



Reachability analysis for nonlinear control validation and synthesis

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Outline

- 1 Control Validation
 - Motivation
 - Problem formulation
 - Illustrative example

- 2 Control Synthesis
 - Problem statement
 - SIVIA algorithm
 - Illustrative example

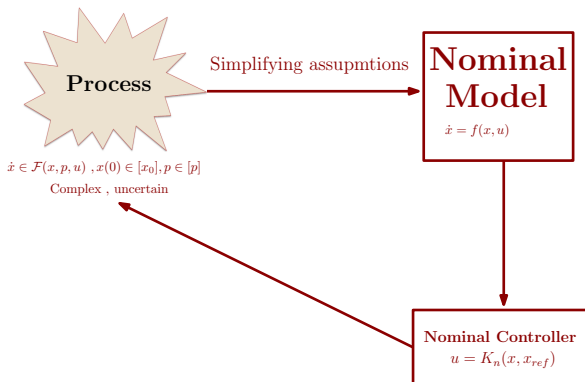
- 3 Conclusion

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Motivation

N. Loukkas, N. Meslem, J.J. Martinez, 'Set-Membership Tests to Evaluate the Performance of Nominal Feedback Control Laws'. IEEE Conference on Control Applications (CCA 2016), pp.1300–1305.



- Are the desired control performance preserved in the actual environment?
- Are the desired control performance preserved for the **comprehensive model** ?

Motivation: Control Validation

Objective: Design a robust and reliable approach for verifying that the specifications of a nominal controller are still satisfied before operation on the actual process.

- **We will rely on numerical proofs** based on **set-value computations**:
 - **Reachability analysis:** To predict all possible system's behaviours.
 - **Set-inclusion tests:** To check the satisfaction of the control specifications.

Problem formulation

- Consider a complex system which dynamics is **poorly-known** but belongs to a **differential inclusion**:

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}, \mathbf{p}, \mathbf{u}), \quad \mathbf{x}(t_0) \in \mathcal{X}_0 \subset \mathbb{R}^n, \quad \mathbf{p} \in \mathcal{P} \subset \mathbb{R}^p$$

- From this differential inclusion a simplified *nominal* model can be derived

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

where the vector field \mathbf{f} is well known and satisfies certain control design assumptions.

- Based on the nominal model, a control law can be designed

$$\mathbf{u} = \mathbf{k}(\mathbf{y}_m, \mathbf{r})$$

where \mathbf{y}_m stands for the available measurements, and \mathbf{r} is the set-point signal.

Problem formulation

Question:

Is the actual system, equipped with the control designed using the nominal model, behaving as expected?

- **Answer:** One has to show that all the possible behaviors of the closed-loop system

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}, \mathbf{p}, \mathbf{k}(\mathbf{y}_m, \mathbf{r})), \quad \mathbf{x}(t_0) \in \mathcal{X}_0 \subset \mathbb{R}^n, \quad \mathbf{p} \in \mathcal{P} \subset \mathbb{R}^p$$

satisfy the desired control specifications.

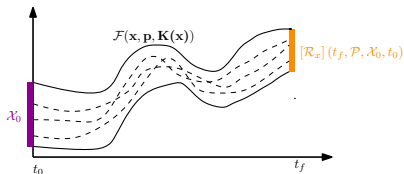
- **How to do that:** We propose a technique mainly based on **reachability analysis** to derive a numerical proof regarding the satisfaction of the control specifications.

Reachability analysis (recall)

Consider the closed-loop system

$$\dot{\mathbf{x}} \in \mathcal{F}(\mathbf{x}, \mathbf{p}, \mathbf{k}(\mathbf{y}_m, \mathbf{r})), \quad \mathbf{x}(t_0) \in \mathcal{X}_0, \quad \mathbf{p} \in \mathcal{P}, \quad \mathbf{y}_m \in \mathcal{Y}$$

- Thanks to reachability analysis, an outer-approximation $[\mathcal{R}_x]([t_0, t_f], \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_0)$ of the reachable set of this system can be computed.



- Here, \mathcal{Y} stands for the feasible domain for measurement \mathbf{y}_m which can be also reconstructed or using interval observer methods.

Reachability analysis (reminder)

To compute the outer-approximation of the reachable set of **nonlinear systems**

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}, \mathbf{p}, \mathbf{k}(\mathbf{y}_m, \mathbf{r})), \quad \mathbf{x}(t_0) \in \mathcal{X}_0, \quad \mathbf{p} \in \mathcal{P}, \quad \mathbf{y}_m \in \mathcal{Y}$$

one can use:



Interval Taylor expansion methods

Interval integration using Taylor expansion:

$$[x_{j+1}] = [x_{j+1}] + \sum_{i=1}^{k-1} h_j^i \mathcal{F}^{[i]}([x_j]) + h_j^k \mathcal{F}^{[k]}([\tilde{x}_j])$$

Comparison theorems for differential inequalities

Transform the uncertain system into two deterministic systems : $x(t) \in [\underline{x}(t), \bar{x}(t)]$

$$\begin{cases} \dot{\bar{x}} = \bar{\mathcal{F}}(\underline{x}, \bar{x}, \underline{p}, \bar{p}, \underline{k}, \bar{k}) \\ \dot{\underline{x}} = \underline{\mathcal{F}}(\underline{x}, \bar{x}, \underline{p}, \bar{p}, \underline{k}, \bar{k}), \dim = 2n \end{cases}$$

- **Interval Taylor expansion methods**, [N.S.Nedialkov],[R.Rihm],[R.J.Lohner],...
- **Comparison theorems for differential inequalities**, [N.Ramdani, N.Meslem],...

Principle of the control validation method

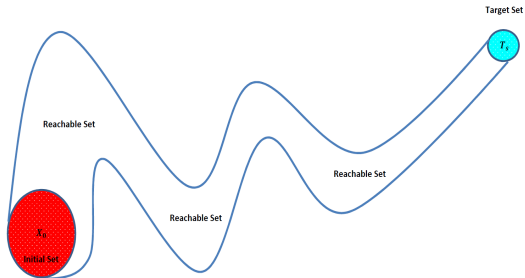
The core idea is summarized in the following three main steps

- 1 **Step 1** : Rewrite the desired control specifications as set-membership criteria
- 2 **Step 2** : Compute an outer-approximation of the set reachable by the closed-loop system (in a worst-case scenario).
- 3 **Step 3** : Check if the desired set-membership criteria are satisfied by all the possible state trajectories of the closed-loop system.

Step 1: Set-membership formulation of the desired specifications

The first specification

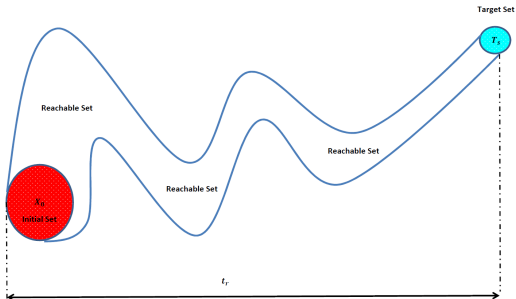
- **Target set \mathcal{T}_s** : The desired behavior of the system at steady state can be characterized by a set of state vectors called target set. Therefore, the ultimate bounds of the closed-loop system must remain in the target set.



Step 1: Set-membership formulation of the desired specifications

The second specification

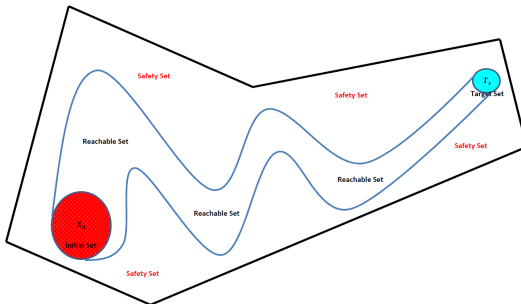
- **Reaching time t_r** : In this framework, the 'rapidity' of the system is measured by its reaching-time t_r , which is equivalent to the classical settling time. More formally, t_r is the time instant when all the possible state trajectories are inside the target set.



Step 1: Set-membership formulation of the desired specifications

The third specification

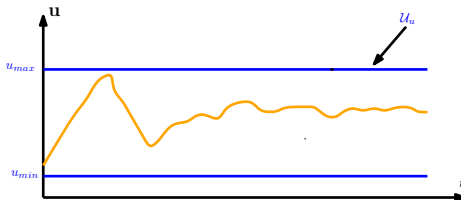
- **Safety set \mathcal{U}_x** : The safety set can be characterized by state constraints and/or by thresholds on authorized overshoot for system outputs, . . .



Step 1: Set-membership formulation of the desired specifications

The fourth specification

- **Feasible set \mathcal{U}_u :** In practice, the working range of the actuators is limited. Therefore, the control law has to satisfy given input constraints.



Step 2: Compute an outer-approximation of the set reachable by the closed-loop

Compute and partition the outer-approximation of the reachable set into three main parts.

- **Transient behavior:** to check safety and feasibility constraints

$$[\mathcal{R}_x]([t_0, t_r], \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_0) \quad t \in [t_0, t_r]$$

- **At the reaching time:** to check the desired settling time

$$[\mathcal{R}_x](t_r, \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_0) \quad t = t_r$$

- **Steady state:** To check the desired performance

$$[\mathcal{R}_x]([t_r, t_f], \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_r) \quad t \in [t_r, t_f]$$

Step 3: Set membership inclusion tests

A nominal controller is **validated** if all the following inclusion tests are true:

- 1 **Specification One:** $[\mathcal{R}_x]([t_r, t], \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_r) \subseteq \mathcal{T}_s$
(The target set is reached at the desired **reaching-time** t_r).
- 2 **Specification Two:** $[\mathcal{R}_x]([t_r, \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_0) \subseteq \mathcal{T}_s$
(The ultimate bound of the closed-loop system is enclosed in the **target set**).
- 3 **Specification Three:** $[\mathcal{R}_x]([t_0, t_r], \mathcal{P}, \mathcal{Y}, \mathcal{X}_0, t_0) \subseteq \mathcal{U}_x$
(The state constraints are not violated).
- 4 **Specification Four:** $\mathbf{k}(\mathcal{Y}, \mathbf{r}) \subseteq \mathcal{U}_u$
(The input constraints are satisfied).

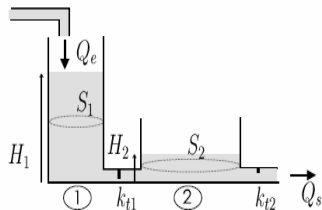
Case Study

- Let's consider a simple hydraulic system described by the following nonlinear model:

$$\begin{aligned}\dot{x}_1 &= \frac{1}{S_1} (u - k_1 \sqrt{x_1 - x_2}) \\ \dot{x}_2 &= \frac{1}{S_2} (k_1 \sqrt{x_1 - x_2} - k_2 \sqrt{x_2})\end{aligned}$$

- where the input $u = Q_e$ of the first tank stands for the liquid inflow and the liquid outflow of the second tank is the output $y = Q_s$ of the system, given by

$$y = k_2 \sqrt{x_2}$$



Point specifications

The desired performance and the physical constraints

Static error

$$|r - y| \leq \epsilon, \text{ where } \epsilon = 0.025m^3/s$$

Settling-time

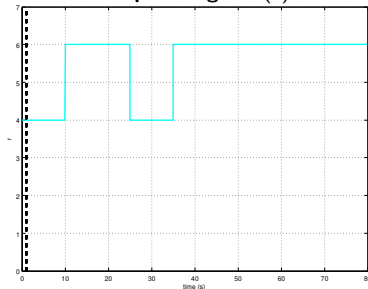
$$t_r \leq 0.9s$$

State constraints

$$0 \leq x_1 \leq 25m \text{ and } 0 \leq x_2 \leq 2.5m$$

Input constraint

$$0 \leq Q_e \leq 16m^3/s$$

The set-point signal $r(t)$ 

- The vertical line shows the desired settling-time.

Case Study

System parameters

$$S_1 = \frac{1}{6} m^2, S_2 = 1.5 m^2, k_1 = \sqrt{2} m^{2.5} s^{-1} \text{ and } k_2 = 4 m^{2.5} s^{-1}$$

Operating point

$$u_{op} = 4 m^3 s^{-1}, \mathbf{x}_{op}^T = (x_{1op} = 9m, x_{2op} = 1m)$$

- An engineer proposes to drive this system by a **linear control law**

$$u = -\mathbf{K}(\mathbf{x} - \mathbf{x}_{op}) + Gr + u_{op}$$

designed from the **linearized system**.

Linearized system

$$\delta \dot{\mathbf{x}} = \begin{bmatrix} -1.5 & 1.5 \\ 0.1667 & -1.5 \end{bmatrix} \delta \mathbf{x} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} \delta u \quad (1)$$

$$\delta y = \begin{bmatrix} 0 & 2 \end{bmatrix} \delta \mathbf{x} \quad (2)$$

Case Study

Linear control law (ideal case)

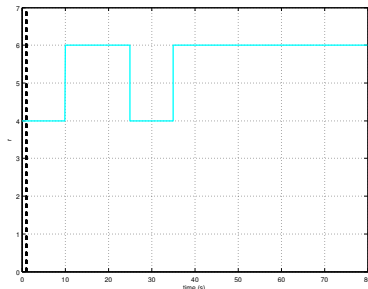
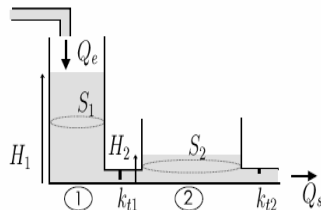
$$u = -\mathbf{K}(\mathbf{x} - \mathbf{x}_{op}) + Gr + u_{op}$$

By applying the pole placement technique one obtains,

- $\mathbf{K} = (0.2 \ 5.2)^T$: the state feedback gain,
- $G = 4.5$: the feedforward gain,

These numerical values for \mathbf{K} and G are computed for:

- settling time = 0.9s
- damping ratio = 0.7



The set-point signal

Case Study

Is it possible to validate a priori this linear controller?

Will this linear controller keep its performance in the real situations?

Case Study

The poorly-known real word

Uncertain initial state

$$\mathbf{x}_0 \in [0, 1]m \times [0, 0.55]m$$

Uncertain measurement

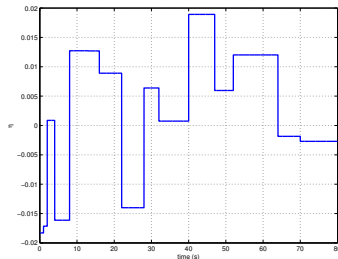
The measurement of the state vector $\mathbf{x}_m = \mathbf{y}_m$ is corrupted by unknown but bounded noise $\eta(t) \in \mathbb{R}^2$ with known bounds $\underline{\eta}$ and $\bar{\eta}$.

$$\mathbf{x} \in \mathbf{y}_m + [\underline{\eta}, \bar{\eta}] = \mathcal{Y}$$

where $[\underline{\eta}, \bar{\eta}] = [-2 \ 2]cm \times [-2 \ 2]cm$.

Nonlinear system

$$\begin{aligned}\dot{x}_1 &= \frac{1}{S_1} (u - k_1 \sqrt{x_1 - x_2}) \\ \dot{x}_2 &= \frac{1}{S_2} (k_1 \sqrt{x_1 - x_2} - k_2 \sqrt{x_2})\end{aligned}$$



- The assumed behavior of the additive measurement noise.

Set-membership specifications

Step 1: Set-membership formulation of the desired specifications

Target set

$$[\mathbf{T}_s] = \{(x_1, x_2) \mid y \in r + [-\epsilon, +\epsilon]\}, \text{ where} \\ \epsilon = 0.025m^3/s$$

Reaching time

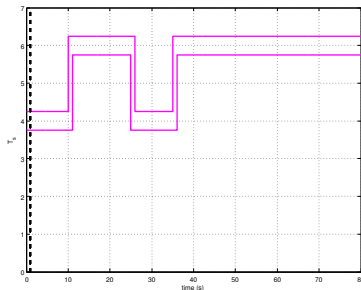
$$t_r \leq 1s$$

Safety set

$$[\mathbf{U}_x] = [0 \ 25]m \times [0 \ 2.5]m$$

Feasible set

$$[\mathbf{U}_u] = [0 \ 16]m^3/s$$



- The desired target set $\mathbf{T}_s(t)$ is the reciprocal image of the above reference output tube.

Bracketing systems (to compute an over approximation of the reachable set)

Step 2 : Compute an outer-approximation of the set reachable by the closed-loop system.

Applying Müller's Theorem

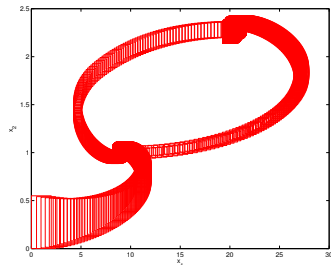
$$\begin{cases} \dot{\bar{x}}_1 &= \frac{1}{S_1} (\bar{u} - k_1 \sqrt{\bar{x}_1 - \bar{x}_2}) \\ \dot{\bar{x}}_2 &= \frac{1}{S_2} (k_1 \sqrt{\bar{x}_1 - \bar{x}_2} - k_2 \sqrt{\bar{x}_2}) \\ \dot{\underline{x}}_1 &= \frac{1}{S_1} (\underline{u} - k_1 \sqrt{\underline{x}_1 - \underline{x}_2}) \\ \dot{\underline{x}}_2 &= \frac{1}{S_2} (k_1 \sqrt{\underline{x}_1 - \underline{x}_2} - k_2 \sqrt{\underline{x}_2}) \end{cases}$$

- The bracketing systems provides this solution

...

The outer-approximation is tight

$$[\mathbf{R}_x](t, \mathcal{Y}, [\mathbf{x}_0], t_0) = [\underline{\mathbf{x}}(t), \bar{\mathbf{x}}(t)]$$



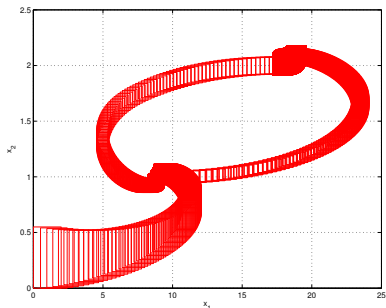
- The outer-approximation of the reachable set.

Simulation results (Test one)

Step 3 : Verify if the desired set-membership criteria are satisfied by all the possible state trajectories of the closed-loop system.

Safety set

$$[\mathbf{U}_x] = [0 \ 25]m \times [0 \ 2.5]m$$



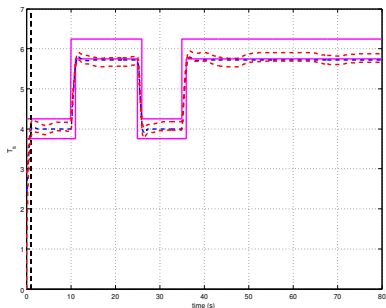
- This set-inclusion test has been successfully checked. The result is true.

Simulation results (Test one)

Step 3 : Verify if the desired set-membership criteria are satisfied by all the possible state trajectories of the closed-loop system.

Target set and Reaching time

$$[\mathbf{T}_s] = \{(x_1, x_2) \mid y \in r + [-\epsilon, +\epsilon]\}, \quad t_r \leq 1s$$



- This set-inclusion test fails. The test returns false.

Simulation results (Test one)

Conclusion

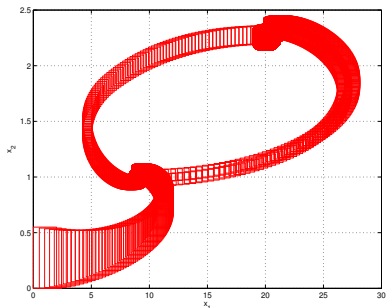
The proposed controller can not be validated.
Some specifications could be not satisfied in a real situation.

Simulation results (Test two, $G = 5.5$)

Step 3 : Verify if the desired set-membership criteria are satisfied by all the possible state trajectories of the closed-loop system.

Safety set

$$[\mathbf{U}_x] = [0 \ 25]m \times [0 \ 2.5]m$$



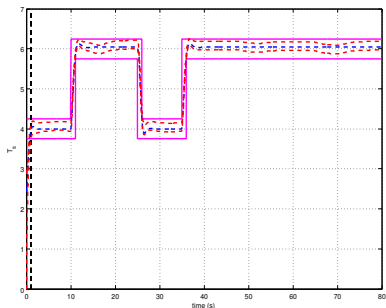
- This set-inclusion test has been successfully checked.

Simulation results (Test two, $G = 5.5$)

Step 3 : Verify if the desired set-membership criteria are satisfied by all the possible state trajectories of the closed-loop system.

Target set and Reaching time

$$[\mathbf{T}_s] = \{(x_1, x_2) \mid y \in r + [-\epsilon, +\epsilon]\}, \quad t_r \leq 1s$$

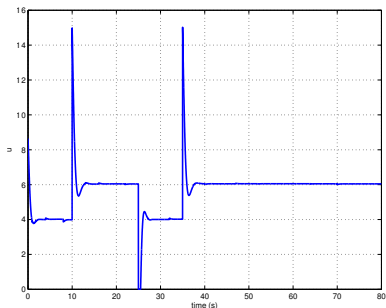


- This set-inclusion test has been successfully checked.

Simulation results (Test Two, $G = 5.5$)

Feasible set

$$[\mathbf{U}_u] = [0 \ 16] m^3/s$$



- This set-inclusion test has been successfully checked.

Simulation results (Test two)

Conclusion

The proposed controller is validated.
It preserves its performance in all the considered actual situations.

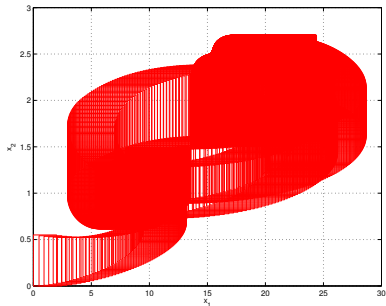
Simulation results (Test Three, $\eta = [-10, 10]cm \times [-10, 10]cm$)

Consider a case where the measurements have poor quality:

$$\eta = [-10, 10]cm \times [-10, 10]cm$$

Safety set

$$[\mathbf{U}_x] = [0 \ 25]m \times [0 \ 2.5]m$$



- This set-inclusion test fails. The result is false.

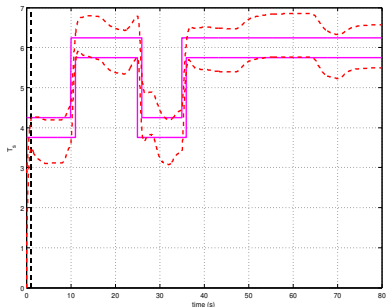
(Test Three, $\eta = [-10, 10]cm \times [-10, 10]cm$)

Consider a case where the measurements have poor quality:

$$\eta = [-10, 10]cm \times [-10, 10]cm$$

Target set and Reaching time

$$[\mathbf{T}_s] = \{(x_1, x_2) \mid y \in r + [-\epsilon, +\epsilon]\}, \quad t_r \leq 1s$$



- This set-inclusion test fails. The result is false.

Simulation results (Test three)

Conclusion

The proposed controller can not be validated.
With very poor quality measurements, the linear controller can not drive the real system.

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Problem Statement

N. Meslem, N. Ramdani. 'Reliable stabilizing controller based on set-value parameter synthesis'. IMA Journal of Mathematical Control and Information, 2017, pp.159–178.

Uncertain system (differential inclusion):

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{u}) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}) \end{cases}$$

where,

- $\mathbf{p} \in [\mathbf{p}] \in \mathbb{R}^P$: the uncertain parameter vector,
- $\mathbf{x}_0 \in [\mathbf{x}] \in \mathbb{R}^n$: the uncertain initial state.

Assumption

- The structure of the control law $\mathbf{u} = \mathbf{h}(\mathbf{x}_m, \mathbf{k})$ is known.

Closed-loop system (differential inclusion):

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{h}(\mathbf{x}_m, \mathbf{k})) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}) \end{cases}$$

Problem Statement

N. Meslem, N. Ramdani. 'Reliable stabilizing controller based on set-value parameter synthesis'. IMA Journal of Mathematical Control and Information, 2017, pp.159–178.

Objective:

Based on reachability analysis, one tries to characterize a set for the tuning parameters, gathered in the vector \mathbf{k} , such that:

- **Test 1:** $[\mathcal{R}_x]([t_r, t], \mathcal{P}, \mathcal{X}_0, t_r) \subset \mathcal{T}_s$
(The target set is reached at the desired time t_r).
- **Test 2:** $[\mathcal{R}_x]([t_r, \mathcal{P}, \mathcal{X}_0, t_0) \subset \mathcal{T}_s$
(The ultimate bound of the closed-loop system is enclosed in the **target set**).
- **Test 3:** $[\mathcal{R}_x]([t_0, t_r], \mathcal{P}, \mathcal{X}_0, t_0) \subset \mathcal{U}_x$
(The state constraints are not violated).
- **Test 4:** $\mathbf{u} = \mathbf{h}([\mathbf{x}_m], \mathbf{k}) \subset \mathcal{U}_u$
(The input constraints are satisfied).

SIVIA-like algorithm

Algorithm: Robust-Tuning-Parameters($f, h, [x_0], [p], [k], \epsilon$)

- $[\mathcal{R}_x] := \text{Reachable-Set}_{f,h}([x_0], [p], [k])$
- **if** :
 - **Test 1** is satisfied (**inclusion test on the reaching time**) and
 - **Test 2** is satisfied (**inclusion test on the target set**) and
 - **Test 3** is satisfied (**inclusion test on the safety set**) and
 - **Test 4** is satisfied (**inclusion test on the feasible control set**)
 - **Return** $[k]$ is a solution
- **else**
 - **if** $w([k]) \geq \epsilon$
 - $([k]^R, [k]^I) := \text{Bisect}([k])$
 - $[\mathcal{R}_x] := \text{Reachable-Set}_{f,h}([x_0], [p], [k]^R)$
 - $[\mathcal{R}_x] := \text{Reachable-Set}_{f,h}([x_0], [p], [k]^I)$
 - **else**
 - **Return** No solution
- **End**

Biological system

- Let's consider the Goldbeter model (Angeli & Sontag, 2008)

Nonlinear dynamical system

$$\begin{aligned} \dot{x}_1 &= u - \frac{v_m x_1}{\gamma_m + x_1} \\ \dot{x}_2 &= \gamma_s x_1 - \frac{v_1 x_2}{\gamma_1 + x_2} + \frac{v_2 x_3}{\gamma_2 + x_3} \\ \dot{x}_3 &= \frac{v_1 x_2}{\gamma_1 + x_2} - \frac{v_2 x_3}{\gamma_2 + x_3} - \frac{v_3 x_3}{\gamma_3 + x_3} + \frac{v_4 x_4}{\gamma_4 + x_4} \\ \dot{x}_4 &= \frac{v_3 x_3}{\gamma_3 + x_3} - \frac{v_4 x_4}{\gamma_4 + x_4} - \Gamma_1 x_4 + \Gamma_2 x_5 - \frac{v_d x_4}{\gamma_d + x_4} \\ \dot{x}_5 &= \Gamma_1 x_4 - \Gamma_2 x_5 \end{aligned}$$

Table: Parameter values

Parameter	Value	Parameter	Value
Γ_2	1.3	Γ_1	1.9
v_1	3.2	v_2	1.58
v_3	5	v_4	2.5
q_1	--	γ_m	0.5
γ_s	0.38	v_d	0.95
γ_d	0.2	n	4
γ_1	2	γ_2	2
γ_3	2	γ_4	2
q_2	--	v_m	0.65

The structure of the proposed stabilizing controller

$$u = \frac{q_1}{q_2^n + x_5^n}$$

with control parameter q_1 , q_2 to tune.

Objective

Aim: Characterize a set of control parameters q_1, q_2 such that all the trajectories of the closed-loop system

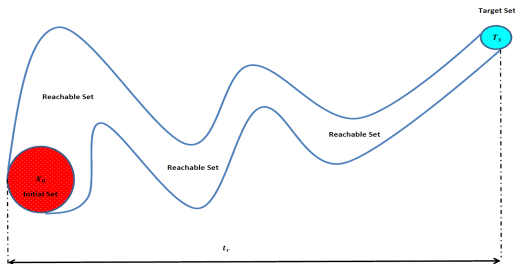
- leave **the box of possible initial conditions**

$$[\mathbf{x}_0] = [0.1, 0.4] \times [0.6, 2.4] \times [0.85, 3.4] \times [0.25, 1] \times [0.5, 2]$$

- to reach **the desired target set**

$$\mathbf{T}_s = [0.78, 0.82] \times [0.29, 0.32] \times [0.15, 0.22] \times [0.08, 0.11] \times [0.10, 0.15]$$

- at **the reaching time** $t_r = 50s$



Bracketing systems

- The dynamics of the maximal solution

$$\left\{ \begin{array}{l} \dot{\bar{x}}_1 = \frac{\bar{q}_1}{\bar{q}_2^n + \bar{x}_5^n} - \frac{v_m \bar{x}_1}{\gamma_m + \bar{x}_1} \\ \dot{\bar{x}}_2 = \gamma_s \bar{x}_1 - \frac{v_1 \bar{x}_2}{\gamma_1 + \bar{x}_2} + \frac{v_2 \bar{x}_3}{\gamma_2 + \bar{x}_3} \\ \dot{\bar{x}}_3 = \frac{v_1 \bar{x}_2}{\gamma_1 + \bar{x}_2} - \frac{v_2 \bar{x}_3}{\gamma_2 + \bar{x}_3} - \frac{v_3 \bar{x}_3}{\gamma_3 + \bar{x}_3} + \frac{v_4 \bar{x}_4}{\gamma_4 + \bar{x}_4} \\ \dot{\bar{x}}_4 = \frac{v_3 \bar{x}_3}{\gamma_3 + \bar{x}_3} - \frac{v_4 \bar{x}_4}{\gamma_4 + \bar{x}_4} - \Gamma_1 \bar{x}_4 + \Gamma_2 \bar{x}_5 - \frac{v_d \bar{x}_4}{\gamma_d + \bar{x}_4} \\ \dot{\bar{x}}_5 = \Gamma_1 \bar{x}_4 - \Gamma_2 \bar{x}_5 \end{array} \right. \quad (3)$$

- The dynamics of the minimal solution

$$\left\{ \begin{array}{l} \dot{\underline{x}}_1 = \frac{q_1}{\bar{q}_2^n + \bar{x}_5^n} - \frac{v_m \underline{x}_1}{\gamma_m + \underline{x}_1} \\ \dot{\underline{x}}_2 = \gamma_s \underline{x}_1 - \frac{v_1 \underline{x}_2}{\gamma_1 + \underline{x}_2} + \frac{v_2 \underline{x}_3}{\gamma_2 + \underline{x}_3} \\ \dot{\underline{x}}_3 = \frac{v_1 \underline{x}_2}{\gamma_1 + \underline{x}_2} - \frac{v_2 \underline{x}_3}{\gamma_2 + \underline{x}_3} - \frac{v_3 \underline{x}_3}{\gamma_3 + \underline{x}_3} + \frac{v_4 \underline{x}_4}{\gamma_4 + \underline{x}_4} \\ \dot{\underline{x}}_4 = \frac{v_3 \underline{x}_3}{\gamma_3 + \underline{x}_3} - \frac{v_4 \underline{x}_4}{\gamma_4 + \underline{x}_4} - \Gamma_1 \underline{x}_4 + \Gamma_2 \underline{x}_5 - \frac{v_d \underline{x}_4}{\gamma_d + \underline{x}_4} \\ \dot{\underline{x}}_5 = \Gamma_1 \underline{x}_4 - \Gamma_2 \underline{x}_5 \end{array} \right. \quad (4)$$

Algorithm

- The considered initial search domain for the control parameters

$$\mathbb{Q} = [0, 4] \times [0, 4] = [q_1] \times [q_2] = [\mathbf{k}]$$

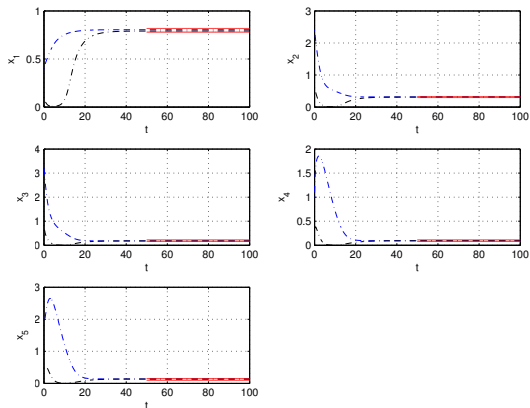
- The considered stopping threshold

$$\epsilon = 0.001$$

Robust-Tuning-Parameters($f, h, [x_0], [k], \epsilon$)

- $[\mathcal{R}_x] := \text{Reachable-Set}_{f,h}([x_0], [k])$ (here, the upper and lower bracketing systems are used)
- if :
 - Test 1 is satisfied (inclusion test on the reaching time) and
 - Test 2 is satisfied (inclusion test on the target set),
 - Return $[k]$ is a solution
- else
 - if $w([k]) \geq \epsilon$
 - $([k]^R, [k]^L) := \text{Bisect}([k])$
 - $[\mathcal{R}_x] := \text{Reachable-Set}_{f,h}([x_0], [k]^R)$
 - $[\mathcal{R}_x] := \text{Reachable-Set}_{f,h}([x_0], [k]^L)$
 - else
 - Return No solution
- End

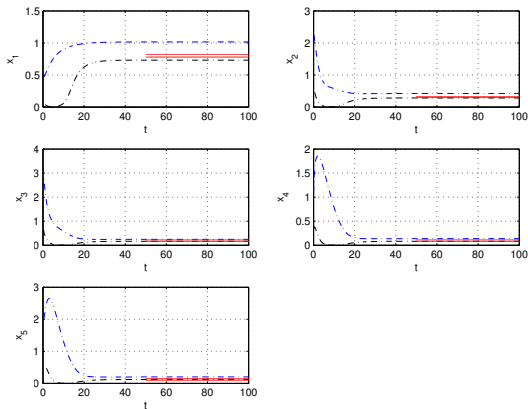
Inclusion Test (accept)



- Inclusion test where the control specifications are satisfied.

Figure: An outer-approximation of the reachable set of the system computed with the box of control parameters $[\mathbf{q}] = [0.4, 0.401] \times [1, 1.001]$.

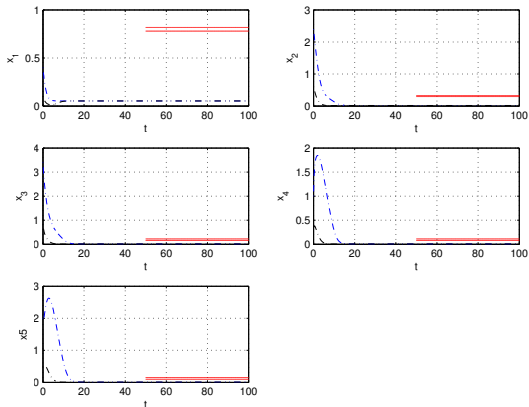
Inclusion Test (bisect)



- Inclusion test where no reliable decision can be taken.

Figure: An outer-approximation of the reachable set of the system computed with the box of control parameters $[\mathbf{q}] = [0.39, 0.402] \times [0.98, 1.002]$.

Inclusion Test (reject)



- Inclusion test which proves that no solution exists.

Figure: An outer-approximation of the reachable set of the system computed with the box of control parameters $[\mathbf{q}] = [1, 1.002] \times [2, 2.002]$.

Result

Inner solution set

The identified set of the control parameters. Any point vector taken in this set and used within the control law will ensure the stability of the nonlinear system.

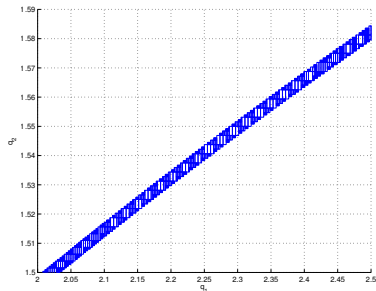
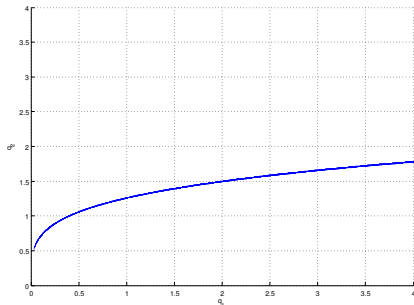


Figure: Inner approximation of the feasible control parameters set. The right picture shows a zoom of the inner approximation over the box $[2 \ 2.5] \times [1.5 \ 1.6]$.

Outline

- 1 Control Validation
 - Motivation
 - Problem formulation
 - Illustrative example
- 2 Control Synthesis
 - Problem statement
 - SIVIA algorithm
 - Illustrative example
- 3 Conclusion

Concluding Remarks

Conclusion

Interval analysis and reachability computation combined with set-inclusion tests can be used to effectively:

- numerically prove the robustness of a given nominal controller,
- design a robust controller for uncertain systems.