to obtain an unknown input-free system:

$$\begin{cases} z_2(k+1) & = A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 z_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{38}$$

2. Second change of coordinates $r_2 = P z_2$ to ensure the cooperativity property of the observation error in the new coordinates:

$$\begin{cases} r_2(k+1) & = R r_2(k) + P B_2 u(k) + M y(k) - M \delta(k) + P \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 P^{-1} r_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{39}$$

where $(A_2 - LC_2)$ is Schur stable and $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

$\Rightarrow$ **The interval observer given in slide 36 can be used to estimate the state r_2 and $\ldots z_2$.**

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1. First change of coordinates $z = H^\top x$ to obtain an unknown input-free system:

$$\begin{cases} z_2(k+1) & = A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 z_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{38}$$

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To sum up the previous steps

1. First change of coordinates $z = H^\top x$ to obtain an unknown input-free system:

$$\begin{cases} z_2(k+1) & = A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2 \delta(k) + \tilde{\omega}_2(k) \\ \tilde{y}_2(k) & = C_2 z_{2(k)} + H_{012}^T \delta(k) \end{cases} \tag{38}$$

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where $(A_2 - LC_2)$ is Schur stable and $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

$\Rightarrow$ **The interval observer given in slide 36 can be used to estimate the state r_2 and ... z_2.**

Step 3: Compute $\underline{x}$ and $\overline{x}$

Step $3a$: Compute $\underline{r}$ and $\overline{r}$

- State estimation is first performed in the coordinates r_2.
- We define $\overline{\Delta}^T = \begin{bmatrix}\overline{\delta} & -\overline{\delta}\end{bmatrix}$, $\underline{\Delta}^T = \begin{bmatrix}-\overline{\delta} & \overline{\delta}\end{bmatrix}$ and $\overline{\Omega}^T = \begin{bmatrix}\overline{\varpi} & -\overline{\varpi}\end{bmatrix}$, $\underline{\Omega}^T = \begin{bmatrix}-\overline{\varpi} & \overline{\varpi}\end{bmatrix}$.

Theorem 5

Assume that $\underline{r}_2(0) \leq r_2(0) \leq \overline{r}_2(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{r}_2(k)$ and $\overline{r}_2(k)$ given by

$$\begin{cases} \overline{r}_2(k+1) & = R\overline{r}_2(k) + PB_2u(k) + My(k) + (-M)^*\overline{\Delta} + P^*\overline{\tilde{\Omega}}_2 \\ \underline{r}_2(k+1) & = R\underline{r}_2(k) + PB_2u(k) + My(k) + (-M)^*\underline{\Delta} + P^*\underline{\tilde{\Omega}}_2 \end{cases} \tag{40}$$

are bounded and verify

$$\underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k) \tag{41}$$

Step 3: Compute $\underline{x}$ and $\overline{x}$

Step $3a$: Compute $\underline{r}$ and $\overline{r}$

- State estimation is first performed in the coordinates r_2.
- We define $\overline{\Delta}^T = \begin{bmatrix} \overline{\delta} & -\overline{\delta} \end{bmatrix}$, $\underline{\Delta}^T = \begin{bmatrix} -\overline{\delta} & \overline{\delta} \end{bmatrix}$ and $\overline{\Omega}^T = \begin{bmatrix} \overline{\varpi} & -\overline{\varpi} \end{bmatrix}$, $\underline{\Omega}^T = \begin{bmatrix} -\overline{\varpi} & \overline{\varpi} \end{bmatrix}$.

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Assume that $\underline{r}_2(0) \leq r_2(0) \leq \overline{r}_2(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{r}_2(k)$ and $\overline{r}_2(k)$ given by

$$\begin{cases} \overline{r}_2(k+1) & = R\overline{r}_2(k) + PB_2u(k) + My(k) + (-M)^*\overline{\Delta} + P^*\overline{\tilde{\Omega}}_2 \\ \underline{r}_2(k+1) & = R\underline{r}_2(k) + PB_2u(k) + My(k) + (-M)^*\underline{\Delta} + P^*\underline{\tilde{\Omega}}_2 \end{cases} \tag{40}$$

are bounded and verify

$$\underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k) \tag{41}$$

Step 3: Compute $\underline{x}$ and $\overline{x}$

Step $3b$: Compute $\underline{z_2}$ and $\overline{z_2}$

- Since $r_2 = Pz_2$, the bounds of $z_2(k)$ are given by:

Corollary

The bounds of the state z_2 is given by:

$$\begin{cases} \overline{z}_2(k) = (P^{-1})^+\overline{r}_2(k) + (P^{-1})^-\underline{r}_2(k) \\ \underline{z}_2(k) = (P^{-1})^+\underline{r}_2(k) + (P^{-1})^-\overline{r}_2(k) \end{cases} \tag{42}$$

with $\underline{z}_2(k) \leq z_2(k) \leq \overline{z}_2(k)$

Step 3: Compute $\underline{x}$ and $\overline{x}$

Step $3c$: Return into the initial coordinates $\underline{x}$ and $\overline{x}$

- Using $x = Hz$ the following theorem ensures the interval estimation of the state vector in the original coordinates.

Theorem 7

Assume $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{x}(k)$ and $\overline{x}(k)$ given by

$$\begin{cases} \overline{x}_1(k) & = H_{11}Ey + (H_{12})^*\overline{Z}_2(k) + (-E_1)^*\overline{Z}_2(k) + (-H_{11}E)^*\overline{\Delta} \\ \underline{x}_1(k) & = H_{11}Ey + (H_{12})^*\underline{Z}_2(k) + (-E_1)^*\underline{Z}_2(k) + (-H_{11}E)^*\underline{\Delta} \\ \overline{x}_2(k) & = H_{21}Ey + (H_{22})^*\overline{Z}_2(k) + (-E_2)^*\overline{Z}_2(k) + (-H_{21}E)^*\overline{\Delta} \\ \underline{x}_2(k) & = H_{21}Ey + (H_{22})^*\underline{Z}_2(k) + (-E_2)^*\underline{Z}_2(k) + (-H_{21}E)^*\underline{\Delta} \end{cases} \tag{43}$$

are bounded and verify

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k) \tag{44}$$

with $E_1 = H_{11}E\tilde{C}_2$ and $E_2 = H_{21}E\tilde{C}_2$.

Step 4: Compute $\underline{d}$ and $\overline{d}$

$$\begin{cases} z_1(k+1) & = \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) & = \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\ y(k) & = \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k) \end{cases} \tag{45}$$

- Expression of d is obtained from (45) with $\tilde{d} = K^\top d$:

$$d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]$$

- Then using

$$z_1(k) = E(y(k) - \tilde{C}_2z_2(k) - \delta(k)) \tag{46}$$

- The expression of the unknown input d is

$$\begin{aligned} d(k) &= KR_0^{-1}[Ey(k+1) - E\tilde{C}_2z_2(k+1) - E\delta(k+1) - \tilde{A}_{11}(Ey(k) \\ &\quad -E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)] \end{aligned} \tag{47}$$

Step 4: Compute $\underline{d}$ and $\overline{d}$

$$\begin{cases} z_1(k+1) & = \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) & = \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\ y(k) & = \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k) \end{cases} \tag{45}$$

- Expression of d is obtained from (45) with $\tilde{d} = K^\top d$:

$$d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]$$

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$$\begin{aligned} d(k) & = KR_0^{-1}[Ey(k+1) - E\tilde{C}_2z_2(k+1) - E\delta(k+1) - \tilde{A}_{11}(Ey(k) \\ & -E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)] \end{aligned} \tag{47}$$

Step 4: Compute $\underline{d}$ and $\overline{d}$

$$\begin{cases} z_1(k+1) & = \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) & = \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\ y(k) & = \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k) \end{cases} \tag{45}$$

- Expression of d is obtained from (45) with $\tilde{d} = K^\top d$:

$$d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) - \tilde{B}_1u(k) - \tilde{\omega}_1(k)]$$

- Then using

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Step 4: Compute $\underline{d}$ and $\overline{d}$

$$\begin{cases} z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1u(k) + R_0\tilde{d}(k) + \tilde{\omega}_1(k) \\ z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2u(k) + \tilde{\omega}_2(k) \\ y(k) &= \tilde{C}_1z_1(k) + \tilde{C}_2z_2(k) + \delta(k) \end{cases} \tag{45}$$

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Step 4: Compute $\underline{d}$ and $\overline{d}$

- The following theorem ensures the interval estimation of the unknown input:

Theorem

for all $k \in \mathbb{Z}_+$ the estimates $\underline{d}(k)$ and $\overline{d}(k)$ given by

$$\begin{cases} \overline{d}(k) & = Qy(k+1) - Q\tilde{A}_{11}Ey(k) - Q\tilde{B}_1 u(k) + G_1^* \overline{Z}_2(k+1) + G_2^* \overline{Z}_2(k) \\ & + G_3^* \overline{\Delta} + G_4^* \overline{\Delta} + G_5^* \overline{\tilde{\Omega}}_1 \\ \underline{d}(k) & = QEy(k+1) - Q\tilde{A}_{11}Ey(k) - Q\tilde{B}_1 u(k) + G_1^* \underline{Z}_2(k+1) + G_2^* \underline{Z}_2(k) \\ & + G_3^* \underline{\Delta} + G_4^* \underline{\Delta} + G_5^* \underline{\tilde{\Omega}}_1 \end{cases} \tag{48}$$

are bounded and verify

$$\underline{d}(k) \leq d(k) \leq \overline{d}(k) \tag{49}$$

With $Q = KR_0^{-1}$, $G_1 = -QE\tilde{C}_2$, $G_2 = Q(\tilde{A}_{11}E\tilde{C}_2 - \tilde{A}_{12})$, $G_3 = -QE$, $G_4 = Q\tilde{A}_{11}E$ and $G_5 = -Q$.

Problem statement

Dynamical systems can be subject to several kinds of faults such as:

- Actuators faults,
- System faults,
- Sensors faults.

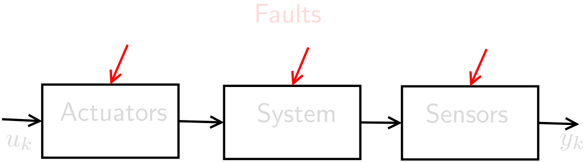

Figure: Faulty system.

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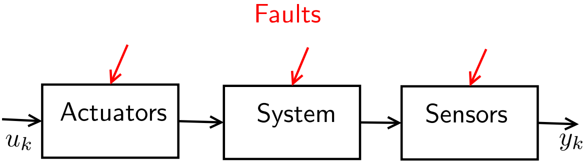

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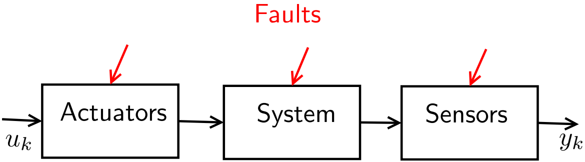

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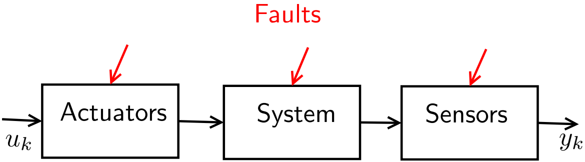

Figure: Faulty system.

Problem statement

Fault Tolerant Control is required to maintain stability and additional performances in presence of faults.

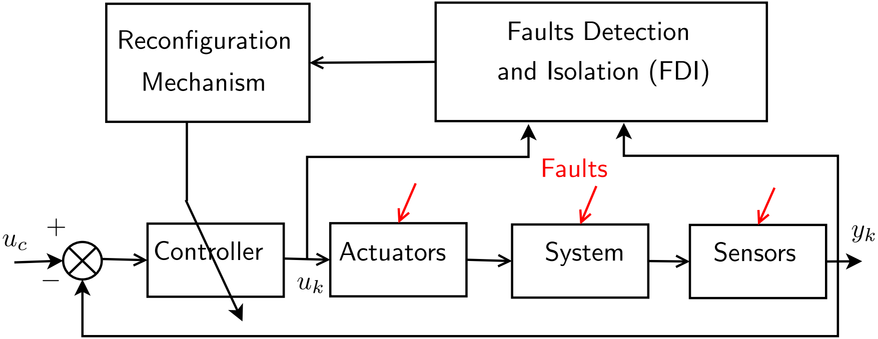

Figure: Active FTC system.

Problem statement

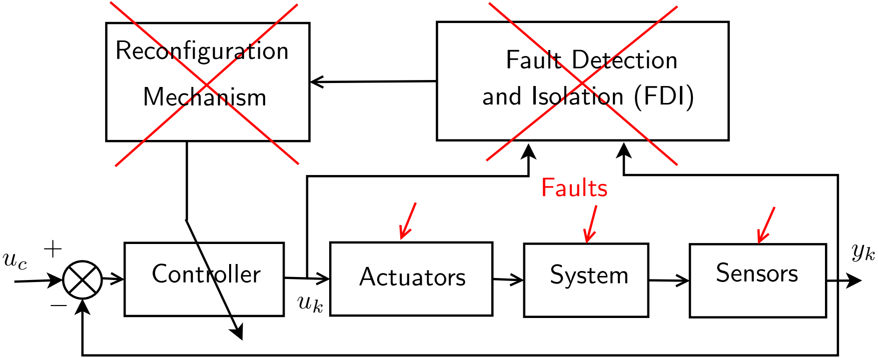

Figure: Passive FTC system.

Problem statement

Consider a discrete-time LTI system:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k \\ y_k = Cx_k + v_k \end{cases} \tag{50}$$

The system dynamics with actuator additive faults can be modeled by:

$$\begin{cases} x_{k+1} = Ax_k + (B + B_{f,k})u_k + w_k \\ y_k = Cx_k + v_k \end{cases} \tag{51}$$

where $B_{f,k} \in \mathbb{R}^{n \times q}$ is a time-varying fault parameter. w and v are respectively disturbance and noise sequences.

Problem statement

Consider a discrete-time LTI system:

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Assumptions and Goal

Assumption

- $\underline{\Delta B} \leqslant B_{f,k} \leqslant \overline{\Delta B} \ \forall \ B_{f,k} \in \mathbb{R}^{n \times q}$.
- $\underline{w}_k \leqslant w_k \leqslant \overline{w}_k$ *are satisfied* $\forall \ k \in \mathbb{N}$. $\|v\| < V < +\infty$.

Goal

The goal of this section is to stabilize the system (50) with a robust feedback control keeping the required performances despite the appearance of actuator faults and external disturbances.

Assumptions and Goal

Assumption

- $\underline{\Delta B} \leqslant B_{f,k} \leqslant \overline{\Delta B} \; \forall \; B_{f,k} \in \mathbb{R}^{n \times q}$.
- $\underline{w}_k \leqslant w_k \leqslant \overline{w}_k$ *are satisfied* $\forall \; k \in \mathbb{N}$. $\|v\| < V < +\infty$.

Goal

The goal of this section is to stabilize the system (50) with a robust feedback control keeping the required performances despite the appearance of actuator faults and external disturbances.

Interval observers design for (51) requires the following assumption.

Assumption

The pair (A, C) is detectable and there exists a matrix $L \in \mathbb{R}^{n \times p}$ such that $A - LC$ is Schur stable.

The proposed interval observer structure for (51) is:

$$\begin{cases} \overline{x}_{k+1} = (A - LC)\overline{x}_k + Bu_k + \overline{\Delta B}u_k^+ - \underline{\Delta B}u_k^- + \overline{w}_k \\ \qquad\quad +Ly_k + |L|VE_p \\ \underline{x}_{k+1} = (A - LC)\underline{x}_k + Bu_k + \underline{\Delta B}u_k^+ - \overline{\Delta B}u_k^- + \underline{w}_k \\ \qquad\quad +Ly_k - |L|VE_p \end{cases} \tag{52}$$

with $u_k^+ = max(u_k, 0)$ and $u_k^- = u_k^+ - u_k$.

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with $u_k^+ = max(u_k, 0)$ and $u_k^- = u_k^+ - u_k$.

Theorem 8

Let the assumptions of this section be satisfied, $A - LC$ is nonnegative and the initial state x_0 verifies $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, then the state x_k solution of (52) satisfies:

$$\underline{x}_k \leq x_k \leq \overline{x}_k, \quad \forall k \in \mathbb{N} \tag{53}$$

In addition if $A - LC$ is Schur stable, it follows that $\underline{e}_k, \overline{e}_k \in \mathcal{L}_n^\infty$ with $\overline{e}_k = \overline{x}_k - x_k$ and $\underline{e}_k = x_k - \underline{x}_k$.

If $\nexists L$ such that $A - LC$ is Schur stable and nonnegative $\Rightarrow$ A change of coordinates $z_k = Rx_k$ with a nonsingular matrix R such that $E = R(A - LC)R^{-1}$ is Schur stable and nonnegative. In the new coordinates z, the interval observer can be written as :

$$\begin{cases} \overline{z}_{k+1} = E\overline{z}_k + RBu_k + \overline{\varphi}(u_k^-, u_k^+) + \overline{\rho}(\overline{w}_k, \underline{w}_k) \\ \qquad\qquad +RLy_k + |F|VE_p \\ \underline{z}_{k+1} = E\underline{z}_k + RBu_k + \underline{\varphi}(u_k^-, u_k^+) + \underline{\rho}(\overline{w}_k, \underline{w}_k) \\ \qquad\qquad +RLy_k - |F|VE_p \end{cases} \tag{54}$$

with $\underline{\varphi}(u_k^-, u_k^+) = (R^+\underline{\Delta B} - R^-\overline{\Delta B})u_k^+ - (R^+\overline{\Delta B} - R^-\underline{\Delta B})u_k^-$, $\overline{\varphi}(u_k^-, u_k^+) = (R^+\overline{\Delta B} - R^-\underline{\Delta B})u_k^+ - (R^+\underline{\Delta B} - R^-\overline{\Delta B})u_k^-$, $\underline{\rho}(\overline{w}_k, \underline{w}_k) = R^+\underline{w}_k - R^-\overline{w}_k$, $\overline{\rho}(\overline{w}_k, \underline{w}_k) = R^+\overline{w}_k - R^-\underline{w}_k$, $F = RL$.

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Theorem 9

Given a nonsingular matrix R such that $E = R(A - LC)S$ is Schur stable and nonnegative. Then, the solutions of (51) and (54) satisfy (in the coordinates z):

$$\underline{z}_k \leq z_k \leq \overline{z}_k, \quad \forall k \in \mathbb{N} \tag{55}$$

provided that $\underline{z}_0 \leq z_0 \leq \overline{z}_0$. In addition, if $A - LC$, then the interval observer errors $\underline{e}_k$ and $\overline{e}_k \in \mathcal{L}_\infty^n$ with $\overline{e}_k = \overline{z}_k - z_k$ and $\underline{e}_k = z_k - \underline{z}_k$.

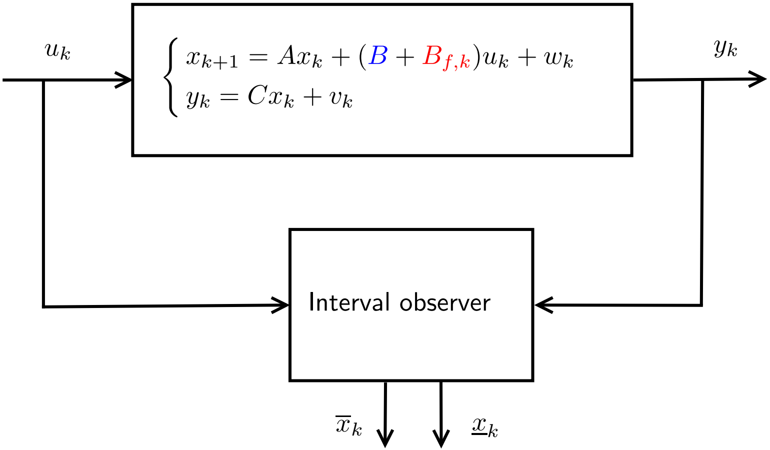


Figure: Faulty system.

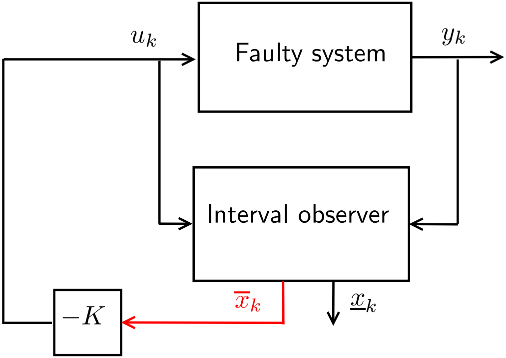
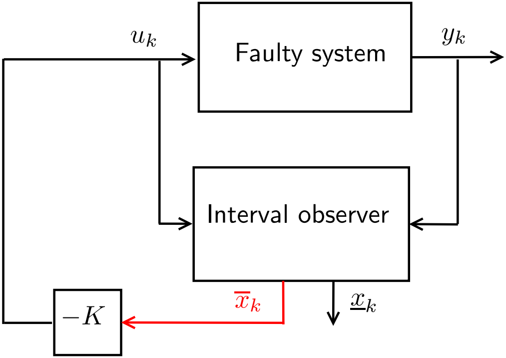


Figure: System with feedback control.

$$u_k = -K\overline{x}_k \tag{56}$$



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Control design

Substituting the control (56) into the interval observer (52), we get:

$$\begin{cases} \overline{x}_{k+1} = (A - LC)\overline{x}_k - BK\overline{x}_k + \overline{\Delta B}\ max\{0, -K\overline{x}_k\} \\ \qquad -\underline{\Delta B}\ max\{0, K\overline{x}_k\} + \overline{w}_k + Ly_k + |L|VE_p \\ \underline{x}_{k+1} = (A - LC)\underline{x}_k - BK\overline{x}_k + \underline{\Delta B}\ max\{0, -K\ \overline{x}_k\} \\ \qquad -\overline{\Delta B}\ max\{0, K\overline{x}_k\} + \underline{w}_k + Ly_k - |L|VE_p \end{cases} \tag{57}$$

Control design

Theorem 10

Let the assumptions of this section be satisfied and $A - LC \geq 0$. If there exist $P \in \mathbb{R}^{2n\times 2n}$, $P = P^T \succ 0$ and $\gamma > 0$, such that the dynamic state feedback K satisfies the following constraint:

$$|K|^2 \leq \frac{1}{8\gamma}\,\frac{1}{10\left|\overline{\Delta B}\right|^2} \tag{58}$$

with $D^T PD - P \leq -I$, $D = A - LC$ and $\gamma = \frac{3}{2}\left|D^T P\right|^2 + |P|$, then, the system (57) is asymptotically stable.

Numerical simulations

Consider the discrete-time LTI system:

$$\begin{cases} x_{k+1} = Ax_k + (B + B_{f,k})u_k + w_k, \\ y_k = Cx_{1,k} + v_k, \end{cases} \tag{59}$$

$$A = \begin{bmatrix} 1.1 & -0.1 & 0.35 \\ 0.9 & 0.2 & -0.2 \\ 0.85 & -0.2 & 0.25 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, B_{f,k} = \begin{bmatrix} 0 \\ sin(k) \\ 0 \end{bmatrix}$$

$\overline{\Delta B} = -\underline{\Delta B} = [0 \quad 1 \quad 0]^T$, $w_k = [0 \quad 0.1\, sin(0.1\, k) \quad 0]^T$, $\overline{w}_k = -\underline{w}_k = [0 \quad 0.1 \quad 0]^T$, $v_k = 0.01\, cos(k)$ and $V = 0.01$.

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Numerical simulations

For $L = [1 \quad 1 \quad 0.5]^T$, the matrix $A - LC$ is not nonnegative. Thus a transformation of coordinates,

$S = \begin{bmatrix} -0.058 & 0.997 & -0.052 \\ 0.134 & -0.044 & -0.99 \\ 0.989 & 0.064 & 0.131 \end{bmatrix}$ is used such that $E = R(A - LC)S$, with $R = S^{-1}$, is nonnegative.

$$\begin{cases} \overline{z}_{k+1} = E\overline{z}_k + RBu_k + \overline{\varphi}(u_k^-, u_k^+) + \overline{\rho}(\overline{w}_k, \underline{w}_k) \\ \qquad\qquad +RLy_k + |F|VE_p \\ \underline{z}_{k+1} = E\underline{z}_k + RBu_k + \underline{\varphi}(u_k^-, u_k^+) + \underline{\rho}(\overline{w}_k, \underline{w}_k) \\ \qquad\qquad +RLy_k - |F|VE_p \end{cases} \tag{60}$$

with $u_k = -K\,\overline{x}_k$ and $K = [0.9140 \quad 0.9128 \quad 0.3999]$.

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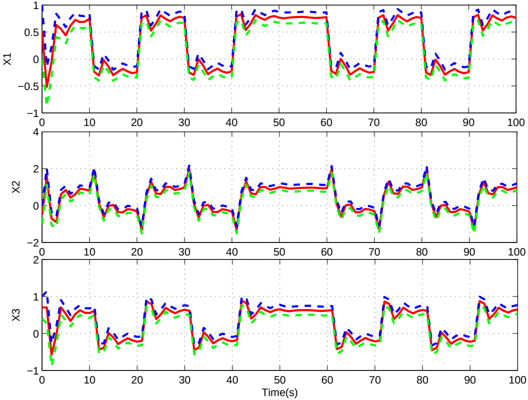
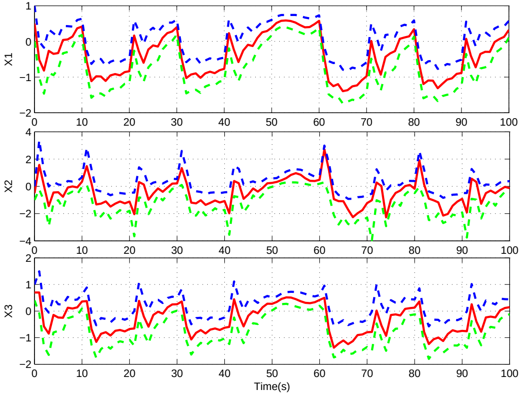


Figure: Simulations results for the case of LTI system without fault.



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More results about Interval estimation at the **Open Invited Track on Interval estimation applied to diagnosis and control of uncertain systems - IFAC-WC'2017 - July 11th.**