Counting the number of connected components of a set and its application to robotics

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Abstract. This paper gives a numerical algorithm able to compute the number of path-connected components of a set $\mathbb{S}$ defined by nonlinear inequalities. This algorithm uses interval analysis to create a graph which has the same number of connected components as $\mathbb{S}$. An example coming from robotics is presented to illustrate the interest of this algorithm for path-planning.

1 Introduction

There exist different kinds of algorithms for path-planning. Most of the approaches are based on the use of potential function introduced by Khatib [3]. This type of methods may be trapped in a local minimum and often fail to give any feasible path.

Interval analysis [7] is known to be able to give guaranteed results (See e.g. [2]). In the first section, the notion of feasible configuration space is recalled and it is shown why its topology can be a powerful tool for path-planning. In the next section, topological definitions and a sufficient condition to prove that a set is star-shaped are given. This sufficient condition is the key result of the `CIA` algorithm presented in the fourth section. This algorithm creates a graph which has the same number of connected components as $\mathbb{S}$ where $\mathbb{S}$ is a subset of $\mathbb{R}^n$ defined by non-linear inequalities. [9] and [8] give algorithms where $\mathbb{S}$ is (closed) semi-algebraic.

Throughout this article, we use a robot to illustrate a new path-planning algorithm.

2 Motivation with an example coming from robotics

2.1 A robot

Consider a 2-dimensional room which contains two walls (represented in gray in the Figure 1. The distance between the walls is y_0. A robotic arm with two degrees of freedom α and β is placed in this room. It is attached to a wall at a point O and has two links OA and AB.

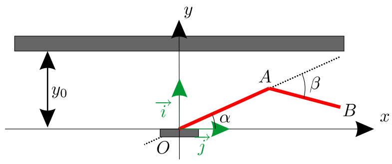

Fig. 1. A robotic arm with two links, $OA = 2$ and $AB = 1.5$.

The Cartesian coordinates of A and B are given by the following equations :

$$\begin{cases} x_A = 2\cos(\alpha) \\ y_A = 2\sin(\alpha) \end{cases} \quad \begin{cases} x_B = 2\cos(\alpha) + 1.5\cos(\alpha+\beta) \\ y_B = 2\sin(\alpha) + 1.5\sin(\alpha+\beta) \end{cases}$$

2.2 Configuration set

Each coordinate of the *configuration space* represents a degree of freedom of the robot (See Figure 2). The number of independent parameters needed to specify an object configuration corresponds to the dimension of the configuration space. In our example, only α and β are necessary to locate the robot configuration, so our configuration space is a 2-dimensional space.

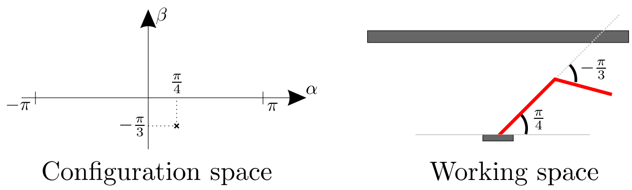

Fig. 2. A point in the configuration space (left) and its corresponding robot configuration.

Since the robot cannot run through the walls, one has the following constraints $y_A \in [0, y_0]$ and $y_B \in]-\infty, y_0]$ and $\alpha \in [-\pi, \pi]$ and $\beta \in [-\pi, \pi]$. When these constraints are satisfied, the robot is said to be in a *feasible configuration*. The feasible configuration set $\mathbb{S}$ is thus defined as :

$$\mathbb{S} = \left\{ (\alpha, \beta) \in [-\pi, \pi]^2 / \begin{cases} 2\sin(\alpha) & \in [0, y_0] \\ 2\sin(\alpha) + 1.5\sin(\alpha+\beta) & \in]-\infty, y_0] \end{cases} \right\}$$

2.3 Connectedness of the feasible configuration set and path-planning

Figure 3 shows how the feasible configuration set is affected by y_0. Three cases are presented :

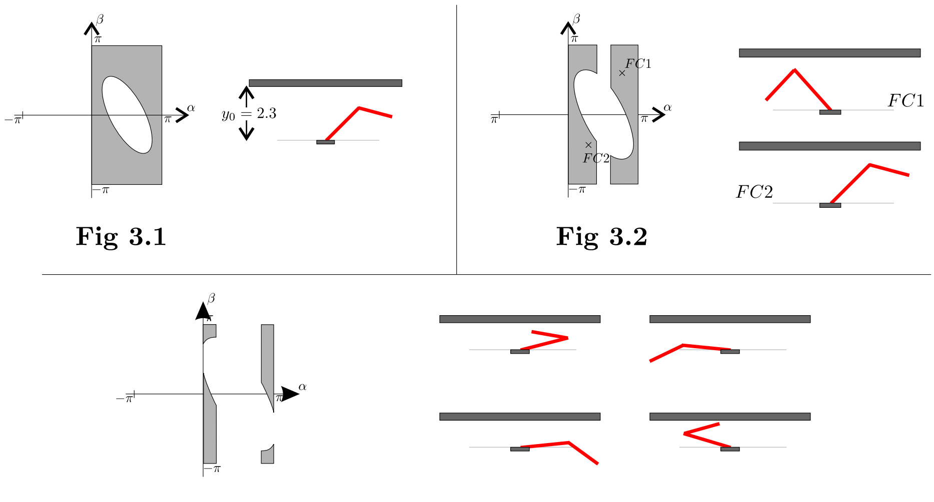

Fig 3.1 **Fig 3.2** **Fig 3.3**

Fig. 3. - **Fig.3.1.** Feasible configuration set when $y_0 = 2.3$. The robot can move from every initial feasible configuration to any goal feasible configuration. In this case, $\mathbb{S}$ has only one connected component. It is said *path-connected* (See Definition 1). - **Fig.3.2.** Feasible configuration set when $y_0 = 1.9$. The configuration set has two path-connected components. It is impossible to move the robot from the first configuration $FC1$ to the second one $FC2$ without violating any constraint. - **Fig.3.3.** Feasible configuration set when $y_0 = 1.1$. The robot can be trapped in four regions. $\mathbb{S}$ has four connected components. In each connected component, the robot can move but cannot reach any another components.

In this article, a reliable method able to count the number of connected components of sets described by inequalities is presented. These sets can be feasible configuration sets. With a couple of configurations, we are able to guarantee that there exists or not a path to connect this ones. Moreover, when we have proven that two configurations are connectable, we are able to propose a path to connect them without violating any constraint.

3 Topological brief overview and a key result leading to discretization

In this section, definitions of a path-connected set and star-shaped set are recalled. Then, it is shown how this notions are linked. The last result is the key result leading to a robust discretization presented in the next section.

3.1 Topological brief overview

Definition 1. *A topological set* $\mathbb{S}$ *is* path-connected *if for every two points* $x, y \in \mathbb{S}$*, there is a continuous function* γ *from* $[0,1]$ *to* $\mathbb{S}$ *such that* $\gamma(0) = x$ *and* $\gamma(1) = y$*. Path-connected sets are also called 0-connected.*

Definition 2. *A point* v^* *is a* star *for a subset* X *of an Euclidean set if* X *contains all the line segments connecting any of its points and* v^**. A subset* X *of an Euclidean set is* star-shaped *or* v^*-star-shaped *if there exists* $v^* \in X$ *such that* v^* *is a star for* X*.*

Proposition 1. *A star-shaped set* $\mathbb{S}$ *is a path-connected set.*

Proof. Since $\mathbb{S}$ is star-shaped, there exists $v \in \mathbb{S}$ such that v is a star for $\mathbb{S}$. Let x and y be in $\mathbb{S}$ and :

$$\begin{aligned} \gamma : [0,1] &\to \mathbb{S} \\ t &\mapsto \begin{cases} (1-2t)x + 2tv & \text{if } t \in [0, \frac{1}{2}[\\ (2-2t)x + (2t-1)v & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

γ is a continuous function from $[0,1]$ to $\mathbb{S}$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 2. *Let* X *and* Y *be two* v^**-star-shaped set, then* $X \cap Y$ *is also* v^**-star-shaped.*

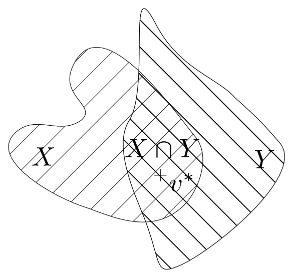

Fig. 4. Intersection stability.

The next result is a sufficient condition to prove that a set defined by only one inequality is star-shaped. This sufficient condition can be checked using interval analysis (An algorithm such as SIVIA, Set Inversion Via Interval Analysis [5], can prove that equations (1) are inconsistent).

Proposition 3. *Let us define* $\mathbb{S} = \{x \in D \subset \mathbb{R}^n | f(x) \leq 0\}$ *where* f *is a* C^1 *function from* D *to* $\mathbb{R}$*, and* D *a convex set. Let* v^* *be in* $\mathbb{S}$*. If*

$$f(x) = 0,\ Df(x) \cdot (x - v^*) \leq 0,\ x \in D \tag{1}$$

is inconsistent then v^* *is a star for* $\mathbb{S}$*.*

Proof. See [1]

Remark 1. Combining this result with the Proposition 2, Proposition 3 can be used to prove that a set is star-shaped even if the set $\mathbb{S}$ is defined by several inequalities.

4 Discretization

The main idea of this disretization is to generate *star-spangled graph* which preserves the number of connected components of $\mathbb{S}$.

Definition 3. *A* star-spangled *graph of a set* $\mathbb{S}$*, noted by* $\mathcal{G}_{\mathbb{S}}$*, is a relation* $\mathcal{R}$ *on a paving* [3] $\mathcal{P} = \{p_i\}_{i \in I}$ *where :*

- *for all* p *of* $\mathcal{P}$*,* $\mathbb{S} \cap p$ *is star-shaped.*
- $\mathcal{R}$ *is the reflexive and symmetric relation on* $\mathcal{P}$ *defined by* $p \, \mathcal{R} \, q \Leftrightarrow \mathbb{S} \cap p \cap q \neq \emptyset$*.*
- $\mathbb{S} \subset \bigcup_{i \in I} p_i$

Proposition 4. *Let* $\mathcal{G}_{\mathbb{S}}$ *be a star-spangled graph of a set* $\mathbb{S}$*.*
$\mathcal{G}_{\mathbb{S}}$ *has the same number of connected components as* $\mathbb{S}$*. i.e.* $\pi_0(\mathbb{S}) = \pi_0(\mathcal{G}_{\mathbb{S}})$[4]*.*

Proof. See [1].

4.1 The algorithm `CIA`

The algorithm called: `CIA` (path-**C**onnected using **I**nterval **A**nalysis) tries to generate a star-spangled graph $\mathcal{G}_{\mathbb{S}}$ (Proposition 4). The main idea is to test a suggested paving $\mathcal{P}$. In the case where the paving does not satisfy the condition that for all p in $\mathcal{P}$, $p \cap \mathbb{S}$ is star-shaped, the algorithm tries to improve this one by bisecting any boxes responsible for this failure.
For a paving $\mathcal{P}$, the algorithm checks for a box p of $\mathcal{P}$ whether $\mathbb{S} \cap p$ is star-shaped or not (Proposition 1 and 2), and to build its associated graph with the relation $\mathcal{R}$ mentioned before.

[3] A paving is a finite collection of non overlapping n-boxes (Cartesian product of n intervals), $\mathcal{P} = \{p_i\}_{i \in I}$
[4] In algebraic topology, $\pi_0(\mathbb{S})$ is the classical notation for the number of connected components of $\mathbb{S}$.

In Alg. 1 `CIA`[5], $\mathcal{P}_*$, $\mathcal{P}_{out}$, $\mathcal{P}_\Delta$ are three pavings such that $\mathcal{P}_* \cup \mathcal{P}_{out} \cup \mathcal{P}_\Delta = \mathcal{P}$, with $\mathcal{P}$ is a paving whose support is a (possibly very large) initial box X_0 (containing $\mathbb{S}$):

- the *star-spangled* paving $\mathcal{P}_*$ contains boxes p such that $\mathbb{S} \cap p$ is star-shaped.
- the *outer* paving $\mathcal{P}_{out}$ contains boxes p such that $\mathbb{S} \cap p$ is empty.
- the *uncertain* paving $\mathcal{P}_\Delta$, nothing is known about its boxes.

Alg. 1 `CIA` - path-Connected using Interval Analysis

Require: $\mathbb{S}$ a subset of $\mathbb{R}^n$, X_0 a box of $\mathbb{R}^n$
1: Initialization : $\mathcal{P}_* \leftarrow \emptyset$, $\mathcal{P}_\Delta \leftarrow \{X_0\}$, $\mathcal{P}_{out} \leftarrow \emptyset$
2: **while** $\mathcal{P}_\Delta \neq \emptyset$ **do**
3: Pull the last element of $\mathcal{P}_\Delta$ into the box p
4: **if** "$\mathbb{S} \cap p$ is proven empty" **then**
5: Push $\{p\}$ into $\mathcal{P}_{out}$, Goto Step 2.
6: **end if**
7: **if** "$\mathbb{S} \cap p$ is proven star-shaped" and if we can guarantee $\forall p_* \in \mathcal{P}^*, p \cap p_* \cap \mathbb{S}$ is empty or not **then**
8: Push $\{p\}$ into $\mathcal{P}_*$, Goto Step 2.
9: **end if**
10: *Bisect*(p) and Push the two resulting boxes into $\mathcal{P}_\Delta$
11: **end while**
12: $n \leftarrow$ Number of connected components of $\mathcal{G}_\mathbb{S}$ (i.e. the relation $\mathcal{R}$ on $\mathcal{P}_*$).
13: return "$\mathbb{S}$ has n path-connected components"

4.2 Application

Consider again the example presented in Section 1, the feasible configuration set $\mathbb{S}$ is :

$$\mathbb{S} = \left\{ (\alpha, \beta) \in [-\pi, \pi]^2 / \left\{ \begin{array}{ll} -2\sin(\alpha) & \leq 0 \\ 2\sin(\alpha) - y_0 & \leq 0 \\ 2\sin(\alpha) + 1.5\sin(\alpha + \beta) - y_0 & \leq 0 \end{array} \right\} \right. \quad (2)$$

When y_0 is equal to 2.3, 1.9 and 1.1, algorithm `CIA` generates these star-spangled graphs presented respectively on Figures 5,6 and 7.

[5] This algorithm has been implemented (CIA.exe) and can be found at http://www.istia.univ-angers.fr/˜delanoue/

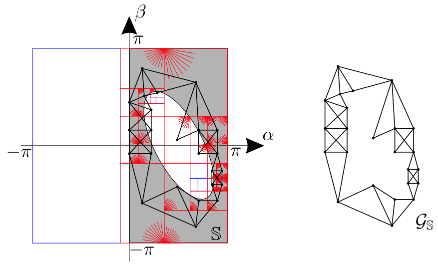

Fig. 5. The feasible configuration set and one of its star-spangled graph generated by `CIA` when $y_0 = 2.3$. The star-spangled graph $\mathcal{G}_{\mathbb{S}}$ is connected. By using Proposition 4, we deduce that from every couple of endpoints, it is possible to create a path to connect this ones. Subsection 4.3 shows how a path can be found.

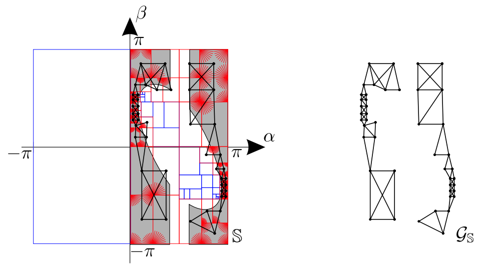

Fig. 6. The feasible configuration set and its star-spangled graph generated by `CIA` when $y_0 = 1.9$. Since $\mathcal{G}_{\mathbb{S}}$ has two connected components, we have a proof that $\mathbb{S}$ has two path-connected components.

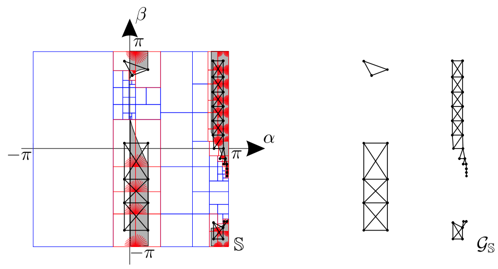

Fig. 7. The feasible configuration set and its star-spangled graph generated by `CIA` when $y_0 = 1.1$. $\mathcal{G}_{\mathbb{S}}$ and $\mathbb{S}$ have 4 connected components.

4.3 Path-planning

A star-spangled graph can be used to create a path between endpoints. Our goal is to find a path from the initial configuration x to the goal configuration y (e.g. Fig. 8).

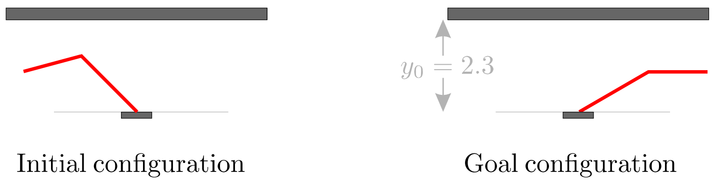

Fig. 8. Initial configuration, $x = (\frac{3\pi}{4}, \frac{\pi}{3})$ and goal configuration, $y = (\frac{\pi}{6}, -\frac{\pi}{6})$

As shown in Section 1, it suffices to find a path which connects x to y in the feasible configuration set. The algorithm `Path-planning with CIA`, thanks to a star-spangled graph, creates a path γ in $\mathbb{S}$. This algorithm uses the `Dijkstra` [6] algorithm which finds the shortest path between two vertices in a graph. Since $\mathcal{G}_{\mathbb{S}}$ is a star-shaped graph, every p in $\mathcal{P}$ is necessary star-shaped and we denote by v_p one of its stars.

Alg. 2 `Path-planning with CIA`

Require: A set $\mathbb{S}$, $x, y \in \mathbb{S}$, $\mathcal{G}_{\mathbb{S}}$ a star spangled graph of $\mathbb{S}$ (The relation $\mathcal{R}$ on the paving $\mathcal{P}$).
Ensure: $\gamma \subset \mathbb{S}$ a path whose endpoints are x and y.

Initialization : $\lambda \leftarrow \emptyset$
for all $p \in \mathcal{P}$ **do**
  if $x \in p$ **then** $p_x \leftarrow p$; **if** $y \in p$ **then** $p_y \leftarrow p$
end for
if `Dijkstra`$(\mathcal{G}_{\mathbb{S}}, p_x, p_y) =$ "Failure" **then**
  Return "x and y are in two different path-connected components"
else
  $(p_k)_{1 \leq k \leq n} = (p_x, \ldots, p_y) \leftarrow$ `Dijkstra`$(\mathcal{G}_{\mathbb{S}}, p_x, p_y)$
end if
$\gamma \leftarrow [x, v_{p_x}]$
for $k \leftarrow 2$ to $n - 1$ **do**
  $w_{k-1,k} \leftarrow$ a point in $p_{k-1} \cap p_k \cap \mathbb{S}$; $w_{k,k+1} \leftarrow$ a point in $p_k \cap p_{k+1} \cap \mathbb{S}$
  $\gamma \leftarrow \gamma \cup [w_{(k-1,k)}, v_{p_k}] \cup [v_{p_k}, w_{(k,k+1)}]$
end for
$\gamma \leftarrow \gamma \cup [v_{p_y}, y]$

Figure 9 shows the path γ created by `Path-planning with CIA`.

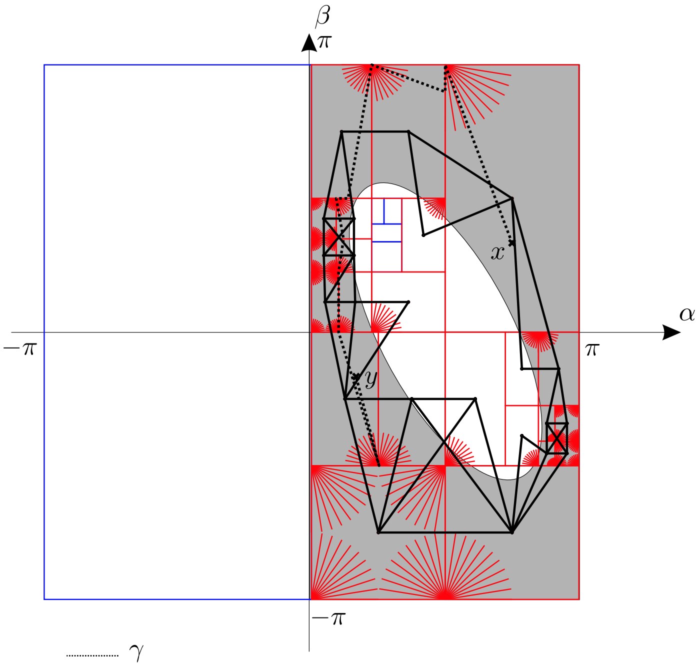

Fig. 9. Path γ generated by `Path-planning with CIA` from x to y when $y_0 = 2.3$.

The corresponding configurations of the path γ are illustrated on Figure 10.

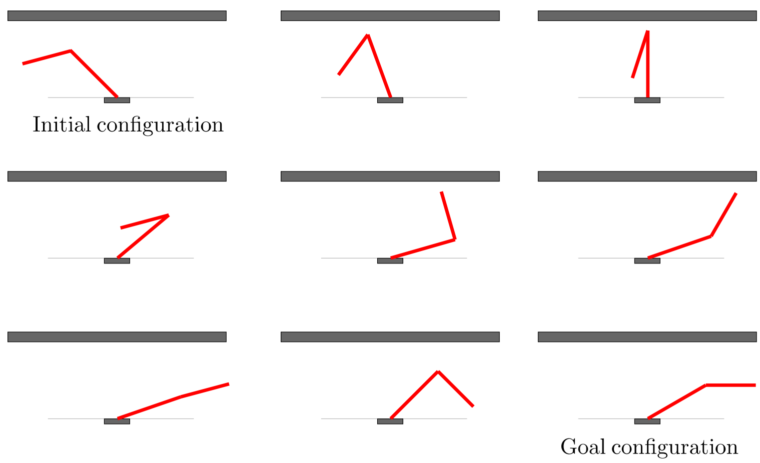

Fig. 10. Corresponding robot motion from the initial to the goal configuration

5 Conclusion

In this article, an algorithm which computes the number of connected components of a set defined by several non-linear inequalities has been presented. This dicretization makes possible to create a feasible path in $\mathbb{S}$ (Alg. 2). One of the main limitations of the proposed approach is that the computing time increases exponentially with respect to the dimension of $\mathbb{S}$. At the moment, we do not have a sufficient condition about f (Proposition 3) to ensure that algorithm `CIA` will terminate.

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