An interval approach to compute Initial Value Problem (IVP)

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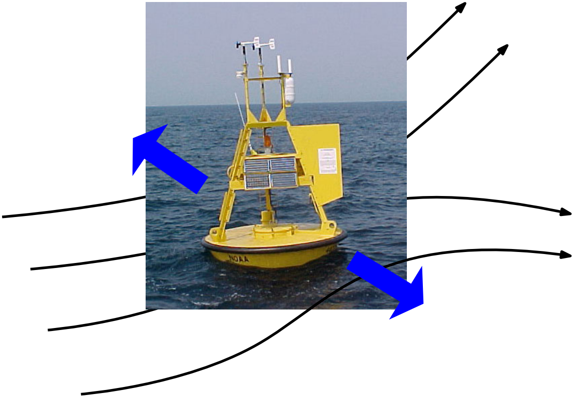
Motivation

→ Robot with under powered propeller



→ Use currents as the main source of energy

Outline

1 Problem Formalization

2 Idea

3 Algorithm for IVP

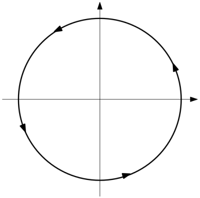
4 Examples

Problem Formalization (1/3)

1. Trajectory: smooth function $\mathbf{x}(\cdot)$ from $\mathbb{R}$ to $\mathbb{R}^n$.
2. Path: set of all $\mathbf{x}(t) \in \mathbb{R}^n$ and an orientation with respect to t.

Example

$$\mathbf{x}(\cdot) : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^2 \\ t & \mapsto & \begin{pmatrix} cos(t) \\ sin(t) \end{pmatrix} \end{array} \right.$$

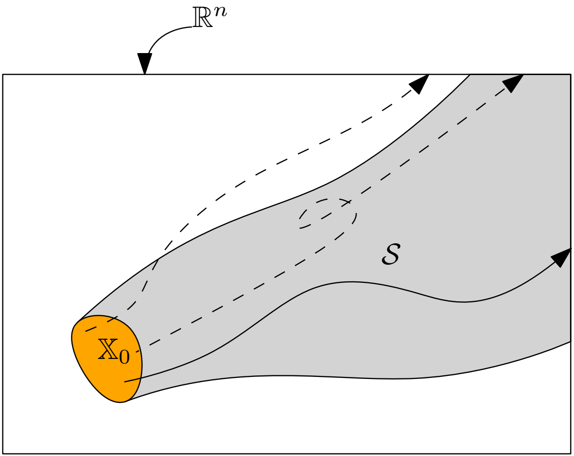


Problem Formalization (2/3)

Problem

Find the set $\mathcal{S}$ which encloses all points of $\mathbb{R}^n$ which belong to a positive path solution of $\begin{cases} \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right) \\ \mathbf{x}(0) \in \mathbb{X}_0 \end{cases}$.

Problem Formalization (3/3)

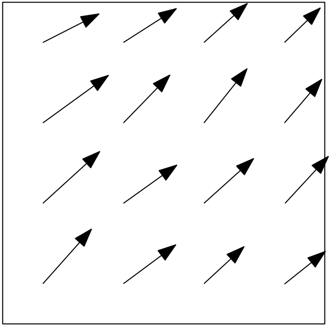


Outline

1. Problem Formalization
2. Idea
3. Algorithm for IVP
4. Examples

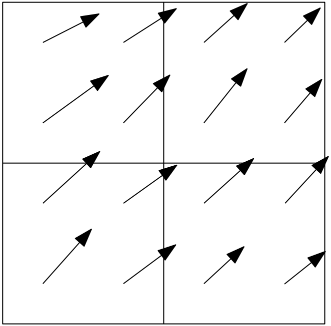
State space and differential inclusion

→ Vector field of the state equation



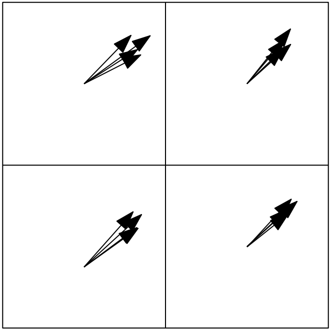
State space and differential inclusion

→ Build a subpaving of state space



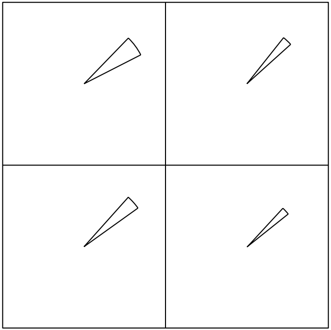
State space and differential inclusion

→ Compute a outer propagation cone with interval arithmetic



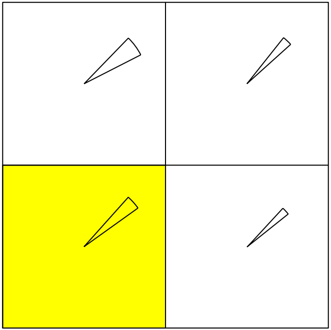
State space and differential inclusion

→ Compute a outer propagation cone with interval arithmetic



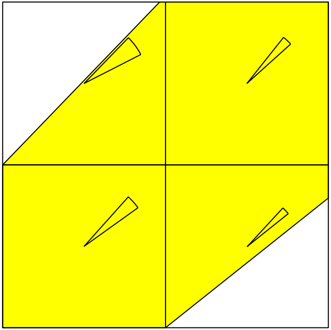
State space and differential inclusion

→ Enclose the initial condition in the subpaving



State space and differential inclusion

→ Propagate the initial condition according to cones across the subpaving



Summary

Main ideas:

1. Build a subpaving of the state space
2. Compute an outer propagation cone: $[\mathbf{f}]([\mathbf{x}])$
3. Propagate the initial condition according to cones

Notes:

1. Smaller pave $\Rightarrow$ better outer cones
2. Only follow directions without considering norms (=time)

Outline

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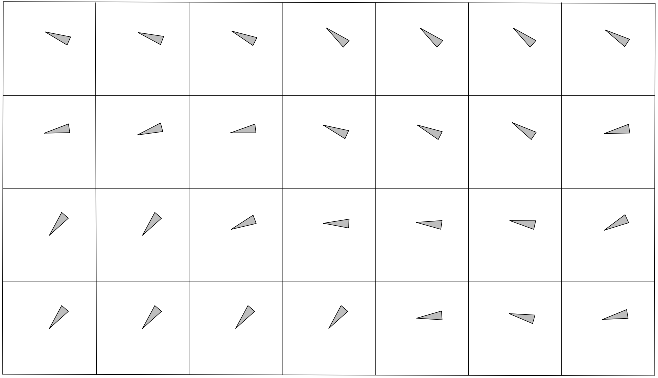
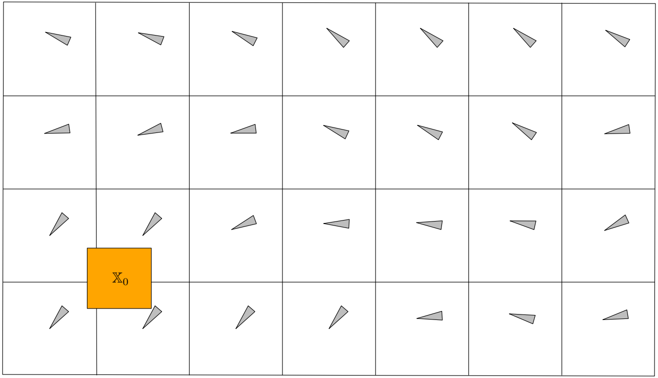
Algorithm

→ Build a subpaving of $\mathbb{R}^n$ and compute cones



Algorithm

→ Intersect the initial condition with the subpaving



Algorithm

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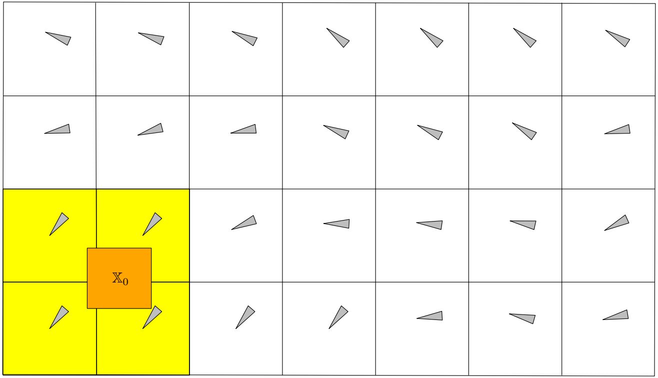
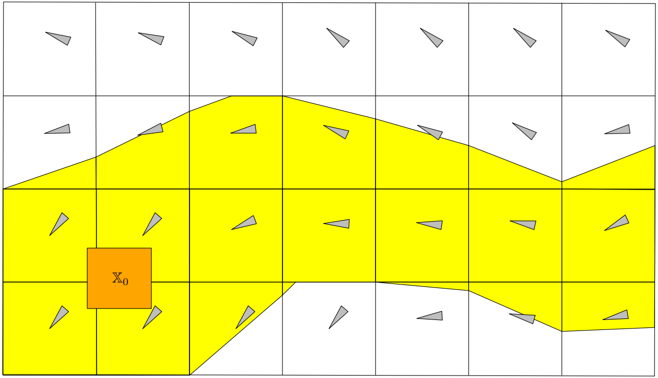


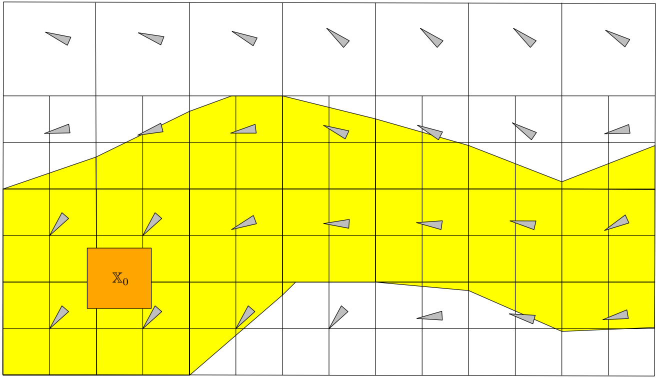
Algorithm

→ Propagate the polyhedron according to the cones



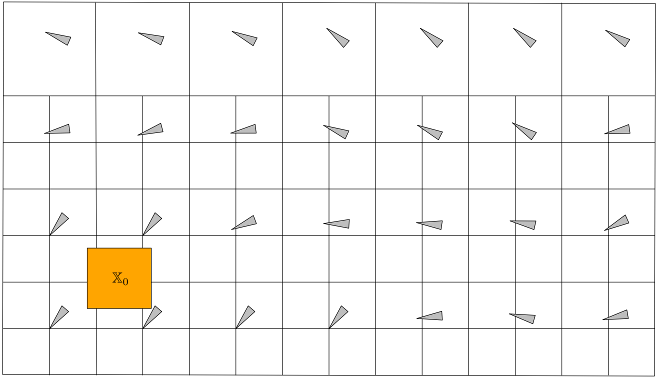
Algorithm

→ Bisect yellow paves



Algorithm

→ Compute cones and intersect the initial condition...



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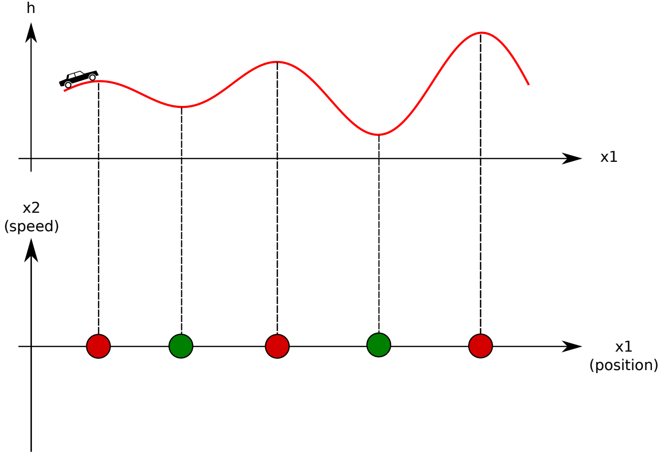
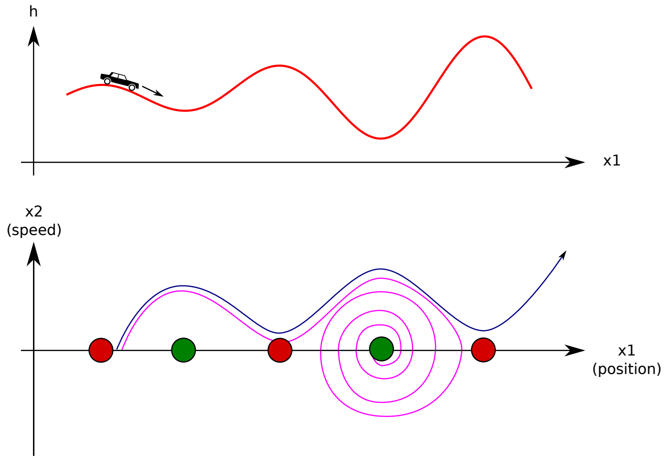
Car On the Hill

$$\begin{cases} \dot{x_1} = & x_2 \\ \dot{x_2} = & -9.81 sin(\frac{dg}{dx_1}(x_1)) - 0.7x_2 + u \end{cases}$$



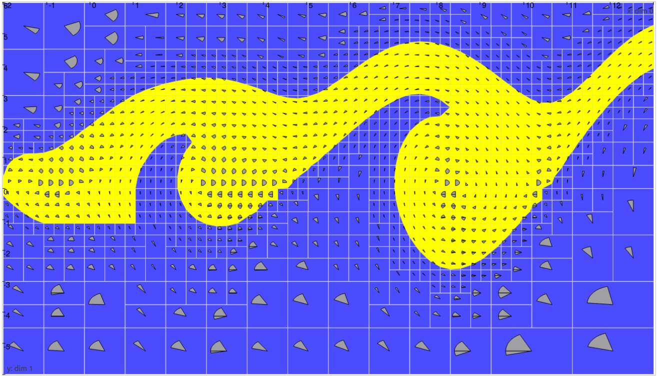
Car On the Hill

$\rightarrow u = 2$



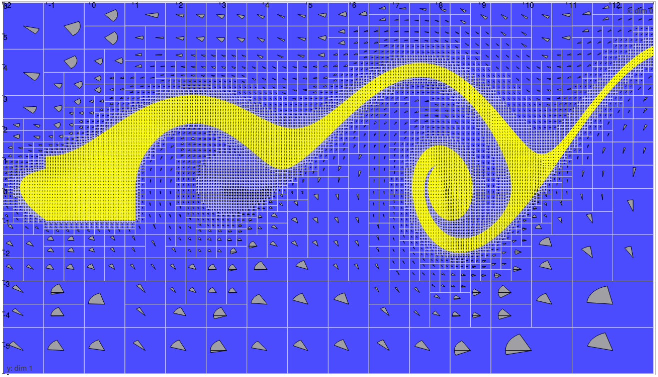
Car On the Hill

$\rightarrow$ Step 10, $\mathbb{X}_0 = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$



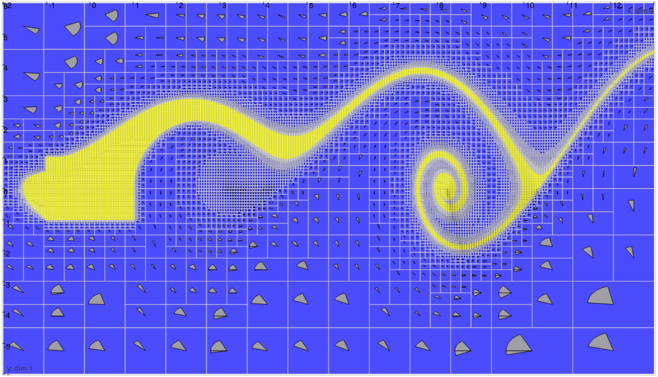
Car On the Hill

$\rightarrow$ Step 15, $\mathbb{X}_0 = \begin{pmatrix} [-1, 1] \\ [-1, 1] \end{pmatrix}$



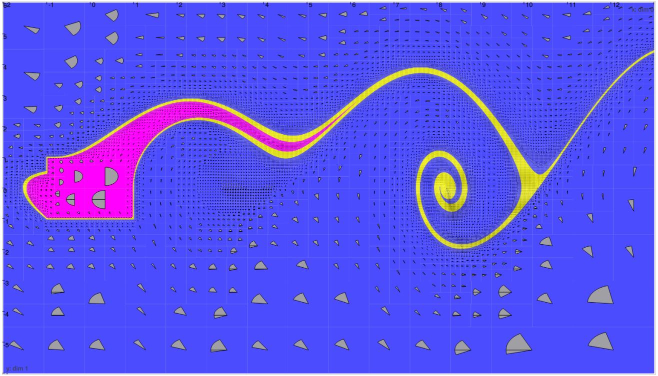
Car On the Hill

$\rightarrow$ Step 18, $\mathbb{X}_0 = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$



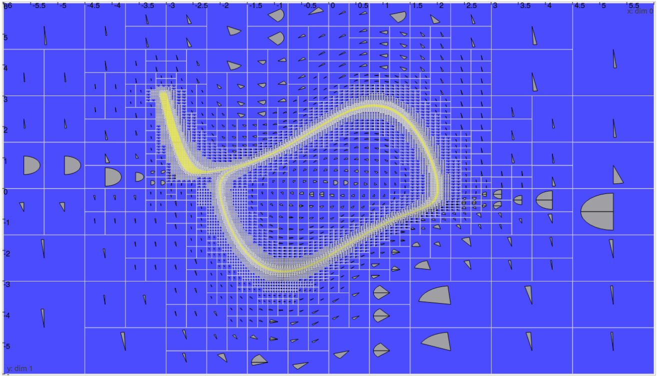
Car On the Hill

$\rightarrow$ Step 18 with inner, $\mathbb{X}_0 = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$



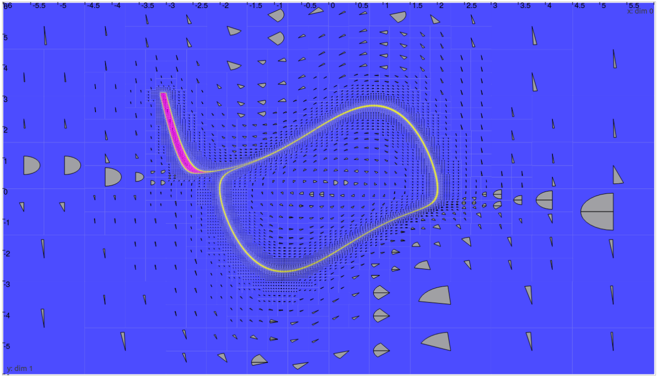
Van Der Pol

$$\begin{cases} \dot{x_1} = & x_2 \\ \dot{x_2} = & (1 - {x_1}^2) \cdot x_2 - x_1 \end{cases}, \mathbb{X}_0 = \begin{pmatrix} [-3.1, -3] \\ [3, 3.1] \end{pmatrix}$$



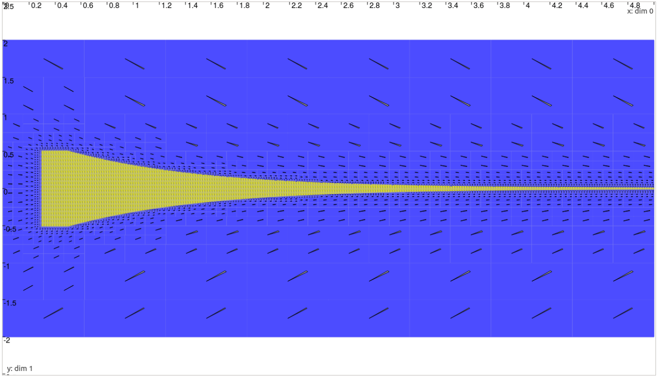
Van Der Pol (with inner)

$$\begin{cases} \dot{x_1} = & x_2 \\ \dot{x_2} = & (1 - {x_1}^2) \cdot x_2 - x_1 \end{cases}, \mathbb{X}_0 = \begin{pmatrix} [-3.1, -3] \\ [3, 3.1] \end{pmatrix}$$



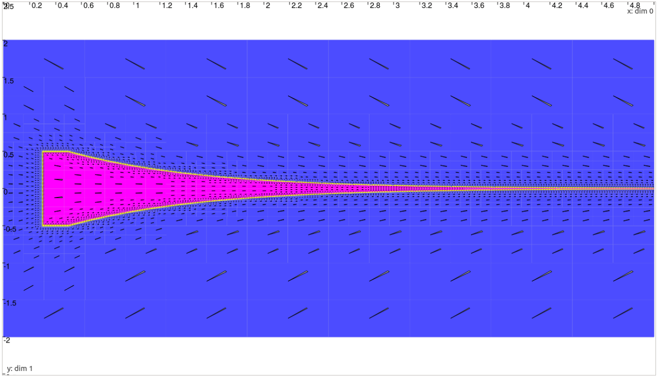
Sinusoidal function

$$\begin{cases} \dot{x_1} = & 1 \\ \dot{x_2} = & -sin(x_2) \end{cases}, \mathbb{X}_0 = \begin{pmatrix} [0.0, 0.5] \\ [-2, 2] \end{pmatrix}$$



Sinusoidal function (with inner)

$$\begin{cases} \dot{x_1} = & 1 \\ \dot{x_2} = & -sin(x_2) \end{cases}, \mathbb{X}_0 = \begin{pmatrix} [0.0, 0.5] \\ [-2, 2] \end{pmatrix}$$



Questions

Thank you

Car On the Hill equations

$$\begin{cases} \dot{x_1} = & x_2 \\ \dot{x_2} = & -9.81 sin(\frac{dg}{dx_1}(x_1)) - 0.7x_2 + u \end{cases}$$

$$g : s \rightarrow \frac{-\frac{1.1}{1.2}cos(s)+\frac{1.2}{1.1}cos(1.1s)}{2}$$