Separators: a new interval tool to bracket solution sets; application to path planning

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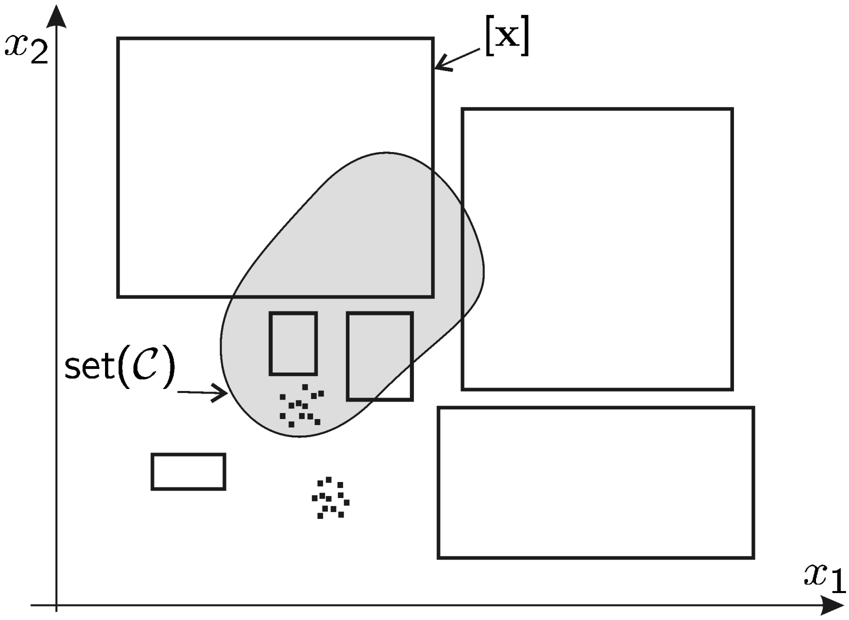
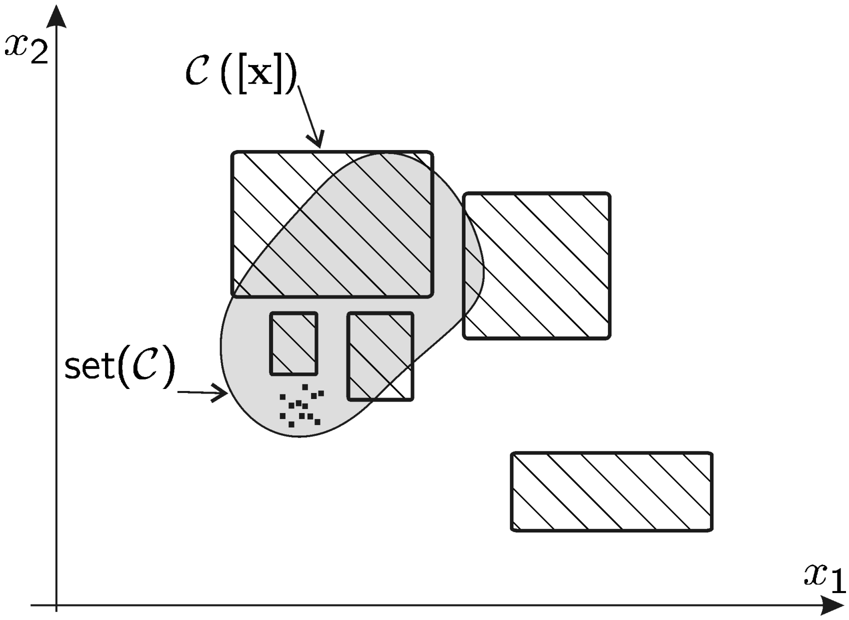
1 Contractors

$$
\begin{array}{ll}
\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\
[\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]) & \text{(monotonicity)}
\end{array}
$$


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
set($\mathcal{C}$)
x_1

Inclusion

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall\, [\mathbf{x}] \in \mathbb{IR}^n,\, \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]).$$

A set $\mathbb{S}$ is *consistent* with the contractor $\mathcal{C}$ (we will write $\mathbb{S} \sim \mathcal{C}$) if

$$\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}.$$

A contractor $\mathcal{C}$ is *minimal* if for any other contractor $\mathcal{C}_1$, we have

$$\mathbb{S} \sim \mathcal{C},\ \mathbb{S} \sim \mathcal{C}_1\ \Rightarrow \mathcal{C} \subset \mathcal{C}_1.$$

We define the *negation* $\neg\mathcal{C}$ of a contractor $\mathcal{C}$ as follows

$$\neg\mathcal{C}\left([\mathbf{x}]\right) = \left\{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{x} \notin \mathcal{C}\left([\mathbf{x}]\right)\right\}.$$

which is not a box in general.

1.1 Separators

A *separator* $\mathcal{S}$ is pair of contractors $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ such that

$$\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}] \qquad \text{(complementarity).}$$

A set $\mathbb{S}$ is *consistent* with the separator $\mathcal{S}$ (we will write $\mathbb{S} \sim \mathcal{S}$), if

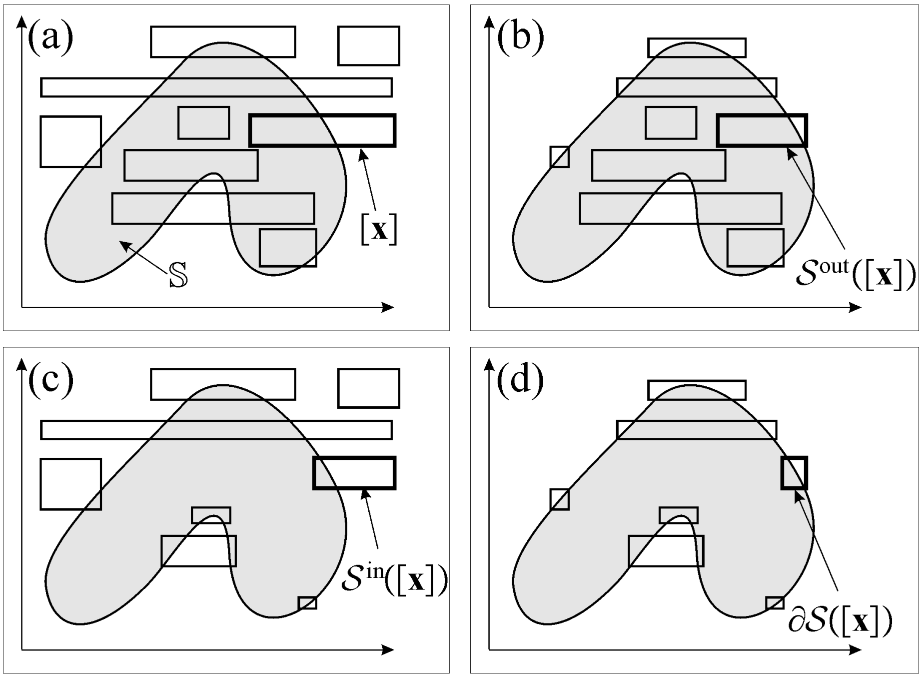
$$\mathbb{S} \sim \mathcal{S}^{\text{out}} \text{ and } \overline{\mathbb{S}} \sim \mathcal{S}^{\text{in}}.$$

We define the *remainder* of a separator $\mathcal{S}$ as

$$\partial\mathcal{S}([\mathbf{x}]) = \mathcal{S}^{\mathsf{in}}([\mathbf{x}]) \cap \mathcal{S}^{\mathsf{out}}([\mathbf{x}]). \tag{1}$$

Note that the remainder is a contractor and not a separator.

Example.

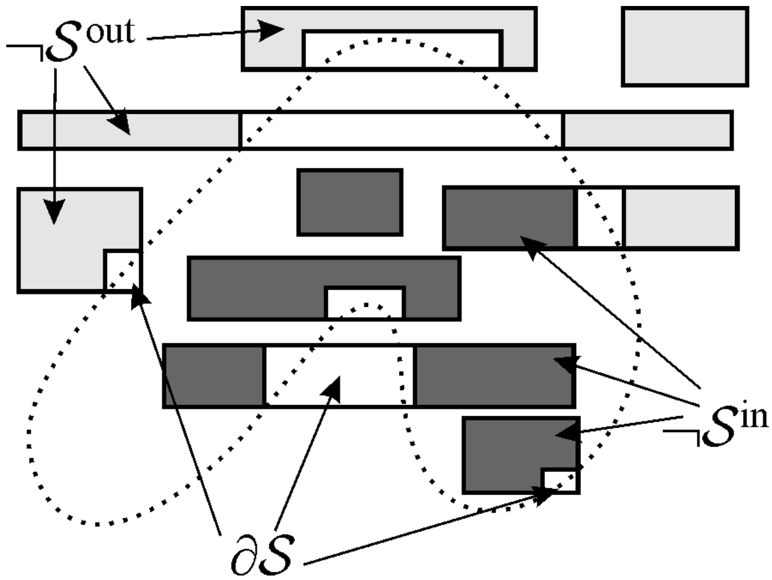


A separator $\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\}$ is a pair of two contractors.

$\neg\mathcal{S}^{\text{in}}([\mathbf{x}])$, $\neg\mathcal{S}^{\text{out}}([\mathbf{x}])$ and $\partial\mathcal{S}([\mathbf{x}])$ cover $[\mathbf{x}]$, i.e.,

$$\neg\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \neg\mathcal{S}^{\text{out}}([\mathbf{x}]) \cup \partial\mathcal{S}([\mathbf{x}]) = [\mathbf{x}]\,.$$

Moreover, they do not overlap each other.



$\neg\mathcal{S}^{\mathsf{in}}([\mathbf{x}])$, $\neg\mathcal{S}^{\mathsf{out}}([\mathbf{x}])$ and $\partial\mathcal{S}([\mathbf{x}])$

Inclusion

$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \mathcal{S}_1^{\text{in}} \subset \mathcal{S}_2^{\text{in}} \text{ and } \mathcal{S}_1^{\text{out}} \subset \mathcal{S}_2^{\text{out}}.$$

A separator $\mathcal{S}$ is *minimal* if

$$\mathcal{S}_1 \subset \mathcal{S} \Rightarrow \mathcal{S}_1 = \mathcal{S}.$$

$\mathcal{S}$ is minimal if and only if $\mathcal{S}^{\text{in}}$ and $\mathcal{S}^{\text{out}}$ are both minimal.

1.2 Paver

We want an enclosure

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+$$

Algorithm Paver(in: $[\mathbf{x}], \mathcal{S}$; out: $\mathbb{X}^-$, $\mathbb{X}^+$)

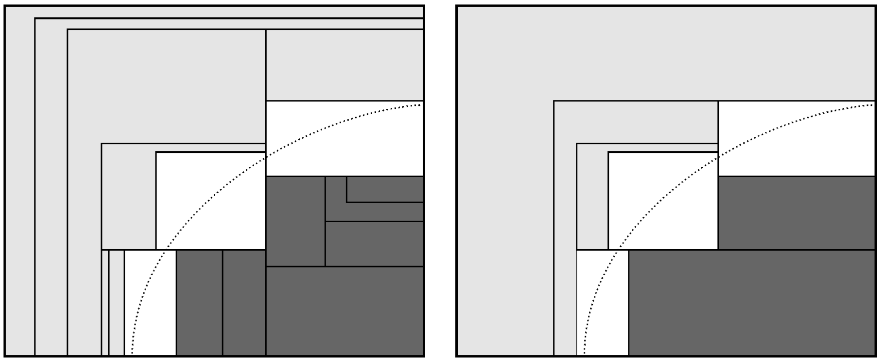
1 $\mathcal{L} := \{[\mathbf{x}]\}$;
2 Pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 $\left\{[\mathbf{x}^{\text{in}}], [\mathbf{x}^{\text{out}}]\right\} = \mathcal{S}([\mathbf{x}])$;
4 Store $[\mathbf{x}] \backslash [\mathbf{x}^{\text{in}}]$ into $\mathbb{X}^-$ and also into $\mathbb{X}^+$;
5 $[\mathbf{x}] = [\mathbf{x}^{\text{in}}] \cap [\mathbf{x}^{\text{out}}]$;
6 If $w([\mathbf{x}]) < \varepsilon$, then store $[\mathbf{x}]$ in $\mathbb{X}^+$,
7 Else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$ the two resulting boxes
8 If $\mathcal{L} \neq \emptyset$, go to 2.

For the implementation, the resulting paving can be represented by a binary tree.

The ith node of the tree contains the two boxes $[\mathbf{x}^{\mathsf{in}}](i)$ and $[\mathbf{x}^{\mathsf{out}}](i)$.

The binary tree is said to be minimal if for any node i_1 with brother i_2 and father j, we have

$$\left\{\begin{array}{l} [\mathbf{x}^{\mathsf{in}}](i_1) \neq \emptyset,\ [\mathbf{x}^{\mathsf{out}}](i_1) \neq \emptyset \\ [\mathbf{x}^{\mathsf{in}}](j) = [\mathbf{x}^{\mathsf{in}}](i_1) \sqcup [\mathbf{x}^{\mathsf{in}}](i_2) \\ [\mathbf{x}^{\mathsf{out}}](j) = [\mathbf{x}^{\mathsf{out}}](i_1) \sqcup [\mathbf{x}^{\mathsf{out}}](i_2) \end{array}\right.$$



left: before simplication, right: after.

2 Algebra

Contractor algebra does not allow decreasing operations, i.e., we should have expression $\mathcal{E}$ such that

$$\forall i, \mathcal{C}_i \subset \mathcal{C}_i^{'} \Rightarrow \mathcal{E}\left(\mathcal{C}_1, \mathcal{C}_2, \dots\right) \subset \mathcal{E}\left(\mathcal{C}_1^{'}, \mathcal{C}_2^{'}, \dots\right).$$

The complementary $\overline{\mathcal{C}}$ of a contractor $\mathcal{C}$ or the restriction $\mathcal{C}_1 \backslash \mathcal{C}_2$ of two contractors cannot be defined.

Separators extend the operations allowed for contractors to non monotonic expressions.

If $\mathcal{S} = \left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ is a separator, we define the *complement* as

$$\overline{\mathcal{S}} = \left\{\mathcal{S}^{\text{out}}, \mathcal{S}^{\text{in}}\right\}.$$

We define the *exponentiation* of a separator $\mathcal{S} = \left\{\mathcal{S}^{\mathsf{in}}, \mathcal{S}^{\mathsf{out}}\right\}$ as:

$$\begin{aligned} \mathcal{S}^{0} &= \{\top, \top\} \\ \mathcal{S}^{k+1} &= \left\{\neg\mathcal{S}^{k\ \mathsf{out}} \sqcup (\mathcal{S}^{\mathsf{in}} \circ \partial\mathcal{S}^{k}), \neg\mathcal{S}^{k\ \mathsf{in}} \sqcup (\mathcal{S}^{\mathsf{out}} \circ \partial\mathcal{S}^{k})\right\}. \end{aligned}$$

Example.

$$
\begin{aligned}
\mathcal{S}^1 \quad = \quad \{ \quad & \neg(\mathcal{S}^0)^{\mathsf{out}} \sqcup \mathcal{S}^{\mathsf{in}} \circ (\left(\mathcal{S}^0\right)^{\mathsf{in}} \cap \left(\mathcal{S}^0\right)^{\mathsf{out}}), \\
& \neg(\mathcal{S}^0)^{\mathsf{in}} \sqcup \mathcal{S}^{\mathsf{out}} \circ (\left(\mathcal{S}^0\right)^{\mathsf{in}} \cap \left(\mathcal{S}^0\right)^{\mathsf{out}}) \quad \} \\
= \; & \left\{ \neg\top\mathcal{S}^{\mathsf{in}} \circ \top \cap \top), \neg\top \sqcup \mathcal{S}^{\mathsf{out}} \circ \top \cap \top) \right\} \\
= \; & \left\{ \mathcal{S}^{\mathsf{in}}, \mathcal{S}^{\mathsf{out}}) \right\} = \mathcal{S}.
\end{aligned}
$$

If $\mathcal{S}_i = \left\{\mathcal{S}_i^{\text{in}}, \mathcal{S}_i^{\text{out}}\right\}, i \in \{1, 2, \dots\}$ are separators, we define

$$\begin{array}{rclc} \mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\right\} & \text{(intersection)} \\ \mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\right\} & \text{(union)} \\ \overset{\{q\}}{\bigcap} \mathcal{S}_i & = & \left\{ \overset{\{m-q-1\}}{\bigcap} \mathcal{S}_i^{\text{in}}, \overset{\{q\}}{\bigcap} \mathcal{S}_i^{\text{out}} \right\} & \text{(relaxed intersection)} \\ \mathcal{S}_1 \backslash \mathcal{S}_2 & = & \mathcal{S}_1 \cap \overline{\mathcal{S}_2}. & \text{(difference)} \end{array}$$

Theorem 1. If $\mathbb{S}_i$ are sets of $\mathbb{R}^n$, we have

$$\begin{array}{cccc}
\text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 & \sim & \mathcal{S}_1 \cap \mathcal{S}_2 \\
\text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 & \sim & \mathcal{S}_1 \cup \mathcal{S}_2 \\
\text{(iii)} & \overline{\mathbb{S}}_i & \sim & \overline{\mathcal{S}_i} \\
\text{(iv)} & \mathbb{S}_i & \sim & \mathcal{S}_i^k,\ k \geq 0 \\
\text{(v)} & \bigcap\limits^{\{q\}} \mathbb{S}_i & \sim & \bigcap\limits^{\{q\}} \mathcal{S}_i \\
\text{(vi)} & \mathbb{S}_1 \backslash \mathbb{S}_2 & \sim & \mathcal{S}_1 \backslash \mathcal{S}_2.
\end{array}$$

3 Inversion of separators

The inverse of a set $\mathbb{Y} \subset \mathbb{R}^n$ by a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) = \{\mathbf{x} \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\}.$$

$\mathbf{f}$ can be a translation, rotation, homothety, projection, ...

We define

$$\mathbf{f}^{-1}\left(\mathcal{S}_{\mathbb{Y}}\right) = \left\{\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}^{\mathsf{in}}), \mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}^{\mathsf{out}})\right\}.$$

Theorem 2. The separator $\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}})$ is a separator associated with the set $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$, i.e.,

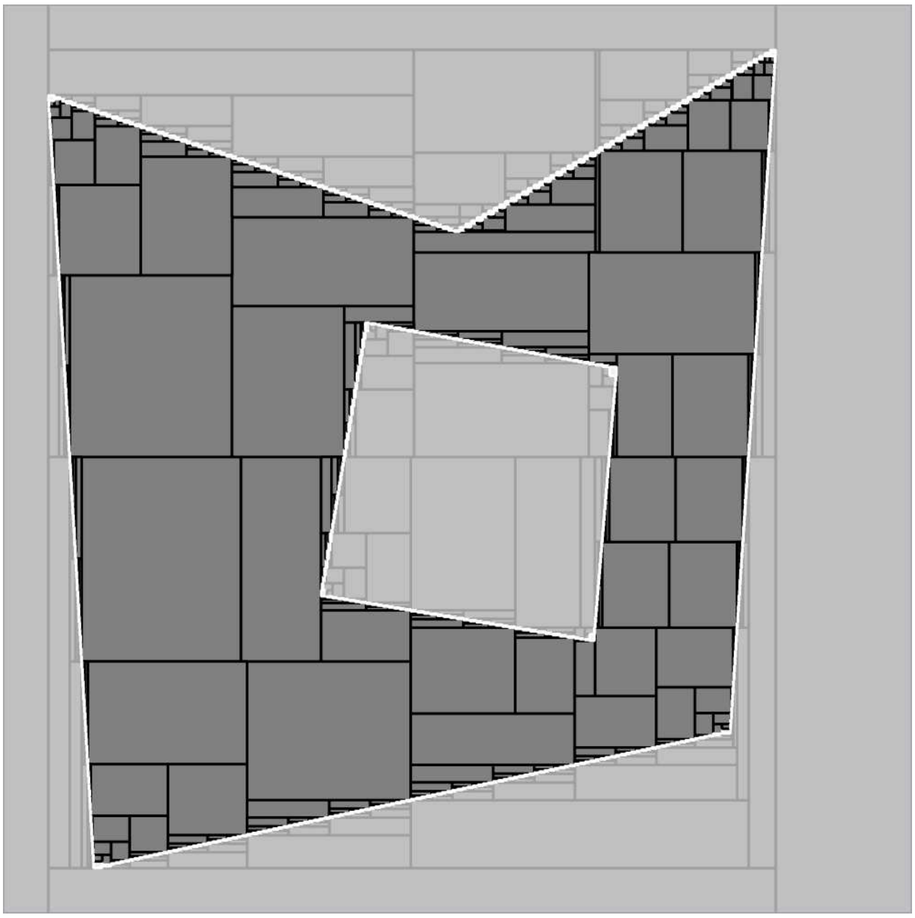
$$\mathbf{f}^{-1}(\mathbb{Y}) \sim \mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}}).$$

Example. If

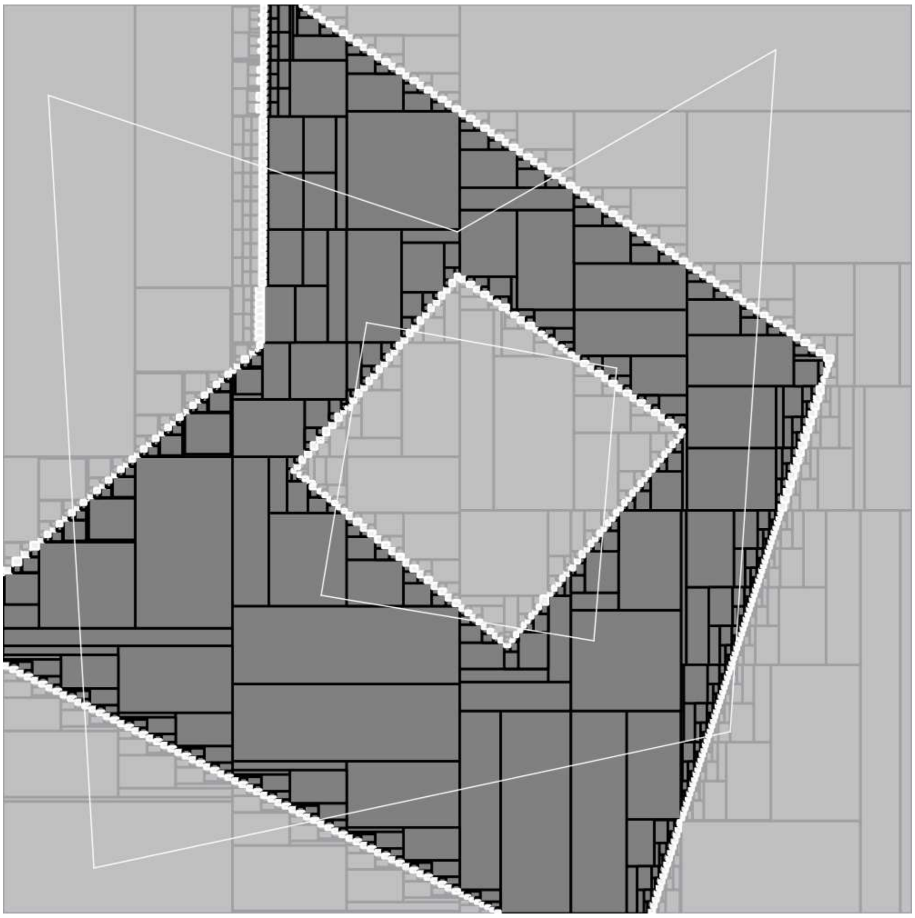
$$\mathbf{f}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix}. \qquad (2)$$

The separator $\mathbf{f}^{-1}(\mathcal{S}_{\mathbb{Y}})$ corresponds to the following algorithm.

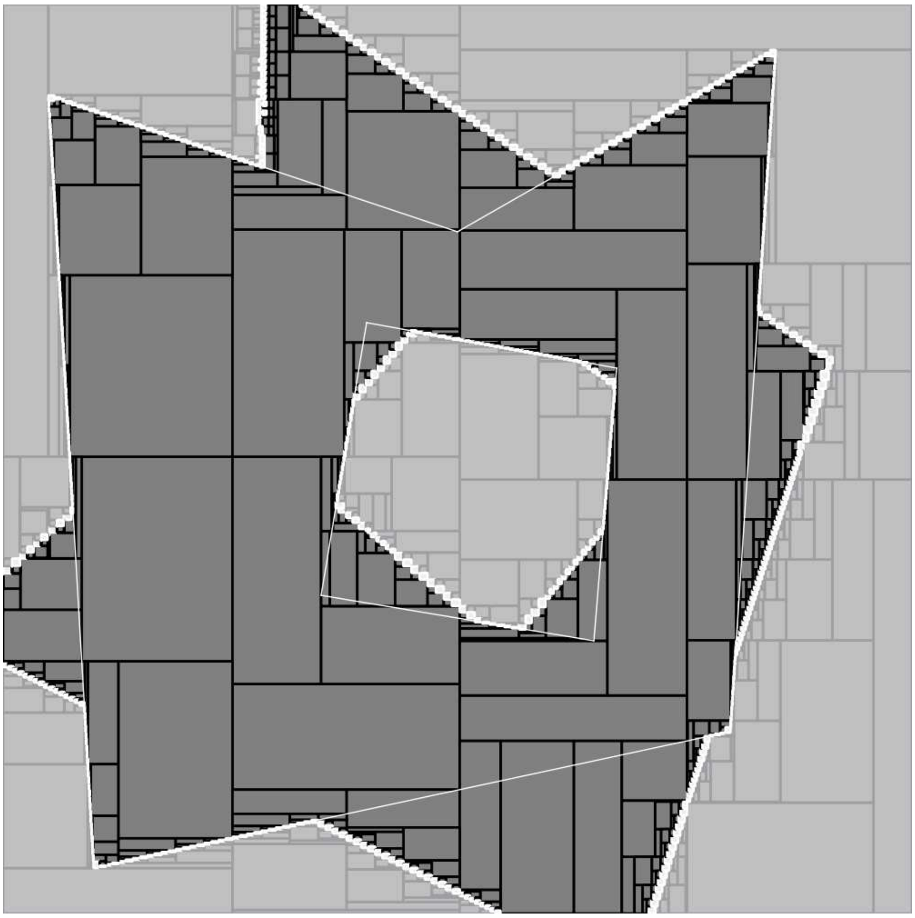
Separator $\mathcal{S}_{\mathbb{X}}$(in: $[\mathbf{x}], \mathcal{S}_{\mathbb{Y}}$; out: $\{[\mathbf{x}^{\text{in}}], [\mathbf{x}^{\text{out}}]\}$)	
1	$[y_1] = [x_1] + 2[x_2]$;
2	$[y_2] = [x_1] - [x_2]$;
3	$[\mathbf{y}^{\text{in}}] = \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}])$; $[\mathbf{y}^{\text{out}}] = \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}])$;
4	$[\mathbf{x}^{\text{in}}] = [\mathbf{x}]$; $[\mathbf{x}^{\text{out}}] = [\mathbf{x}]$;
5	$[x_1^{\text{in}}] = [x_1^{\text{in}}] \cap \left([y_2^{\text{in}}] + [x_2^{\text{in}}]\right)$;
6	$[x_2^{\text{in}}] = [x_2^{\text{in}}] \cap \left([x_1^{\text{in}}] - [y_2^{\text{in}}]\right)$;
7	$[x_1^{\text{out}}] = [x_1^{\text{out}}] \cap \left([y_2^{\text{out}}] + [x_2^{\text{out}}]\right)$;
8	$[x_2^{\text{out}}] = [x_2^{\text{out}}] \cap \left([x_1^{\text{out}}] - [y_2^{\text{out}}]\right)$;
9	$[x_1^{\text{in}}] = [x_1^{\text{in}}] \cap [y_1^{\text{in}}] - 2[x_2^{\text{in}}]$;
10	$[x_2^{\text{in}}] = [x_2^{\text{in}}] \cap \frac{1}{2}\left([y_1^{\text{in}}] - [x_1^{\text{in}}]\right)$;
11	$[x_1^{\text{out}}] = [x_1^{\text{out}}] \cap [y_1^{\text{out}}] - 2[x_2^{\text{out}}]$;
12	$[x_2^{\text{out}}] = [x_2^{\text{out}}] \cap \frac{1}{2}\left([y_1^{\text{out}}] - [x_1^{\text{out}}]\right)$.



A map $\mathbb{M}$



$\mathsf{Rot}(\mathbb{M})$



$\mathsf{Rot}(\mathbb{M}) \cap \mathbb{M}$

4 Atomic separators

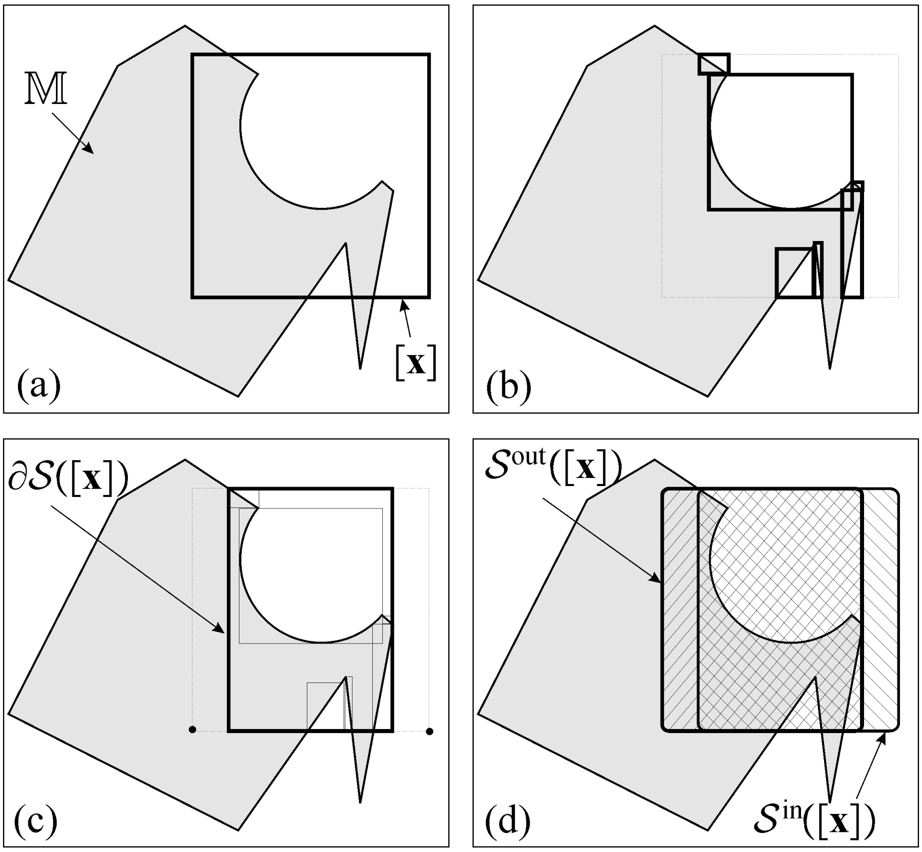
4.1 Equation-based separators

For instance,

$$\mathbb{X} = \{f(\mathbf{x}) \leq 0\},$$

we can easily build a contractor $\mathcal{S}^{\text{out}}$: $f(\mathbf{x}) \leq 0$ and a contractor $\mathcal{S}^{\text{in}}$: $f(\mathbf{x}) \geq 0$. The pair $\left\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\right\}$ is then a separator associated with $\mathbb{X}$.

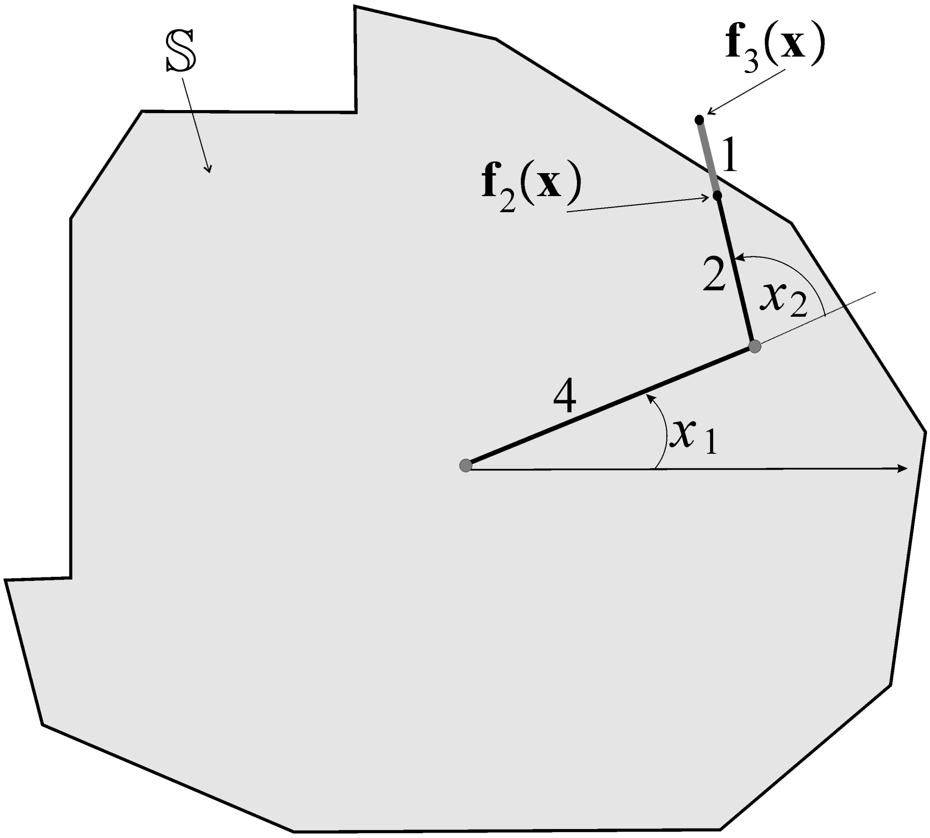
4.2 Database-based separators



Optimal separator using boundaries

5 Path planning

Wire loop game : a metal loop on a handle and a curved wire. The player holds the loop in one hand and attempts to guide it along the curved wire without touching.



Wire loop game. Is it possible to perform to complete circular path ?

The feasible configuration space is

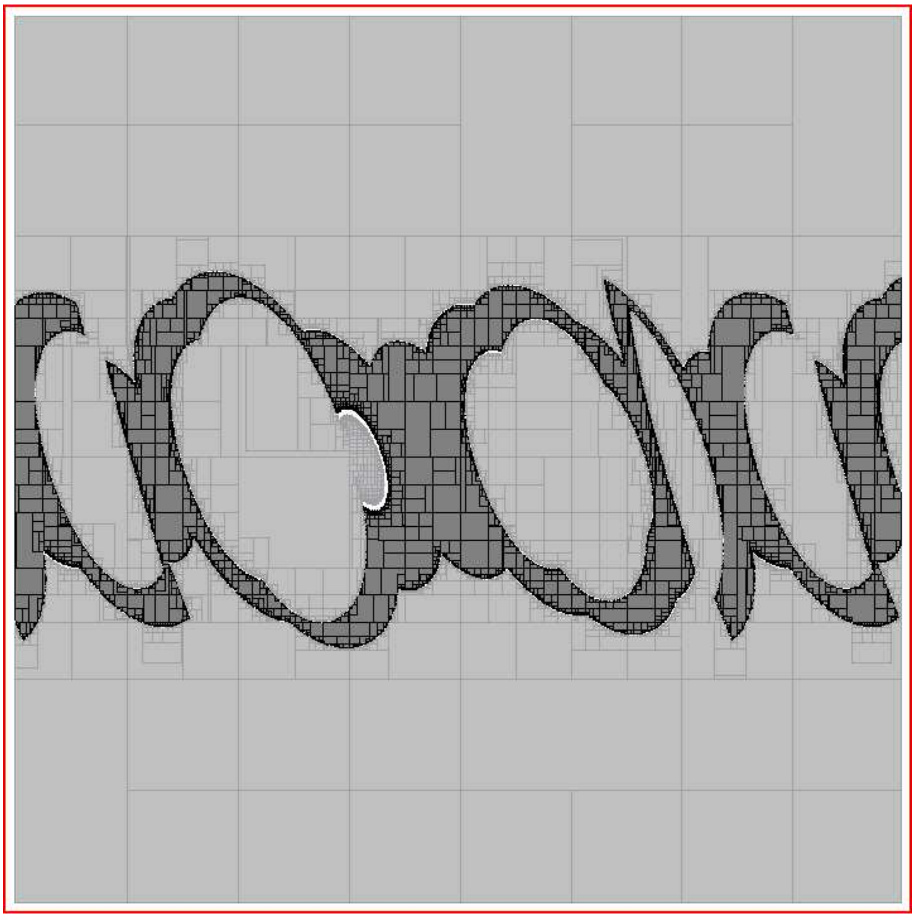
$$\mathbb{M} = \{(x_1, x_2) \in [-\pi, \pi] \mid \mathbf{f}_2(\mathbf{x}) \in \mathbb{Y} \text{ and } \mathbf{f}_3(\mathbf{x}) \notin \mathbb{Y}\}$$

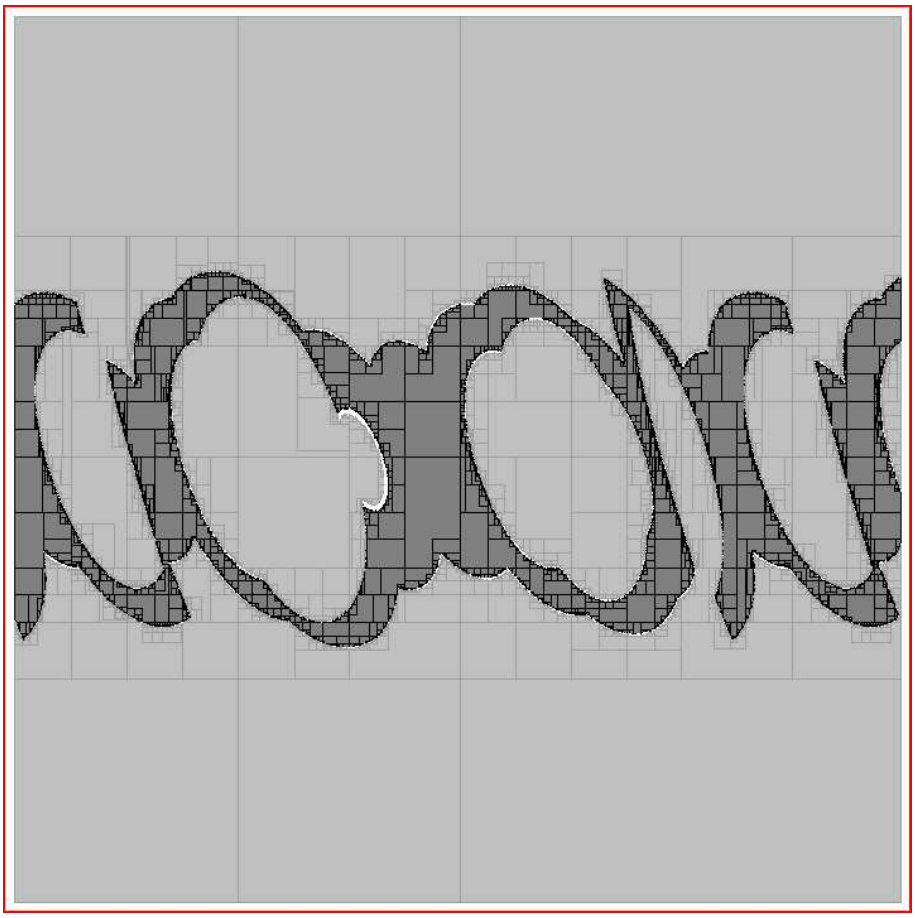
where

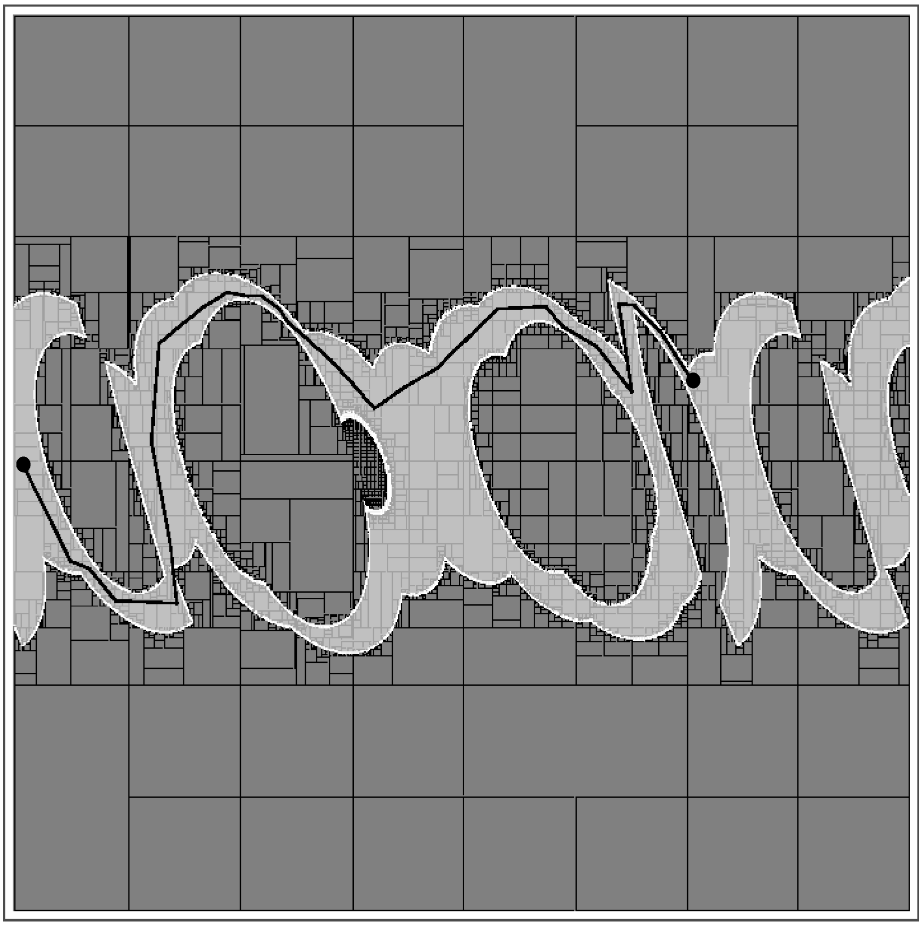
$$\mathbf{f}_\ell(\mathbf{x}) = 4 \begin{pmatrix} \cos x_1 \\ \sin x_1 \end{pmatrix} + \ell \begin{pmatrix} \cos(x_1 + x_2) \\ \sin(x_1 + x_2) \end{pmatrix}.$$

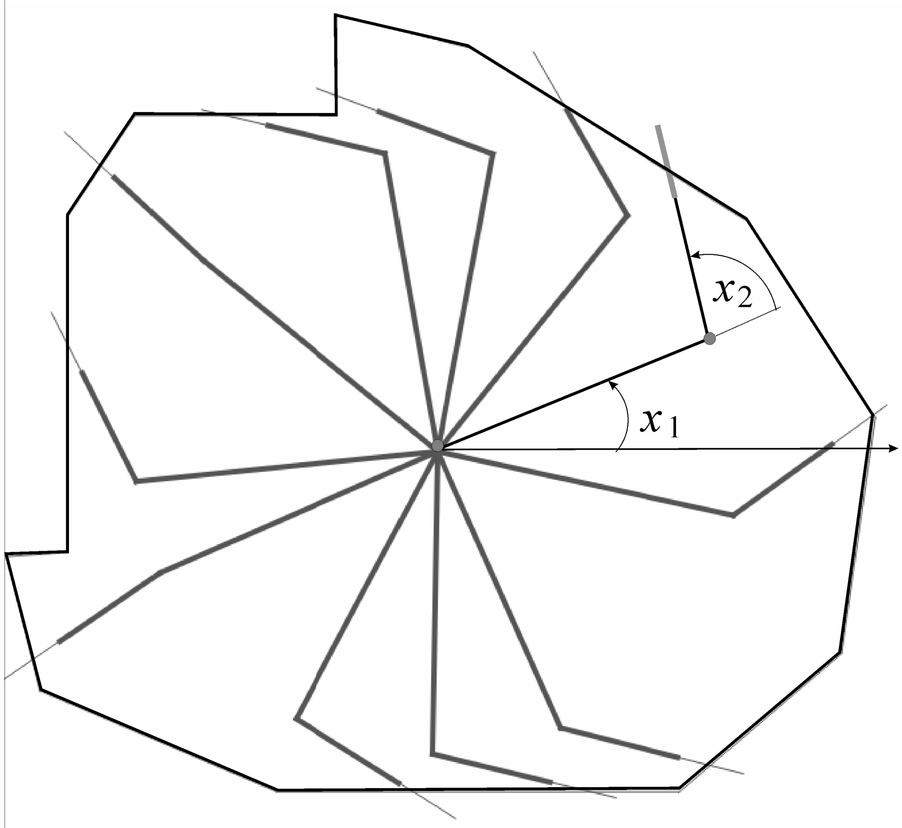
If $\mathcal{S}_{\mathbb{Y}}$ is a separator associated with $\mathbb{Y}$, then a separator for $\mathbb{M}$ is

$$\mathcal{S}_{\mathbb{M}} = \mathbf{f}_2^{-1}(\mathcal{S}_{\mathbb{Y}}) \cap \mathbf{f}_3^{-1}\left(\overline{\mathcal{S}_{\mathbb{Y}}}\right).$$









x_2
x_1