Some stories about small octagons

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ENSEEIHT-LAPLACE, Toulouse, France

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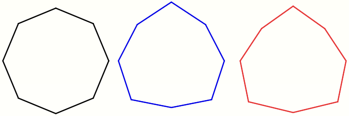

Journée Intervalle et Géométrie 2013 - ENSTA - Brest

Presentation Outline

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Some Definitions

Definition

A $n-$gon is a polygon with n sides and n vertices.

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The diameter of a $n-$gon is longest distance between two vertices.

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A small $n-$gon is a $n-$gon with a diameter 1.

We address in this work: isodiametric problems and questions about perimeter and area.

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A_n denote the area of a small $n-$gon and P_n its perimeter.

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$A_n^=$ denote the area of an equilateral small $n-$gon and $P_n^=$ its perimeter.

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The regular $n-$gons have all the properties of maximal perimeter and area, for n odd.

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For all n, a bound for the perimeter is

$$P_n \leq 2n \sin \frac{\pi}{2n}$$

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For all n, a bound for the area is

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Small Quadrilateral Polygons: Maximal Area

- Maximal area $n = 4$:

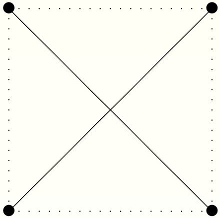

Figure : $A_4^{=} = A_4 = \frac{1}{2}$.

With $\frac{b \times h_1}{2} + \frac{b \times h_2}{2} = \frac{b \times (h_1 + h_2)}{2} = \frac{1}{2}$, $b = 1$ and $h_1 + h_2 = 1$.

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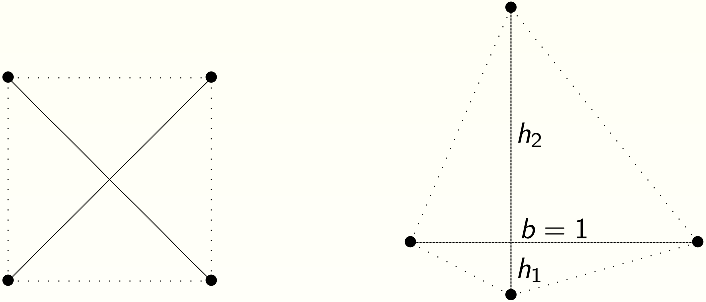

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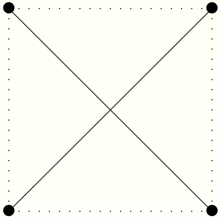

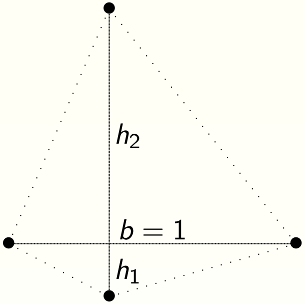

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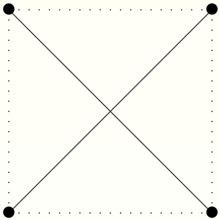

$P_4^{=} = 2\sqrt{2} \approx 2.8284$

$P_4 = 2 + 4\sin\frac{\pi}{12} \approx 3.0353$

Result from Tamvakis 1987 (and Datta 1997).

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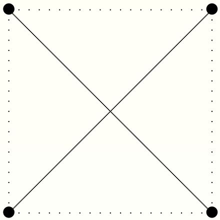

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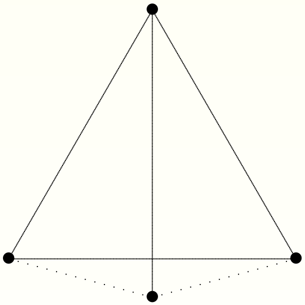

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Problem of P_∞ and Reuleaux Polygons

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P_\infty := & \max & \text{Perimeter} \\
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SOLUTION : The disk

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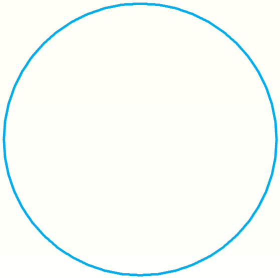

Disk

$P_\infty = \pi$

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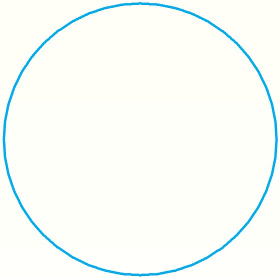

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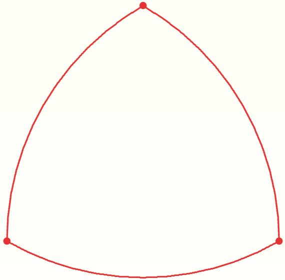

Reuleaux triangle

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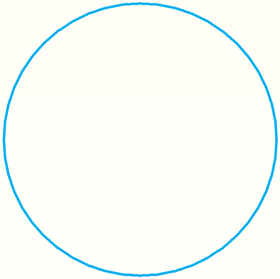

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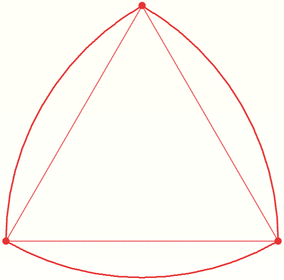
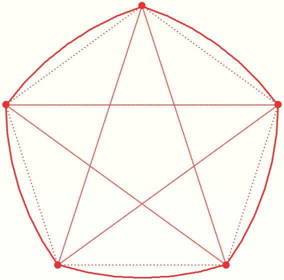


Reuleaux triangle

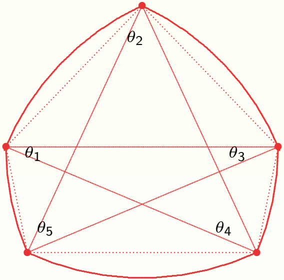
$P_\infty = 3(\frac{\pi}{3}) = \pi$

Odd Reuleaux polygons : figures of constant width



Reuleaux regular pentagon

$P_\infty = 5(\frac{\pi}{5}) = \pi$



Reuleaux irregular pentagon

$P_\infty = \sum_{i=1}^{5} \theta_i = \pi$

Reuleaux Polygons Main Property and Applications

Same width in every directions:

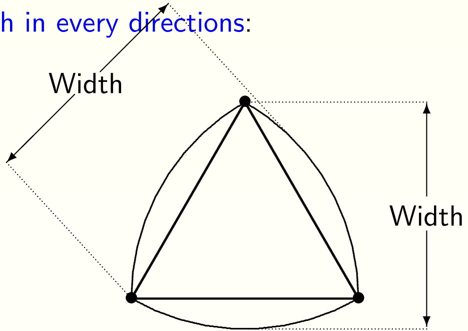


Figure : Example of width of a Reuleaux triangle.

Applications:

- A Reuleaux triangle is used in the SMART car (for the injection system)!
- For the design of a dollar: 1$ Canadian is a Reuleaux polygon with eleven sides.
  See "A $1 Problem" paper in AMM of Mossinghoff.

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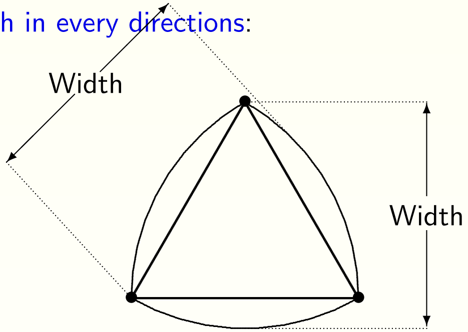

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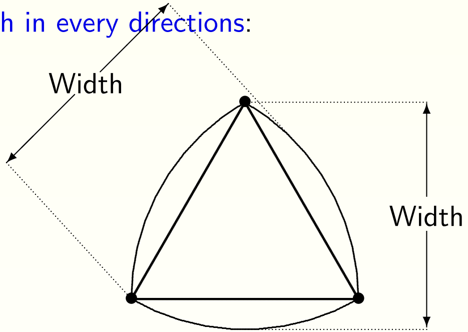

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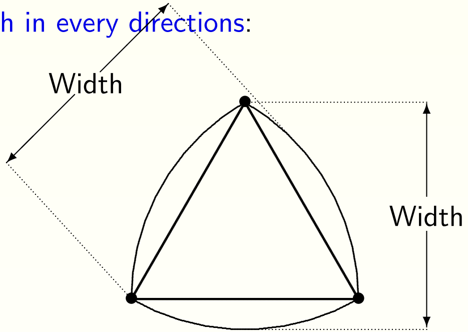

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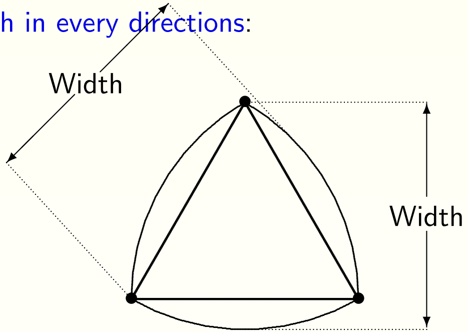

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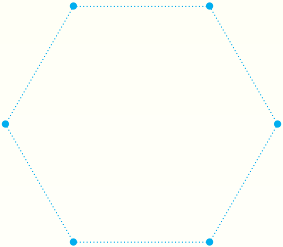
Examples of coins

Hexagon with Maximal Perimeter

The upper bound of $2n\sin(\frac{\pi}{2n})$ is attained for irregular n-gons.



Regular hexagon

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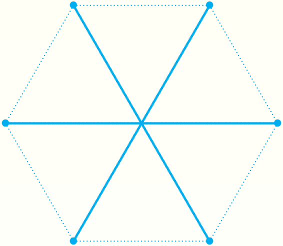

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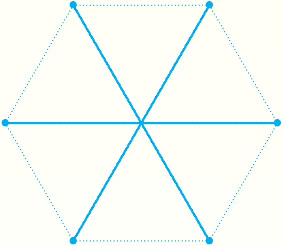

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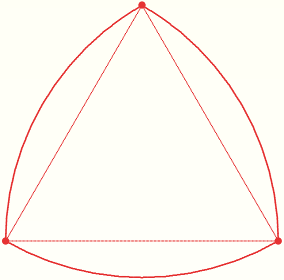

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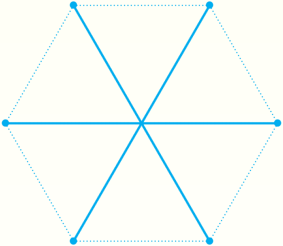

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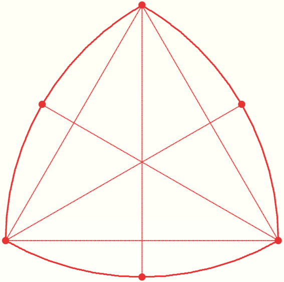

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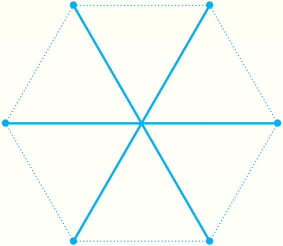

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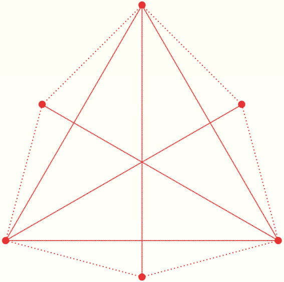

Optimal hexagon

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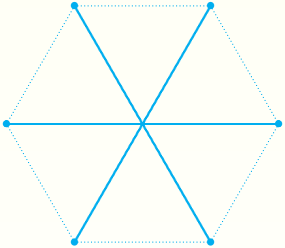

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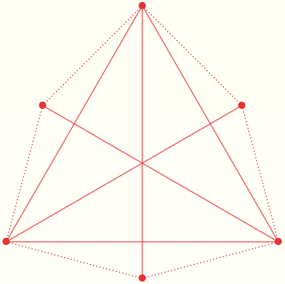

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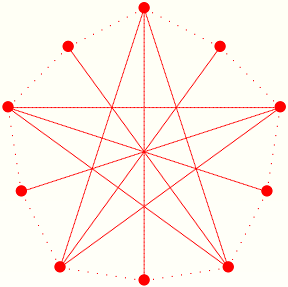
When n is not a power of 2,

$2n\sin(\frac{\pi}{2n}) \leq \max P_n^{=} \leq \max P_n \leq 2n\sin(\frac{\pi}{2n})$.

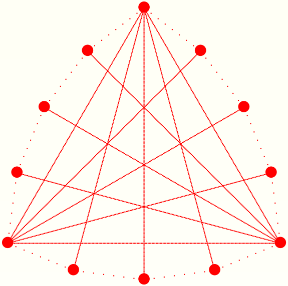
This result is due to Vincze 1952.

Examples of Maximal Perimeter Solutions when n is not a Power of 2



Optimal Decagon

$P_{10} = P_{10}^{=} \approx 3.1287$

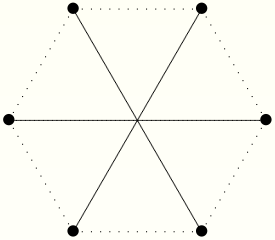


Optimal Dodecagon

$P_{12} = P_{12}^{=} \approx 3.1326$

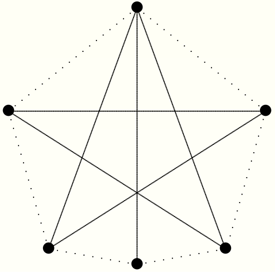
Figure : Examples of polygons with maximal perimeter when n is even but $n \neq 2^s$.

Graham's Hexagon with Maximal Area, 1975



Regular Hexagon

$A_6^{=} = \frac{3\sqrt{3}}{8} \approx 0.6495$



Graham's Hexagon

$A_6 \approx 0.6750$

Figure : Two hexagons with maximal area.

Gain about 3.9% (comparing to the regular hexagon).

Diameter Graph and Geometric Reasoning

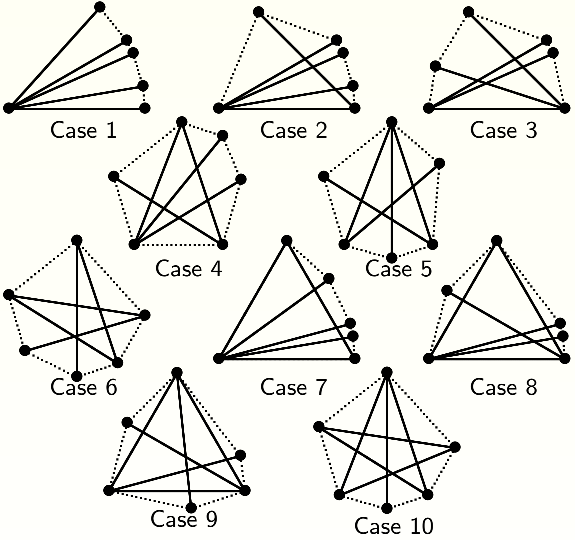


Figure : The ten possible diameter configurations for the hexagon

Case 10 with Symmetry Hypothesis

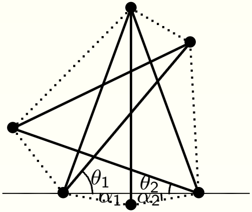


Figure : Configuration 10 of area R_1.

Hypothesis of symmetry ? Graham wrote in his paper:

> *"It is immediate that in order to maximize area R_1, it is necessary that $\alpha_1 = \alpha_2$. It is slightly less immediate (but equally true) that it is also necessary that $\theta_1 = \theta_2$. (The details are not particularly interesting and are omitted)."*

$\Longrightarrow$ solve a global optimization problem in one variable.

Graham's, Bieri's or Yuan's Hexagon ?

Retrospectively with hypothesis of symmetry, Bieri answers to this question in 1961. 14 years before Graham!
Title: "Ungelöste Probleme: Zweiter Nachtrag zu Nr. 12" (Open Problem, second supplement to number 12) answering to Lenz: "Ungelöste Probleme Nr. 12" posed in 1956 in *Elemente der Mathematik*.
This remark come from Mossinghoff: "a 1$ problem", AMM.

Bao Yuan give a complete proof in his Report of Master Degree in 2004: "The Largest Small Hexagon".

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Solved problems and open problems

- The solutions to P_∞ are the disk and all odd Reuleaux polygons.
- When n is odd, the regular n-gon solves the four problems of maximal perimeter and area.
- When $n = k2^s$ for k odd and s integer, then the k-gon with extra vertices solves the both problems for the perimeter.
- When $n = 4$, the square solves $P_4^= = 2\sqrt{2} \approx 2.828427$ and the following solves $P_4 = 2 + 4\sin(\frac{\pi}{12}) \approx 3.035276$.
- When $n = 6$, the Graham's hexagon solved max A_6 and the regular hexagon solve max $A_6^=$; Vincze's hexagon (based on a Reulaux triangle) solved max P_6 and max $P_6^=$.

Open cases, when $n = 8$: the Octagons.

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- When $n = 6$, the Graham's hexagon solved max A_6 and the regular hexagon solve max $A_6^{=}$; Vincze's hexagon (based on a Reulaux triangle) solved max P_6 and max $P_6^{=}$.

Open cases, when $n = 8$: the Octagons.

Solved problems and open problems

- The solutions to P_∞ are the disk and all odd Reuleaux polygons.
- When n is odd, the regular n-gon solves the four problems of maximal perimeter and area.
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Optimal Small Octagons

- For the area and the perimeter $\Longrightarrow$ 31 diameter graphs.
- About 10 cases can be discarded by geometric reasoning.
- Algorithms: Branch and Cut algorithm for quadratic non-convex programs for the area -CPU—times: about 100h- and IBBA for the perimeter -56h (44h for discarding case 18)-

Solutions:

Optimal Octagon (31)
$A_8 \approx 0.726867$

Optimal Octagon (29)
$P_8 \approx 3.121147\ldots$

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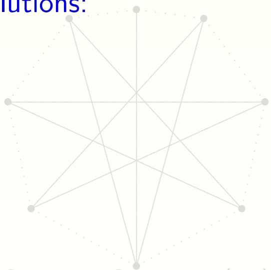

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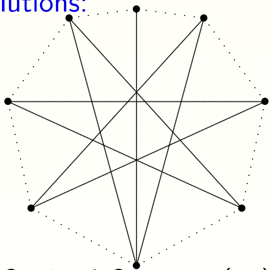

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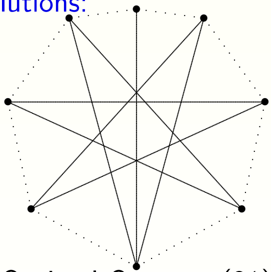

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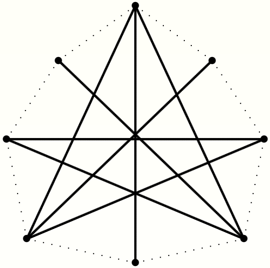

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Vcinze's Wife's Octagon

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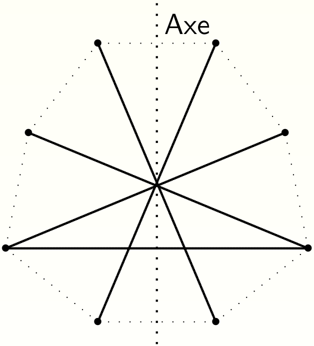

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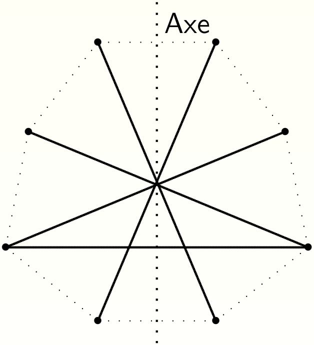

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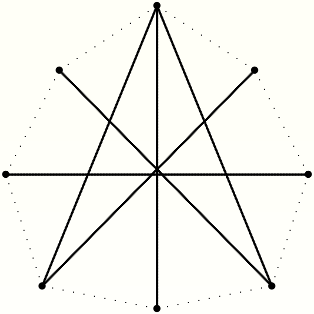

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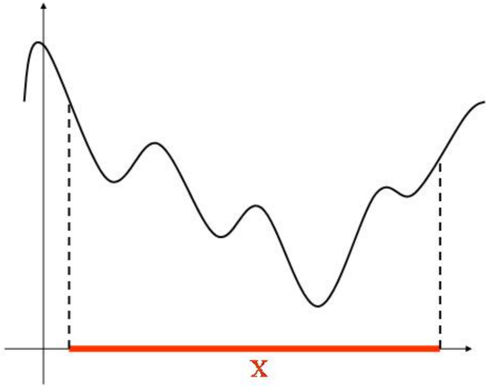
Presentation Outline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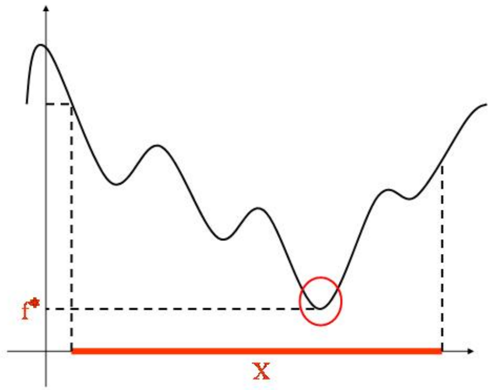

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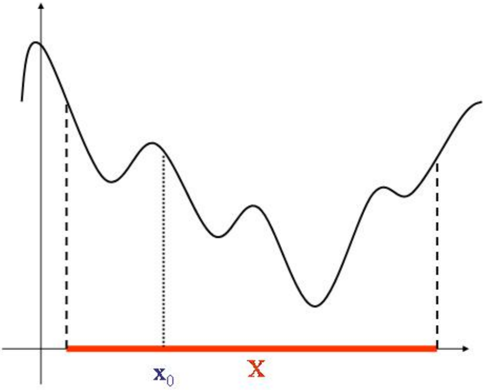

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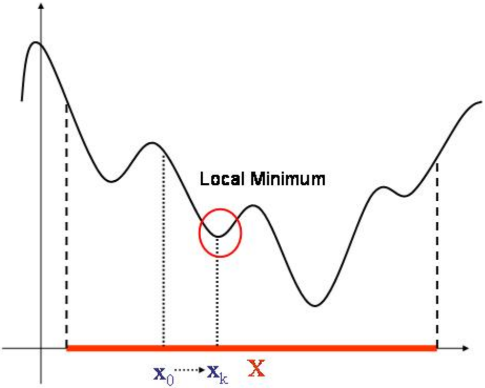

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  - Taboo Research, (Glover and Hansen),
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  - Simulated Annealing,
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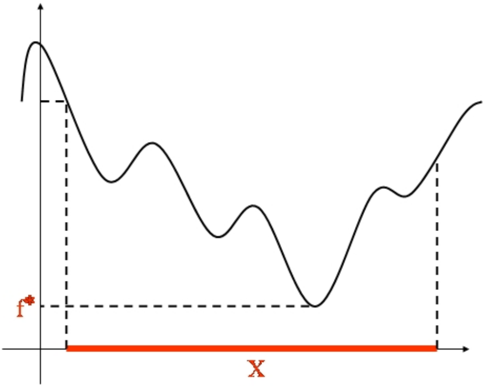

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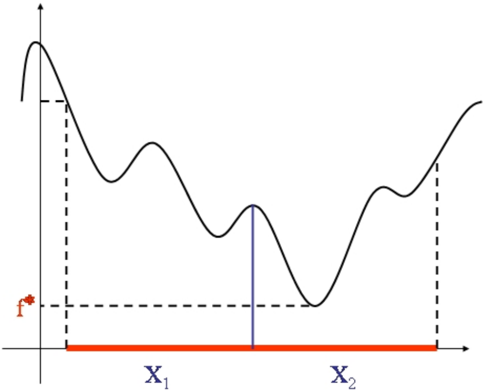

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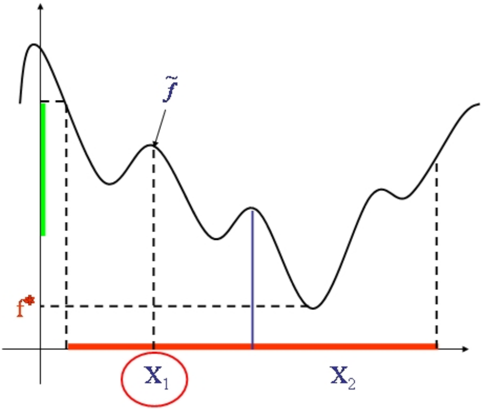

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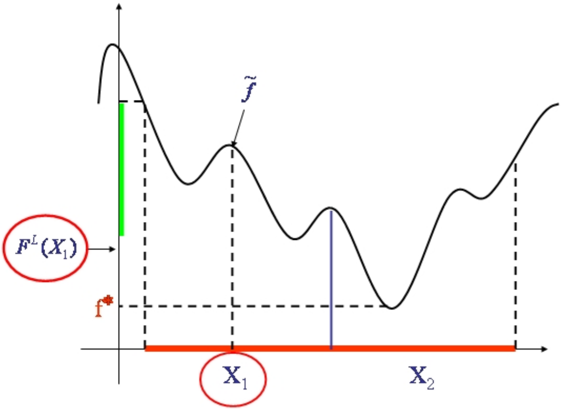

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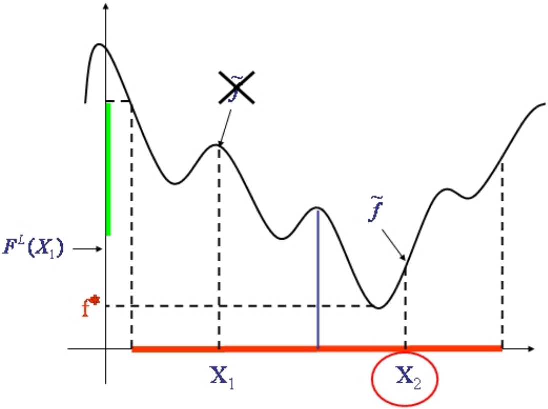

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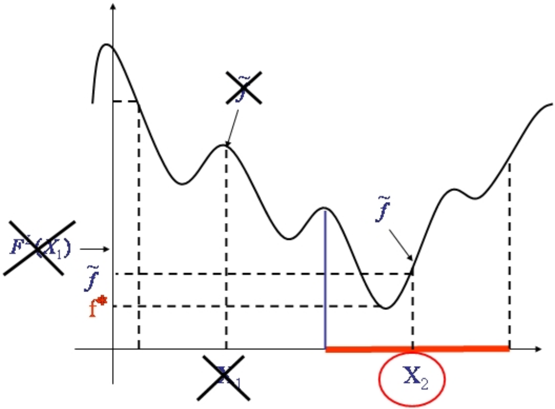

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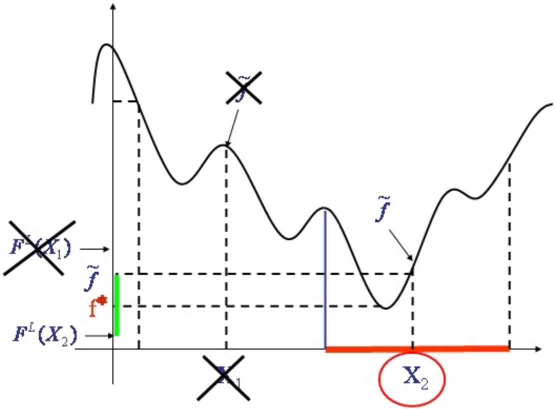

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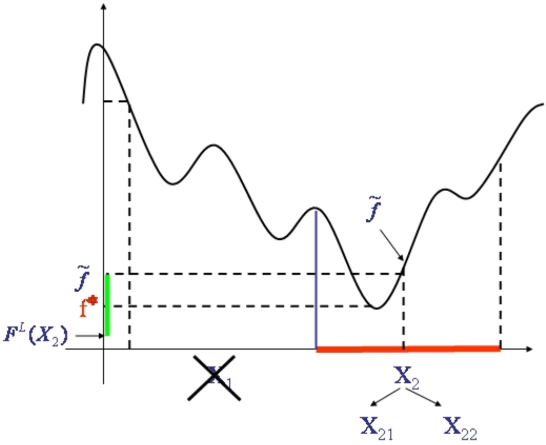

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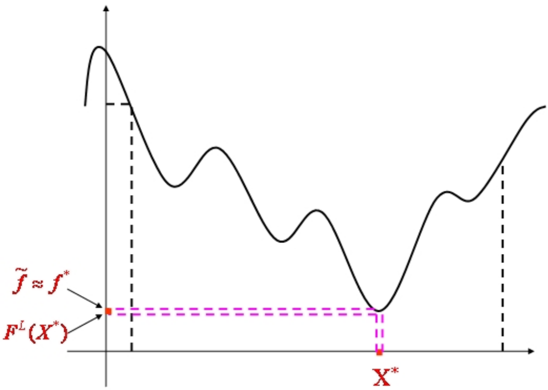


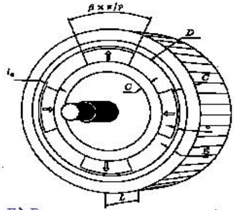
Principle of a Branch and Bound Algorithm for a problem with constraints

- Choice and Subdivision of the box X, (in 2 parts by step): list of possible solutions,
- Reduction of the sub-boxes, by using a constraint propagation technique,
- Computation of bounds of the functions F, G_j, H_j on the sub-boxes, - inclusion functions -
- Elimination of the sub-boxes which cannot contain the global optimum: $F^L(X) > \tilde{f}$ or $G_i^L(X) > 0$ or $0 \notin H(X)$, where $\tilde{f}$ denotes the current solution,
- STOP when accurate enclosures of the optimum are obtained.

Rotating Machines with Magnetic Effects

- **Criteria :**

$$\begin{cases} V_{ap} = \pi \dfrac{D}{\lambda}(D+E-e-l_a)(2C+E+e+l_a) \\ V_m = \pi \beta l_a \dfrac{D}{\lambda}(D-2e-l_a) \\ P_j = \pi \rho_{cu} \dfrac{D}{\lambda}(D+E)E_{ch} \end{cases}$$



Constraints :

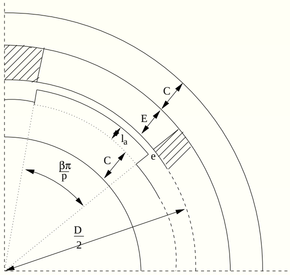
$$C_{em} = \frac{\pi}{2\lambda}(1-K_f)\sqrt{k_r \beta E_{ch} E}\, D^2 (D+E) B_e$$

$$E_{ch} = AJ_{cu} = k_r E J_{cu}^2, K_f \approx 1.5\, p\beta \frac{e+E}{D}, B_e = \frac{2 l_a P}{D \log\left(\dfrac{D+2E}{D-2(l_a+e)}\right)}$$

$$C = \frac{\pi\beta B_e}{4 p B_{fer}} D, p = \frac{\pi D}{\Delta_p}, e_{min} - e \le 0, K_f - K_{f\,max} \le 0$$

Example for the Dimensioning of an Electrical Motor

Electrical Slotless Rotating Machines with Permanent Magnet:



- IBBA standard (defined by Ratschek and Rokne 1988) $\longrightarrow$ 1h35,
- IBBA + propagation due to E. Hansen $\longrightarrow$ 41.5s,
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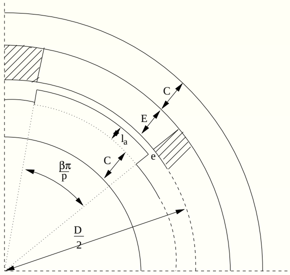

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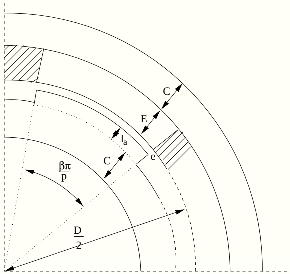

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Problem Formulation for the Largest Small Octagon

Formulation by a nonconvex quadratic program:

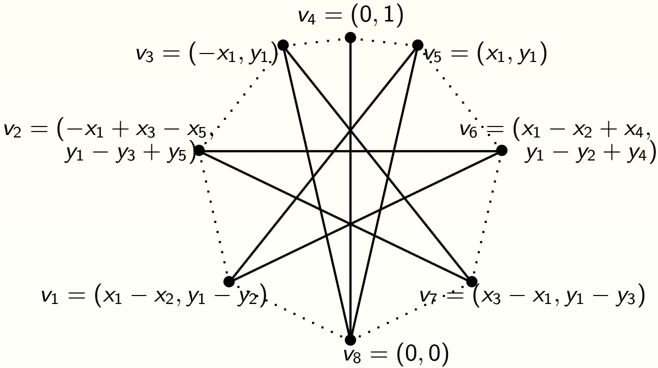

Figure : Case of $n = 8$ vertices. Definition of variables.

A nonconvex quadratic program

$$
\left\{
\begin{array}{ll}
\max\limits_{x,y} & \frac{1}{2}\{(x_2 + x_3 - 4x_1)y_1 + (3x_1 - 2x_3 + x_5)y_2 \\
& +(3x_1 - 2x_2 + x_4)y_3 + (x_3 - 2x_1)y_4 \\
& +(x_2 - 2x_1)y_5\} + x_1 \\
& \\
\text{s.t.} & (2x_1 - x_2 - x_3 + x_4 + x_5)^2 + (y_2 - y_3 + y_4 - y_5)^2 = 1 \\
& (x_3 - 2x_1 + x_2)^2 + (x_7 - x_8)^2 \leq 1 \\
& x_i^2 + y_i^2 = 1, i = 1, 2, 3, 4, 5 \\
& x_2 - x_3 \geq 0 \\
& y \geq 0 \\
& 0 \leq x_1 \leq 0.5 \\
& 0 \leq x_i \leq 1, \quad i = 2, 3, 4, 5.
\end{array}
\right.
$$

Numerical solutions

Soft.	Year	Accuracy	CPU-time
QP	1997	10^{-4}	100h
Gloptipoly	2010	$10^{-7}*$	5s
IBBA	2013	$10^{-8}*$	171s

- QP : $A_8^* \approx 0.726867$
- Gloptipoly: $A_8^* \in [0.72686845, 0.72686849]$
- IBBA: $A_8^* \in [0.72686848979732928, 0.7268684897329281]$

Solution: $x_1 = 0.26214172, x2 = 0.67123417, x3 = 0.67123381, x4 = 0.90909242, x5 = 0.90909213$

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Solvers

- QP : (RLT) $x_i x_j \to w_{ij}$ + McCormick constraints ($w_{ij} \leq .\& \geq .$).
- Gloptipoly:
  SDP relaxation (find polynomial bases - 2nd relaxation)
  VSDP - rigorous upper bounds.
- IBBA:

$$\left\{ \begin{array}{ll} \max\limits_{x \in X \subseteq \mathbb{R}^n} & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \\ & h_j(x) = 0, \end{array} \right. \to \left\{ \begin{array}{ll} \max\limits_{x \in X^{\mathcal{F}} \subseteq \mathcal{F}^n} & f^{\mathcal{F}}(x) \\ \text{s.t.} & g_i^{\mathcal{F}}(x) \leq \epsilon_g^{\mathcal{F}}, \\ & h_j^{\mathcal{F}}(x) \in [-\epsilon_f^{\mathcal{F}}, \epsilon_f^{\mathcal{F}}], \end{array} \right.$$

$$(P) \leq (P_R)$$

Lower bounds : $QP = 0.726867$, $Gloptipoly = 0.72686845$, $IBBA = 0.726868479732928$

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About bounds ?

UB (decreases by its) $\geq (P_R) \geq (P) \geq LB?$

LB:

$A_8^{\mathcal{F}}$: floating point solution of (P_R).

$A_8^{\mathcal{F}} \leq (P_R)$, but

$A_8^{\mathcal{F}} \simeq (P)$?

$A_8^{\mathcal{F}} > (P)$ or $A_8^{\mathcal{F}} >> (P)$?

About bounds ?

UB (decreases by its) $\geq (P_R) \geq (P) \geq LB?$

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Lower bound: a formulation

Remark: The solution is almost symmetric!

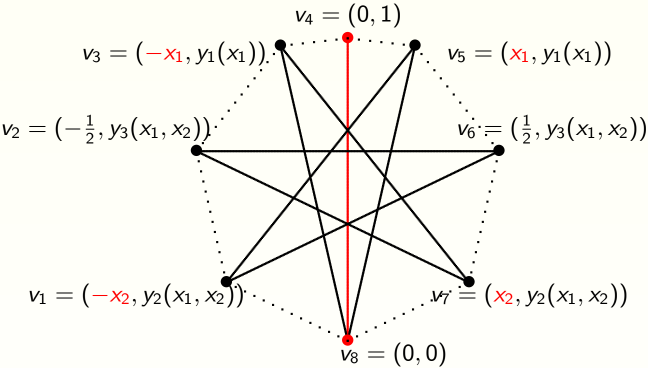

Figure : Symmetric case. Definition of variables.

A nonconvex program for LB

$$\begin{cases} \max\limits_{x_1,x_2} & x_2 y_3 - \frac{1}{2} y_2 + \frac{1}{2} y_1 - x_1 y_3 + x_1 \\ & \\ & 0 \leq x_1 \leq 0.5 \\ & 0 \leq x_2 \leq 0.5. \end{cases}$$

Where

$$y_1(x_1) = \sqrt{1 - x_1^2}$$
$$y_2(x_1, x_2) = y_1(x_1) - \sqrt{1 - (x_1 + x_2)^2}$$
$$y_3(x_1, x_2) = y_2(x_1, x_2) + \sqrt{1 - \left(\frac{1}{2} + x_2\right)^2}$$

IBBA $\longrightarrow A_8^S \in 0.7268684827516[265, 365]$

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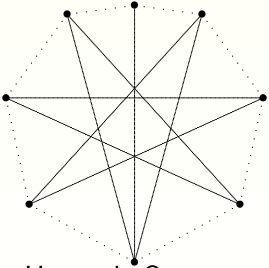
Conclusion on the Hansen's Octagon

- 31 diameter graphs: Graham's conjecture is proved $\longrightarrow$ 1 case (31). Foster and Szabo Results (2007).
- Bounds:

$$A_8^* \in 0.72686848275[16265, 26265]$$

$$\mathit{Gloptipoly} = 0.726868849, \ \mathit{IBBA} = 0.726868489 7329$$

Solutions:



Hansen's Octagon

Area $\approx$ 0.72686848275

Application of the Hansen's Octagon