Thick Set Inversion

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Outlines

Problem statement

Classical set inversion

With $\mathbb{Y} \in \mathbb{R}^p$, and $\mathbf{f} : \mathbb{R}^n \longmapsto \mathbb{R}^p$.
Set inversion algorithm aims at characterizing the set :

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

In our case :

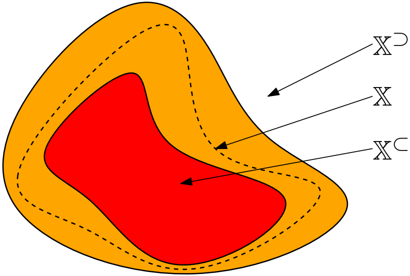
1. $\mathbb{Y}$ is an uncertain set (*thick set*)
2. $\mathbf{f}$ is an uncertain function (*thick function*)

Thick set

Definition

A *thick set* $[\![\mathbb{X}]\!]$ of $\mathbb{R}^n$ is an interval of $(\mathcal{P}(\mathbb{R}^n), \subset)$. such as :

$$\begin{aligned} [\![\mathbb{X}]\!] &= [\mathbb{X}^{\subset}, \mathbb{X}^{\supset}] \\ &= \{\mathbb{X} \in \mathcal{P}(\mathbb{R}^n) \mid \mathbb{X}^{\subset} \subset \mathbb{X} \subset \mathbb{X}^{\supset}\} \end{aligned}$$



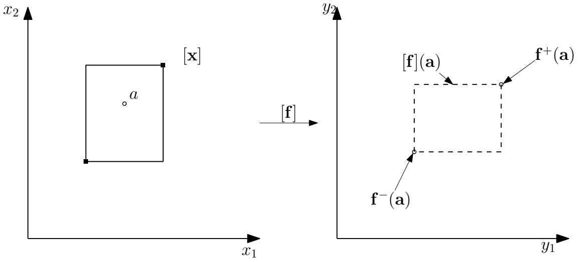
Thick Set

Thick function

Definition

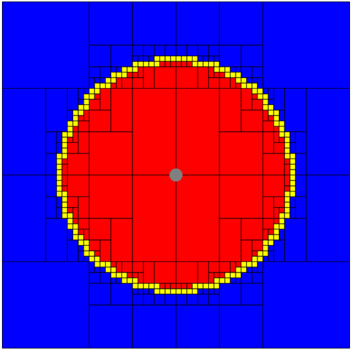
A *thick function* $[\mathbf{f}] : \mathbb{R}^n \longrightarrow \mathbb{IR}^m$ is defined by:

$$\begin{aligned}[\mathbf{f}] &= [\mathbf{f}^-, \mathbf{f}^+] \\ &= \{\mathbf{f} \in \mathcal{F}(\mathbb{R}^n, \mathbb{IR}^m) \mid \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{f}^-(\mathbf{x}) \leq \mathbf{f}(\mathbf{x}) \leq \mathbf{f}^+(\mathbf{x})\}\end{aligned}$$



Example of thick inversion problem

$$\begin{cases} f(\mathbf{x}) & = & \sqrt{(x_1 - m_1)^2 + (x_2 - m_2)^2} \\ \mathbb{Y} & = & [0, 2] \end{cases}$$



$\mathbf{m} = (1, -2)$

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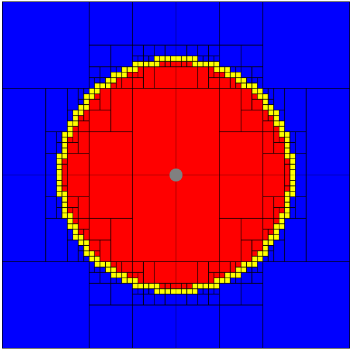

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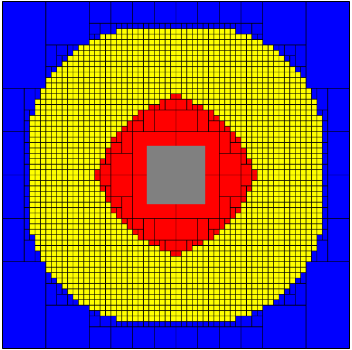

$\mathbf{m} \in [0.5, 1.5] \times [-2.5, -1.5]$

Thick set inversion

Theorem

Given $[\mathbf{f}] \subset \mathcal{F}(\mathbb{R}^n, \mathbb{IR}^m)$ and $[\![\mathbb{Y}]\!] = [\![\mathbb{Y}^{\subset}, \mathbb{Y}^{\supset}]\!]$, the smallest thick set solution of $[\![\mathbb{X}]\!] = [\mathbf{f}]^{-1}([\![\mathbb{Y}]\!])$ is:

$$\begin{aligned} [\![\mathbb{X}]\!] &= [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!] \\ &= \left[\bigcap_{\mathbf{f} \in [\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\subset}), \bigcup_{\mathbf{f} \in [\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\supset}) \right] \end{aligned}$$

Outlines

Thick interval

Definition

A *thick interval* $[\![x]\!]$ is a subset of $\mathbb{IR}$ which can be written under the form

$$\begin{aligned} [\![x]\!] &= [\![[x^-],[x^-]]\!] \\ &= \{[x^-,x^+] \in \mathbb{IR} \mid x^- \subset [x^-] \text{ and } x^+ \subset [x^+]\} \end{aligned}.$$

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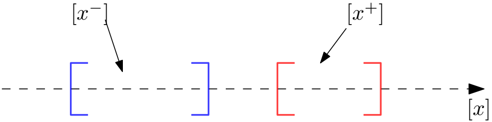

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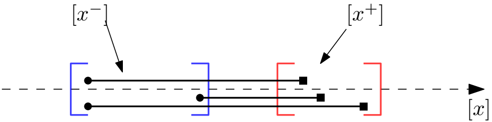
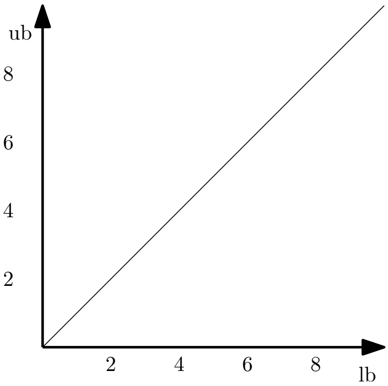
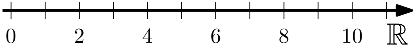


Endpoints diagram

Representation of classical intervals





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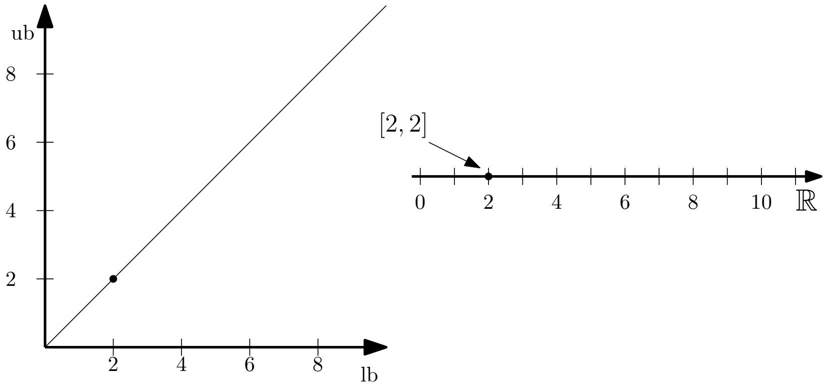

Endpoints diagram
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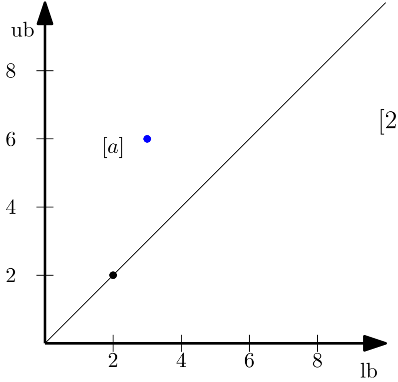

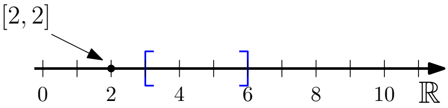

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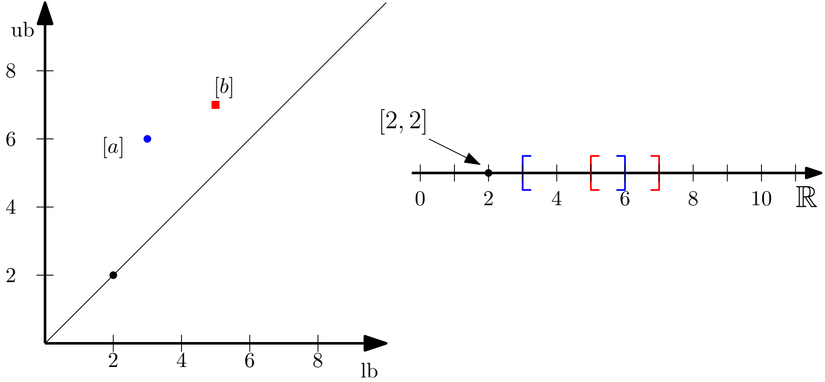

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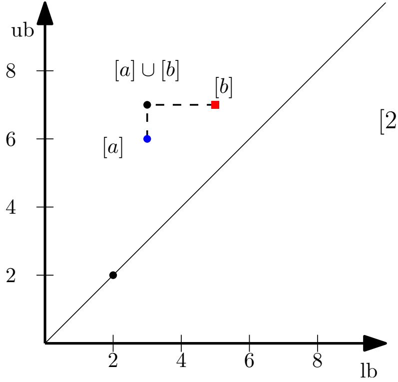

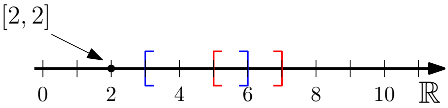

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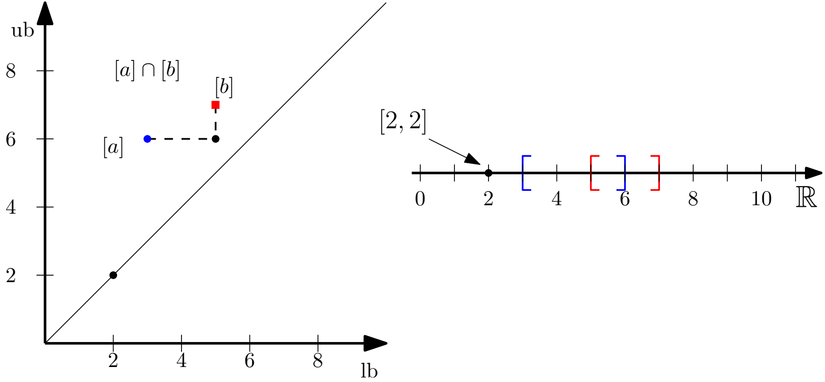
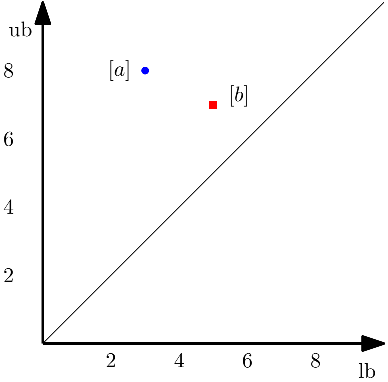
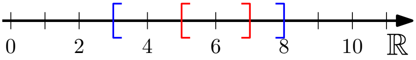


Endpoints diagram

Comparison $[b] \subset [a]$





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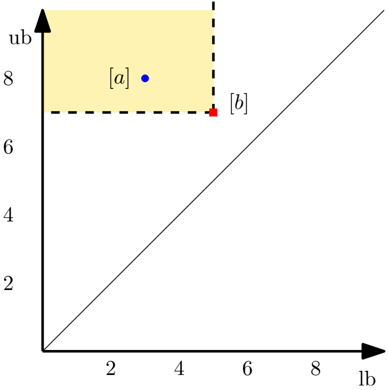
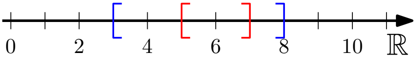
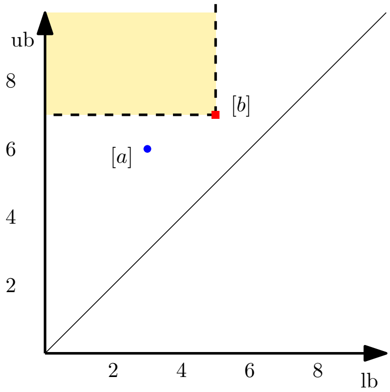
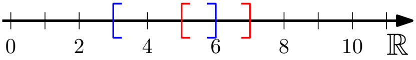




Endpoints diagram
Comparison $[b] \cap [a] \neq \emptyset$





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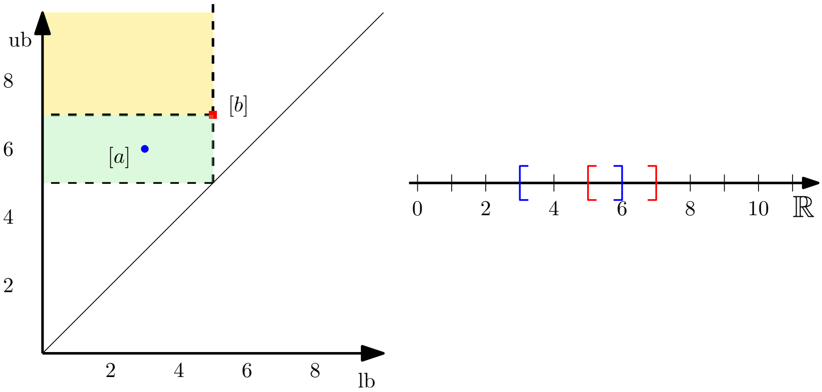

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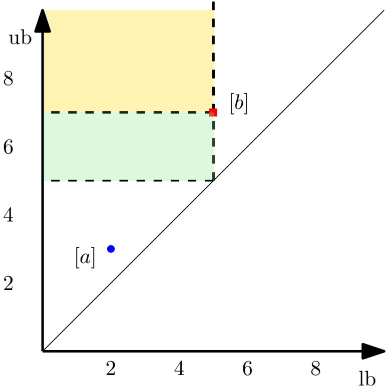

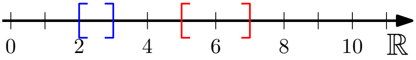

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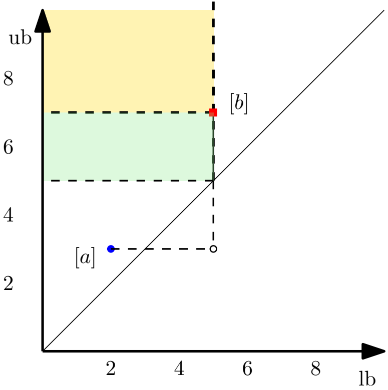

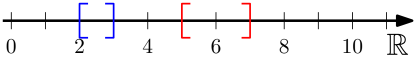

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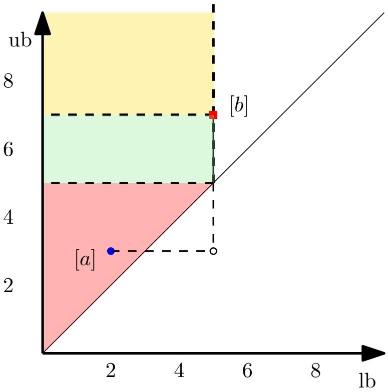
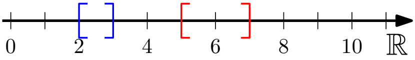
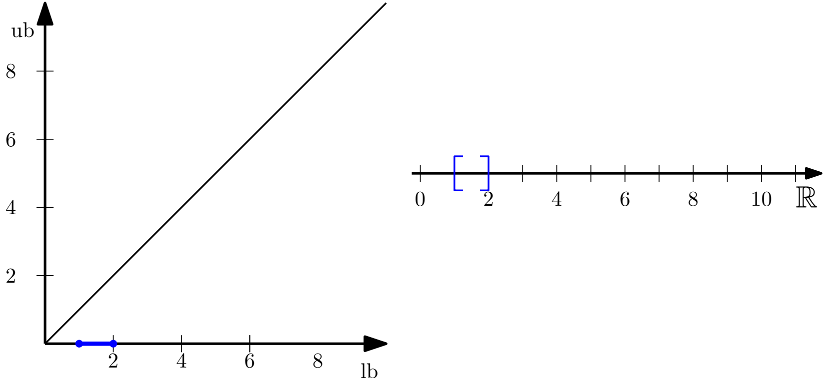




EndPoints Diagram
Thick Interval



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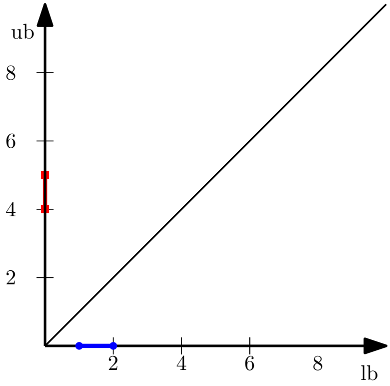

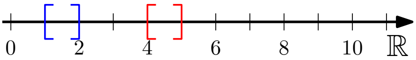

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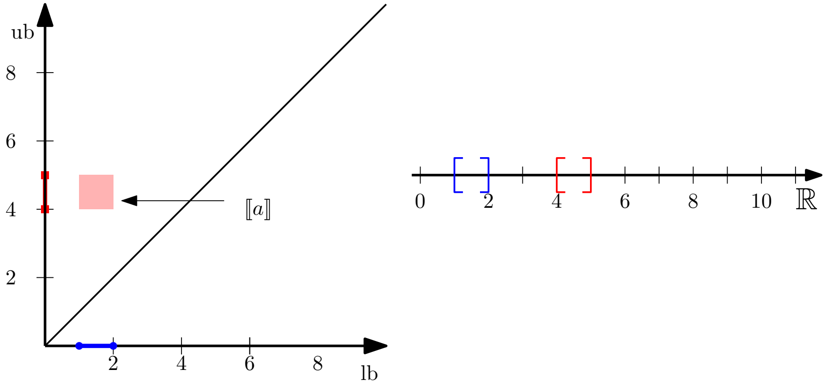

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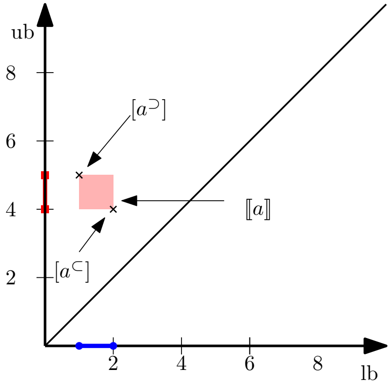

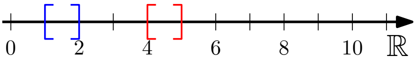

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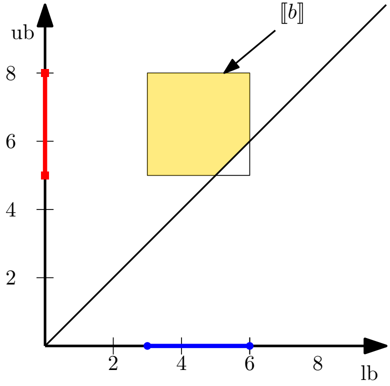

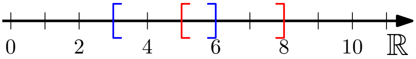

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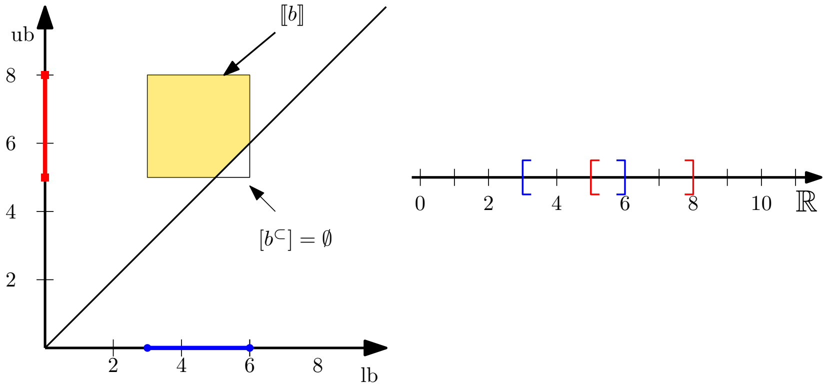

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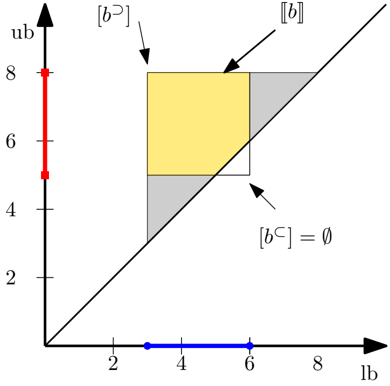
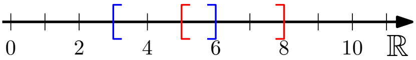




$$\emptyset \subset [\![b]\!] \subset [b^{\supset}]$$

Test between two thick intervals

Properties:

Given $[\![a]\!] = [\![[a^-], [a^+]]\!]$ and $[\![b]\!] = [\![[b^-], [b^+]]\!]$, we have:

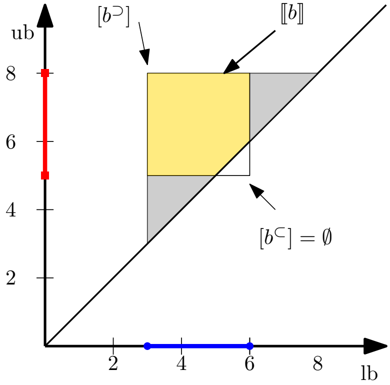
- $([\![a]\!] \cap [\![b]\!] \neq \emptyset)^\forall \Leftrightarrow \forall [a] \in [\![a]\!], \forall b \in [\![b]\!], [a] \cap [b] \neq \emptyset$
- $([\![a]\!] \not\subset [\![b]\!])^\forall \Leftrightarrow \forall [a] \in [\![a]\!], \forall b \in [\![b]\!], [a] \not\subset [b]$

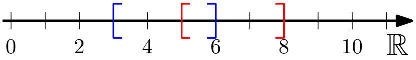
Example

Thick Intervals are more accurate than thick set representation.

Endpoints diagram

Example $([a] \cap [\![b]\!] \neq \emptyset)^{\forall}$





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Endpoints diagram

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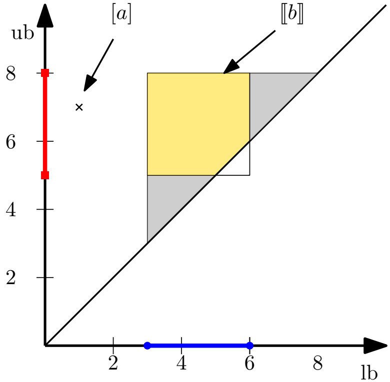

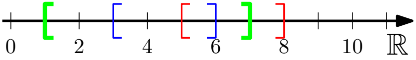

Endpoints diagram

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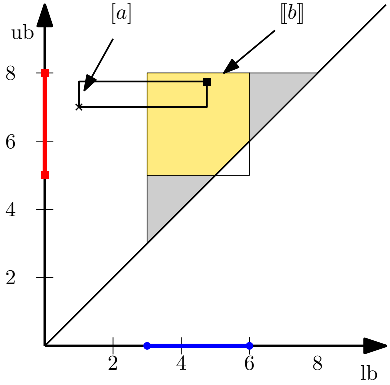

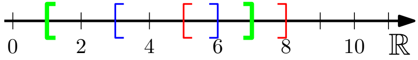

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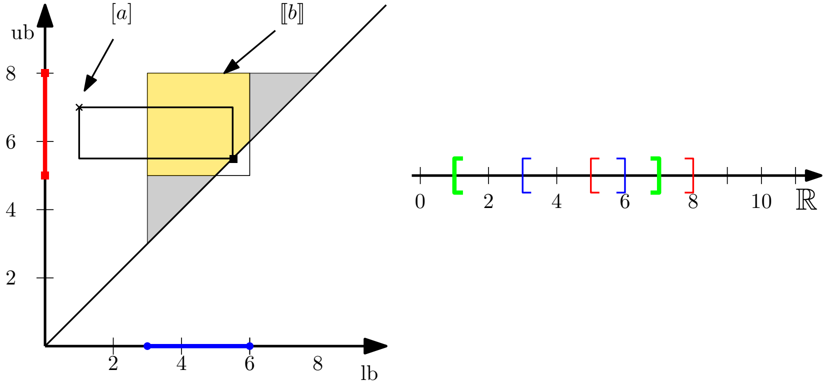

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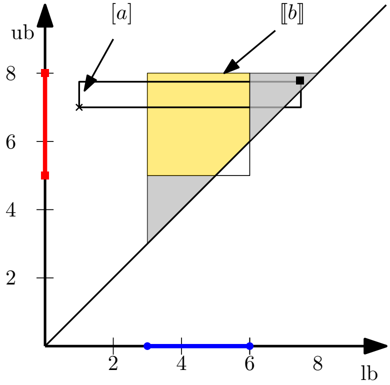
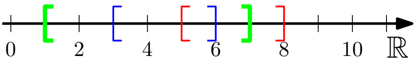




Thick Box

Definition

A *thick box* $[\![\mathbf{x}]\!]$ is a set of boxes of $\mathbb{IR}^n$ which can be defined as

$$[\![\mathbf{x}]\!] = \left\{ \left[\mathbf{x}^-, \mathbf{x}^+\right] \in \mathbb{IR}^n \mid \mathbf{x}^- \subset \left[\mathbf{x}^-\right] \text{ and } \mathbf{x}^+ \subset \left[\mathbf{x}^+\right] \right\}$$

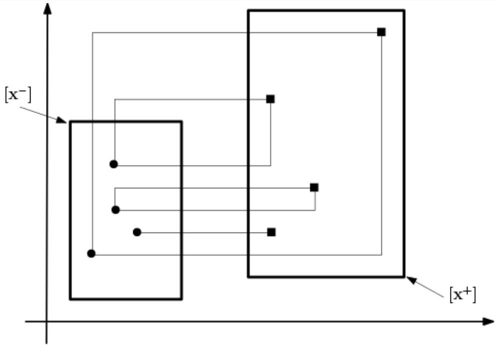
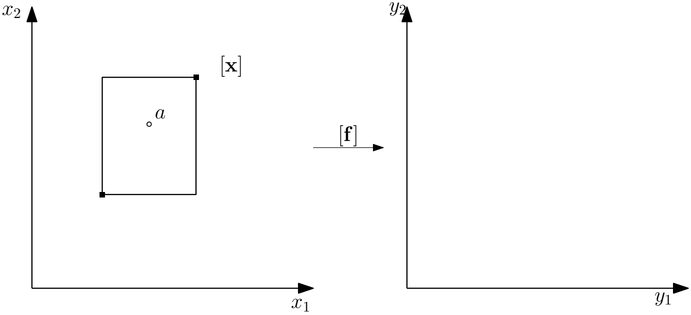


Image of a box through a thick function

Definition

Given a thick function $\mathbb{IR}^n$ to $\mathbb{IIR}^m$, we have :

$$[\![\mathbf{f}]\!]([\mathbf{x}]) = [\![[\mathbf{f}^-]([\mathbf{x}]), [\mathbf{f}^+]([\mathbf{x}])]\!], \quad (1)$$



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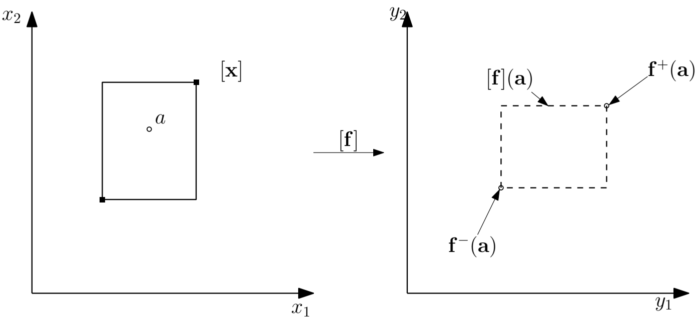

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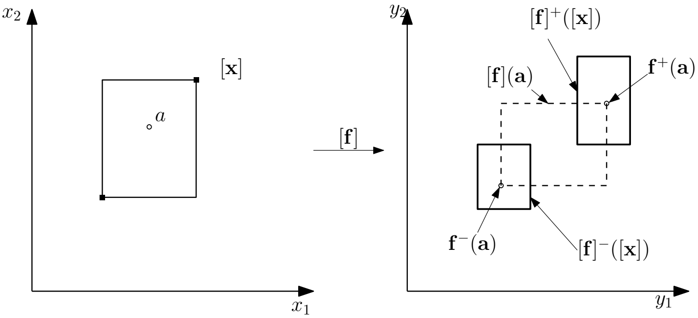

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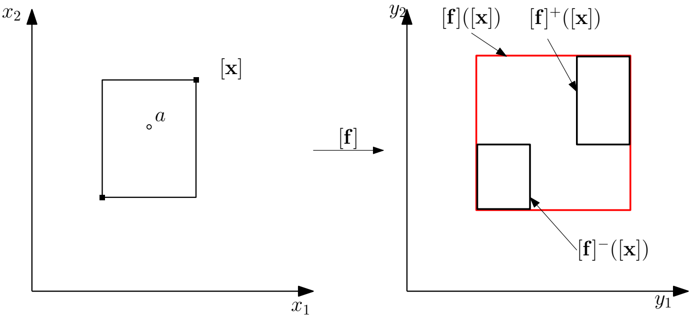


Algorithm

Given a thick function [f] from $\mathbb{R}^n$ to $\mathbb{IR}^m$ and a thick set $[\![\mathbb{Y}]\!] \in \mathbb{IP}(\mathbb{R}^n)$. We want to characterize the thick set

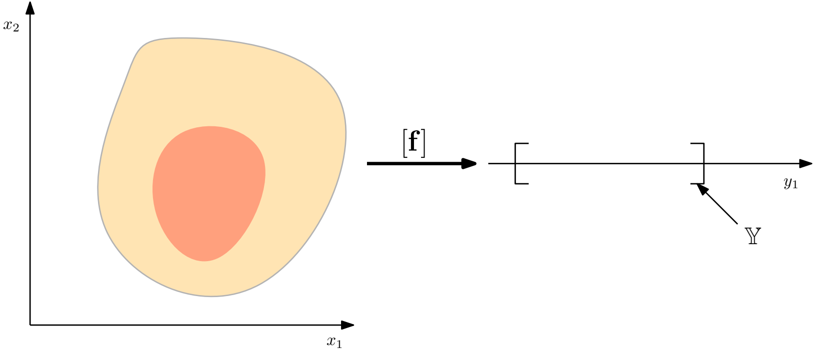
$$[\![\mathbb{X}]\!] = [\mathbf{f}]^{-1}([\![\mathbb{Y}]\!]).$$

For a box [x] we have

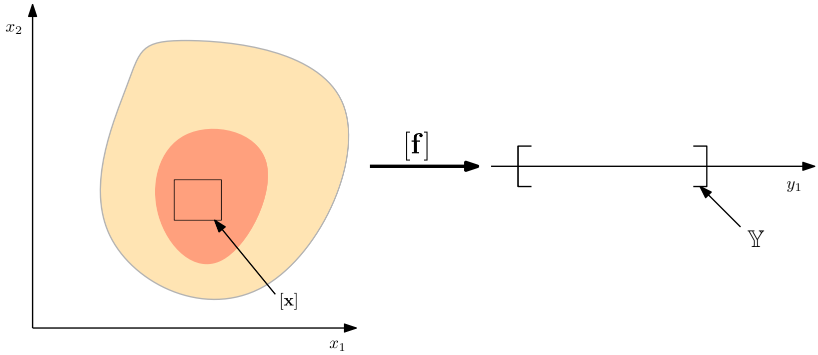
$$\begin{array}{lcll} (i) & \left([\![\mathbf{f}]\!]([\mathbf{x}]) \subset \mathbb{Y}^{\subset}\right)^{\forall} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X}^{\subset} \\ (ii) & \left([\![\mathbf{f}]\!]([\mathbf{x}]) \cap \mathbb{Y}^{\supset} = \emptyset\right)^{\forall} & \Rightarrow & [\mathbf{x}] \cap \mathbb{X}^{\supset} = \emptyset \\ (iii) & \begin{cases} \left([\![\mathbf{f}]\!]([\mathbf{x}]) \not\subset \mathbb{Y}^{\subset}\right)^{\forall} \\ \left([\![\mathbf{f}]\!]([\mathbf{x}]) \cap \mathbb{Y}^{\supset} \neq \emptyset\right)^{\forall} \end{cases} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset}. \end{array}$$

Outlines

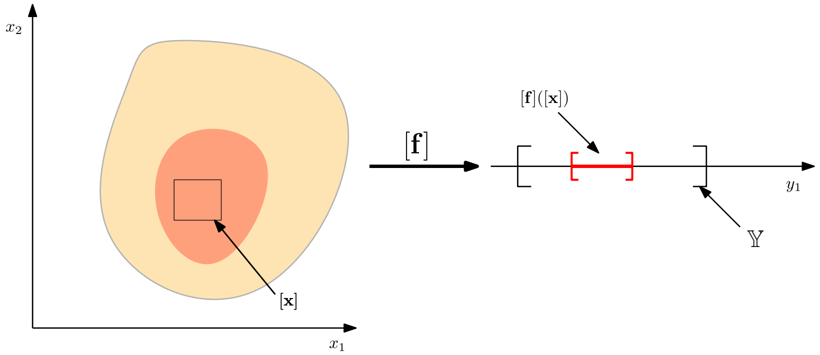
Illustration



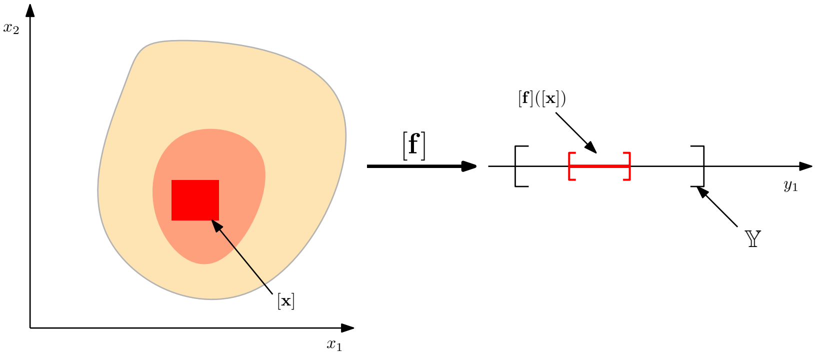
Illustration



Illustration



Illustration



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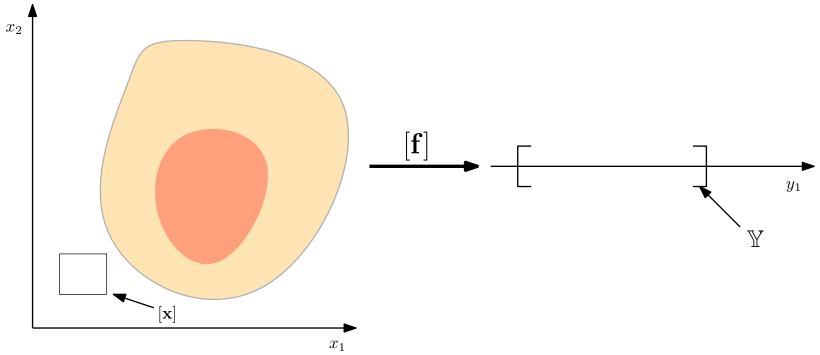

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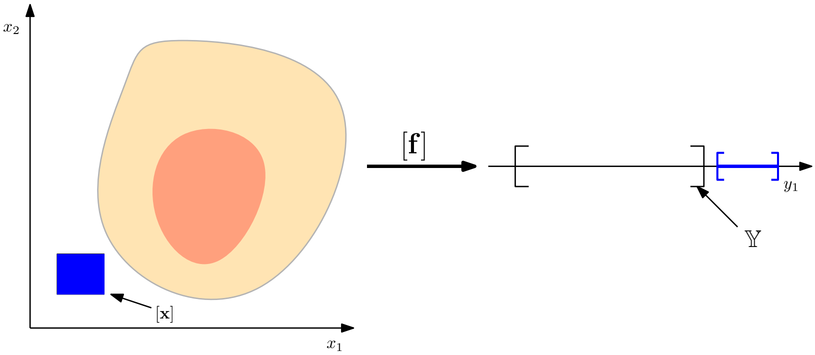

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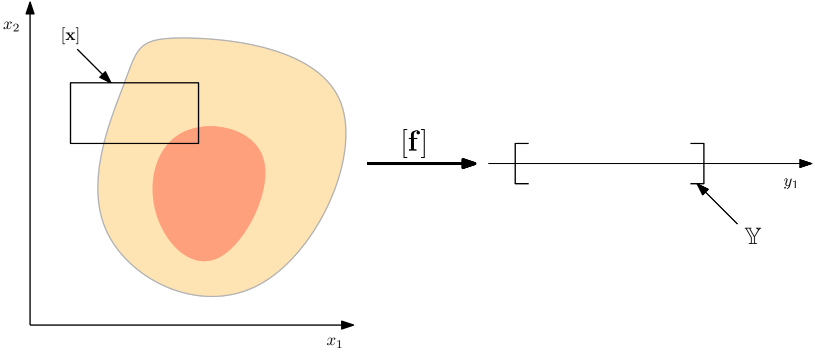

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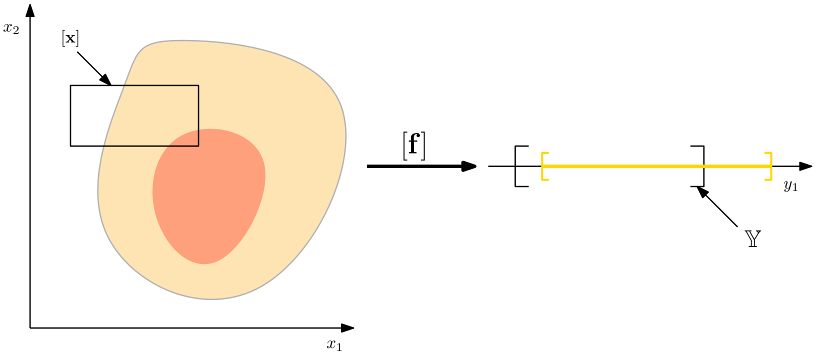

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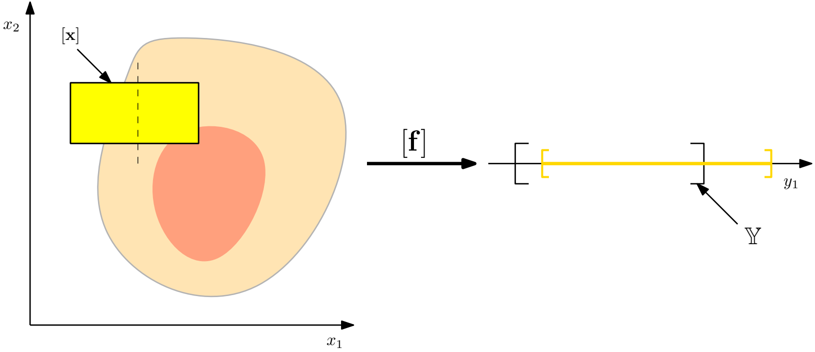

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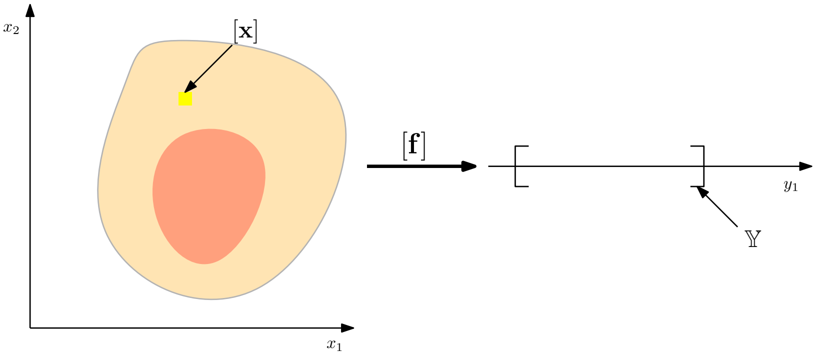
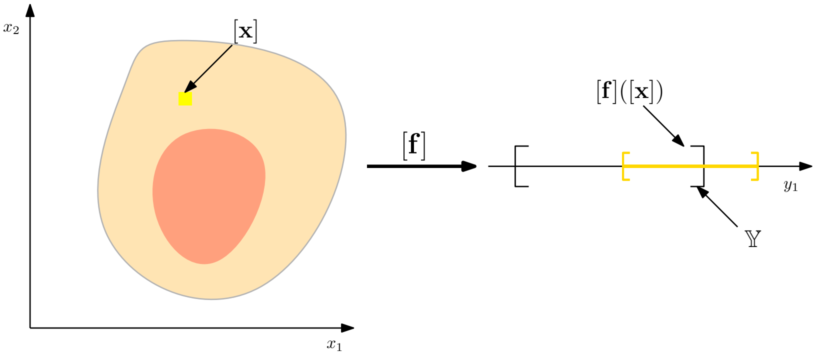


Illustration



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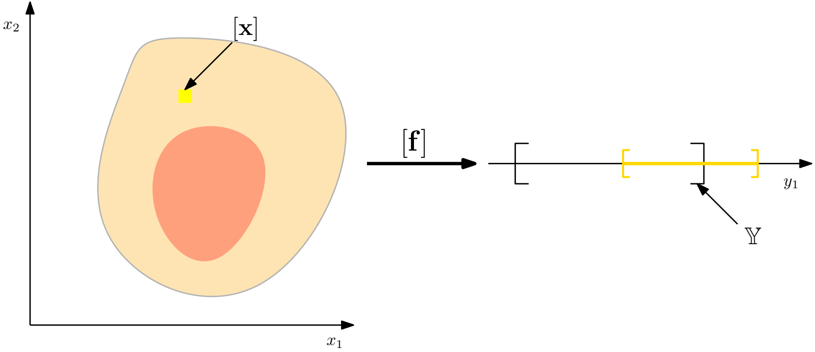

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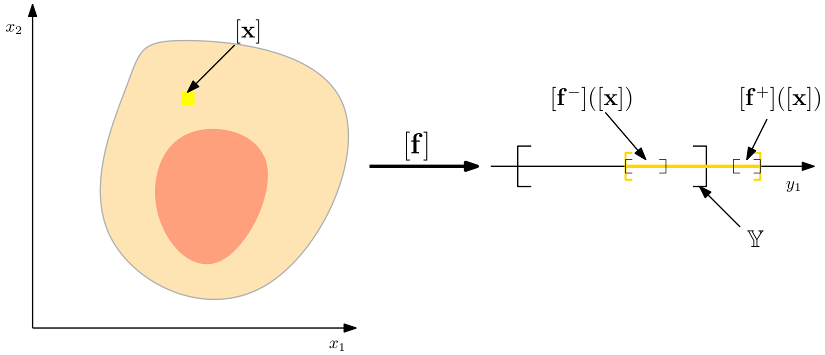

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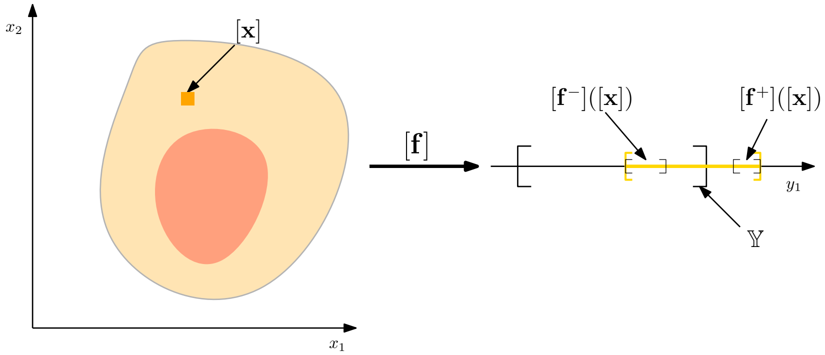
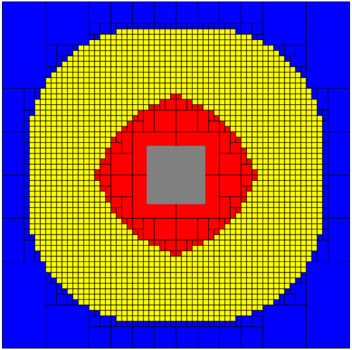


Example

$$\begin{cases} f(\mathbf{x}) & = & \sqrt{(x_1 - m_1)^2 + (x_2 - m_2)^2} \\ \mathbb{Y} & = & [0, 2] \end{cases}$$



$\mathbf{m} \in [0.5, 1.5] \times [-2.5, -1.5]$

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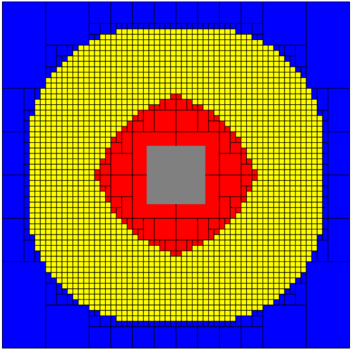

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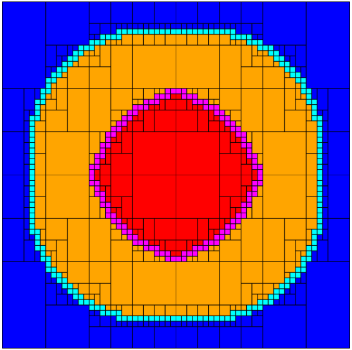


$\mathbf{m} \in [0.5, 1.5] \times [-2.5, -1.5]$

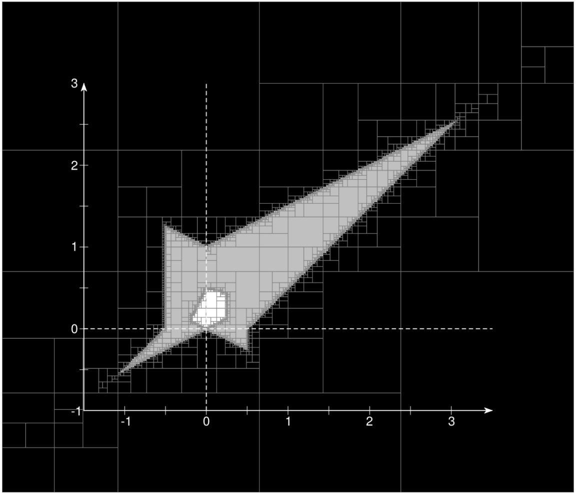
Application

Tolerable-United solution sets.

Consider:

$$\begin{pmatrix} [2,4] & [-2,0] \\ [-1,1] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \begin{pmatrix} [-1,1] \\ [0,2] \end{pmatrix}$$

Application
Tolerable-United solution sets.



Summary

Introduction of tools and algorithm to solve set inversion problem involving uncertain function with :

- the introduction of thick intervals
- an efficient manipulation of uncertainties

The thick set inversion algorithm :

- requires $\mathbf{f}^{-}$ and $\mathbf{f}^{+}$
- is independent of the shape of $[\![\mathbb{Y}]\!]$