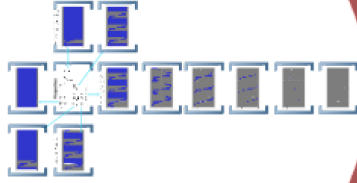
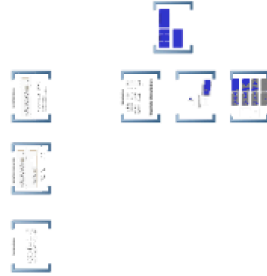




4 - Simple example



5 - Spring-Mass example



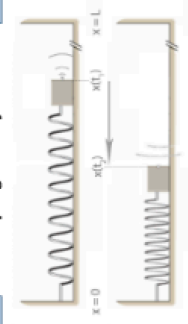
Solving non-linear constraint satisfaction problem involving time-dependant function.

Ayméric BETHENCOURT



1 - Introduction

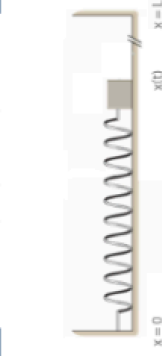
Mass-Spring-Sonar System



$$(t - x(t_1)) + (t - x(t_2)) = c(t_2 - t_1)$$

$$\Rightarrow x(t_2) = x(t_1) - 2L - c(t_2 - t_1)$$

Mass-Spring-Sonar System



$$m\ddot{x} + \gamma\dot{x} + kx - \beta x^3 = 0$$

Non-linear spring
No info on initial conditions

Problem

Non-linear constraint satisfaction problem involving time-dependant function.

- Simple solving with EKF ?
- We are going to use interval of functions.

Plan

1. Introduction.
2. Interval of functions (Tubes).
3. Contractors on tubes.
4. Simple example.
5. Spring-Mass-Sonar example.
6. Conclusion

Mass-Spring-Sonar System

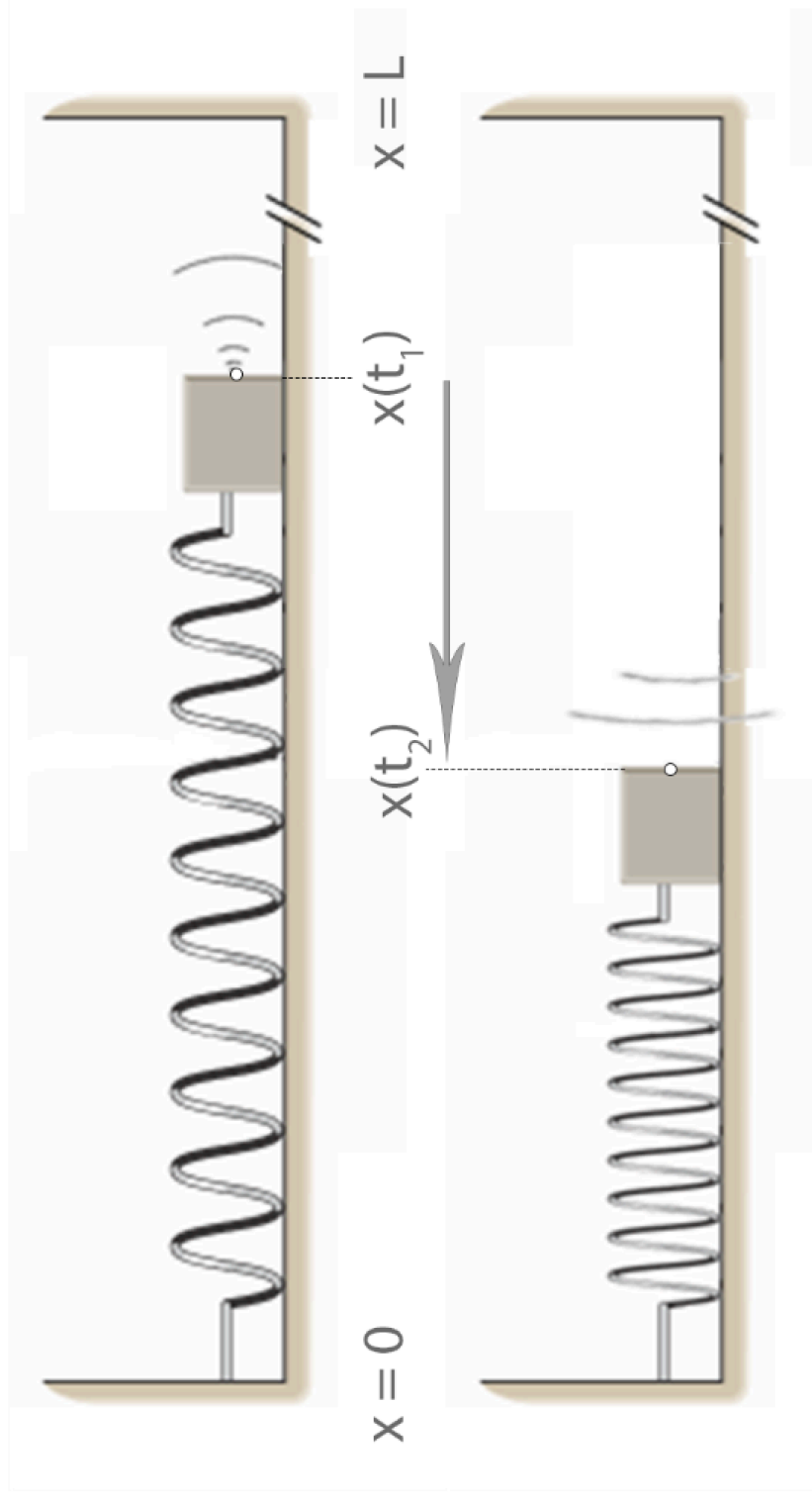


$$m.\ddot{x} + \gamma.\dot{x} + \kappa.x - \beta x^3 = 0$$

Non-linear spring

No info on initial conditions

Mass-Spring-Sonar System



$$\begin{aligned} (L - x(t_1)) + (L - x(t_2)) &= c(t_2 - t_1) \\ \Leftrightarrow x(t_2) + x(t_1) &= 2L - c(t_2 - t_1) \end{aligned}$$

Problem

Non-linear constraint satisfaction problem involving time-dependant function.

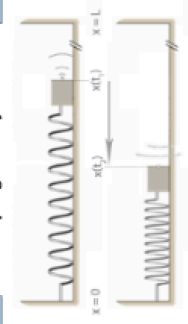
- Simple solving with EKF ?
- We are going to use interval of functions.

Plan

1. Introduction.
2. Interval of functions (Tubes).
3. Contractors on tubes.
4. Simple exemple.
5. Spring-Mass-Sonar example.
6. Conclusion

1 - Introduction

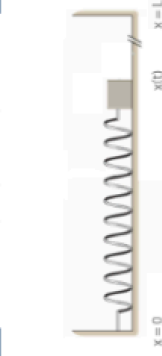
Mass-Spring-Sonar System



$$(t - x(t_1)) + (t - x(t_2)) = c(t_2 - t_1)$$

$$\Rightarrow x(t_2) = x(t_1) - 2L + c(t_2 - t_1)$$

Mass-Spring-Sonar System



$$m\ddot{x} + \gamma\dot{x} + kx - \beta x^3 = 0$$

Non-linear spring
No info on initial conditions

Problem

Non-linear constraint satisfaction problem involving time-dependant function.

- Simple solving with EKF ?
- We are going to use interval of functions.

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6. Conclusion

2 - Intervals of fonctions

Lattice

When basic interval analysis
is performed on a function, the
result is a set of intervals.
These intervals are called
"intervals of the lattice".

Lattice

A lattice L is a partially ordered
set $(L, \leq)$ with a least element
and a greatest element.

Example: $\mathbb{Z} \times \mathbb{Z}$
Signature: $\dots, -1, 0, 1, \dots$

Values

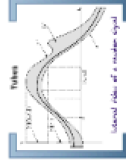
Interval: $[a, b]$ is a subset of $\mathbb{R}$
if $a \leq b$ and $a, b \in \mathbb{R}$.

Operations

Extended operations on lattice to
intervals of the lattice.
The extension is also called
interval extension of the
operation.

Theorem

The interval extension of a
operator f is:

$$f([a, b]) = [f(a), f(b)]$$


Operators

Extremum of a function f on an interval I is:

$$f(I) = [f(a), f(b)]$$

Lattice

- We use classical interval analysis where the unknown variables are trajectories (function from $\mathbb{R}$ to $\mathbb{R}^n$).
- Trajectories belong to a lattice.

Lattice

A lattice $(\mathcal{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds.

Infimum $x \vee y$
(The least upper bound)

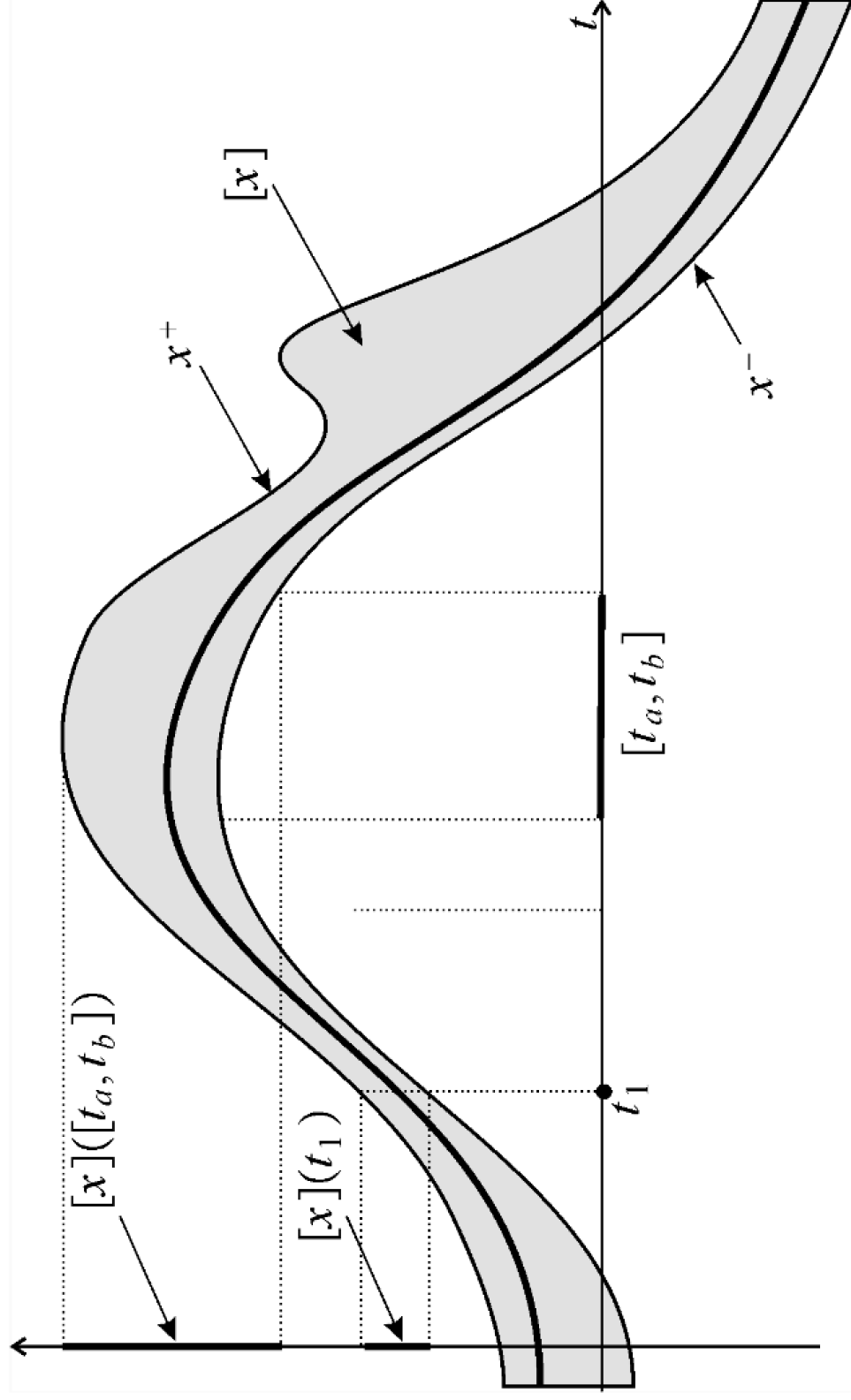
Supremum $x \wedge y$
(The greatest lower bound)

Tubes

The tube $[\mathbf{x}]$ is an interval $[\mathbf{x}^-, \mathbf{x}^+]$ of $\mathcal{F}^n$

$$\mathbf{x}^-(t) \leq \mathbf{x}(t) \leq \mathbf{x}^+(t)$$

Tubes



Interval vision of a random signal

Operations

Extend operators on lattices to intervals of this lattice.

The extension is the smallest interval enclosing all possible values.

Operations

Theorem

The integral (monotonic operator) verify

$$\int_{t_1}^{t_2} [\mathbf{x}] (\tau) d\tau = \left[\int_{t_1}^{t_2} \mathbf{x}^{-} (\tau) d\tau, \int_{t_1}^{t_2} \mathbf{x}^{+} (\tau) d\tau \right]$$

Operators

Extensions of binary operators in the Minkowski sense.

$$[x] + [y] = [x^- + y^-, x^+ + y^+]$$

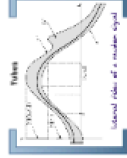
2 - Intervals of fonctions

Lattice
When basic interval analysis
uses binary functions from $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$,
the operators belong to a lattice.

Lattice
A lattice $(L, \leq)$ is a partially ordered
set with least upper and
greatest lower bounds.
Infimum: $\inf$
Supremum: $\sup$

Takes
Interval functions are $\mathbb{R}^n \rightarrow \mathbb{R}^m$
 $\mathbf{x}^* \in \mathbb{R}^n, \mathbf{y}^* \in \mathbb{R}^m$

Operations
Extend operators on lattice to
intervals of the lattice.
This extension is also needed
interval extending all possible
values.



Theorem
The interval of a function
operator verify:
$$f([a, b]) = \left[\inf_{x \in [a, b]} f(x), \sup_{x \in [a, b]} f(x) \right]$$

Operators
Extremum of binary functions
on the interval $[a, b]$
 $f(a) = \inf_{x \in [a, b]} f(x), f(b) = \sup_{x \in [a, b]} f(x)$

3 - Contractors on tubes

Definitions

Contract: (T, φ)
 Associated
 contract: (φ, φ)
 Contract: (φ, φ)
 Conditions: $(\varphi, \varphi) = \varphi$

Proposition 1
 The contract associated with
 $\varphi \in \mathcal{C}(T)$

Proposition 2
 The contract associated with
 $\varphi \in \mathcal{C}(T)$

Proposition 3
 The contract associated with
 $\varphi \in \mathcal{C}(T)$

Definitions

Constraint

$$\mathcal{L}(x).$$

Associated
contractor

$$[x] \xrightarrow{C_{\mathcal{L}}} [y]$$

Contracting
Completeness

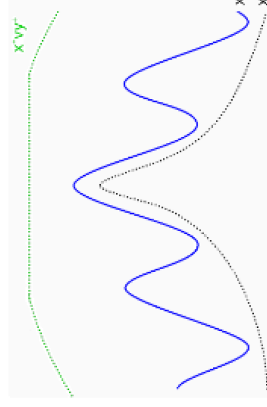
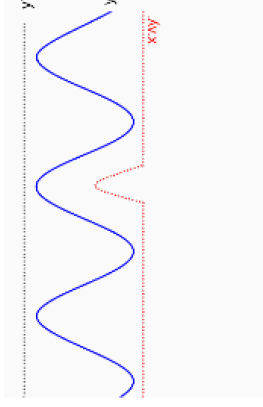
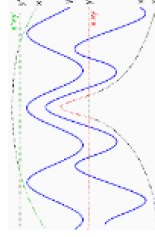
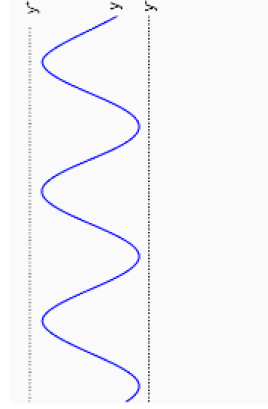
$$\left(\begin{array}{l} \forall t, [x](t) \subset [y](t) \\ \mathcal{L}(x) \\ x \in [x] \end{array} \right) \Rightarrow x \in [y]$$

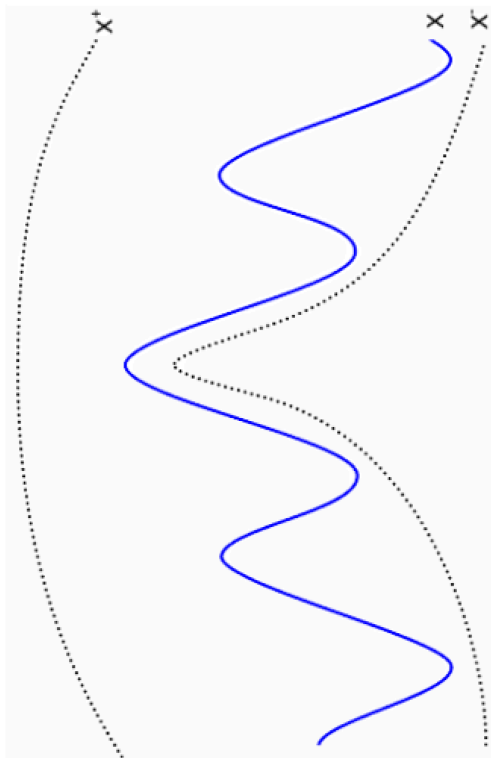
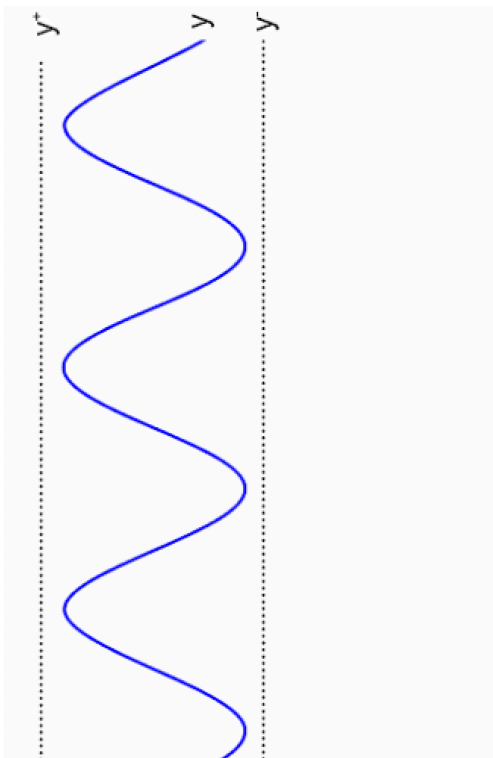
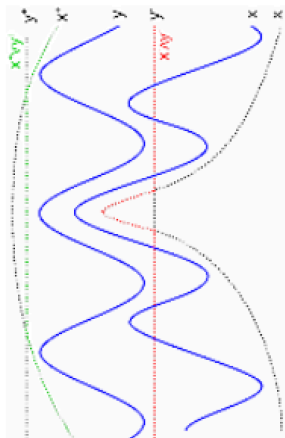
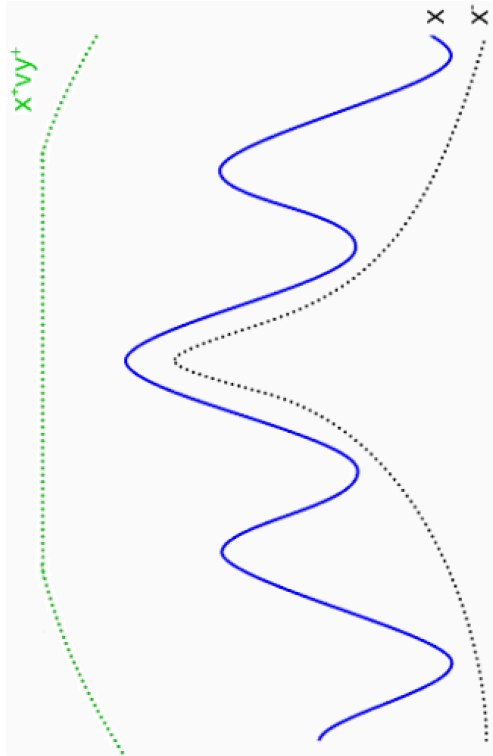
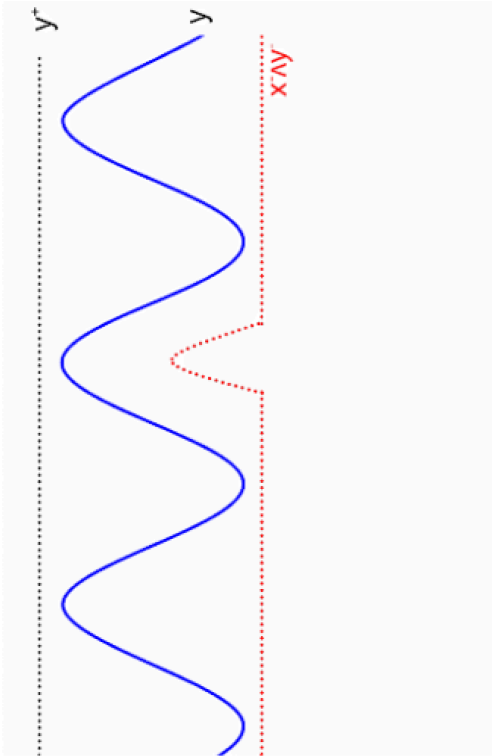
Proposition 1

The minimal contractor associated with

$$x \leq y$$

$$C_{\leq} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \left(\begin{bmatrix} x^- , x^+ \vee y^+ \\ x^- \wedge y^- , y^+ \end{bmatrix} \right)$$



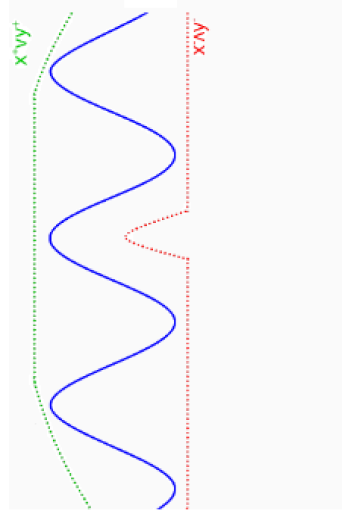
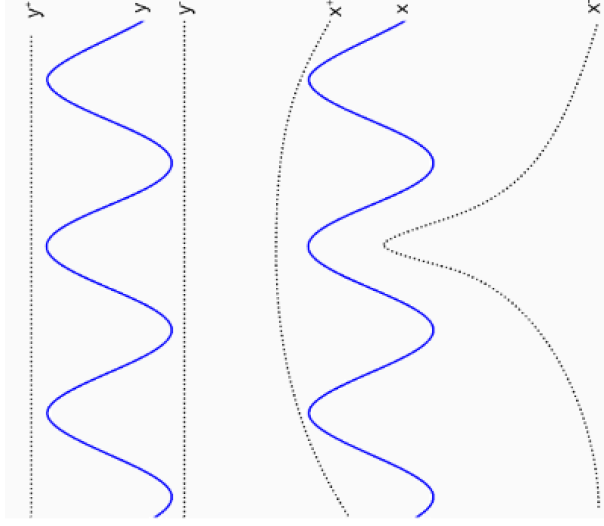


Proposition 2

The minimal contractor associated with

$$x = y$$

$$C_{=} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \left(\begin{bmatrix} x^- \wedge y^-, x^+ \vee y^+ \\ x^- \wedge y^-, x^+ \vee y^+ \end{bmatrix} \right)$$



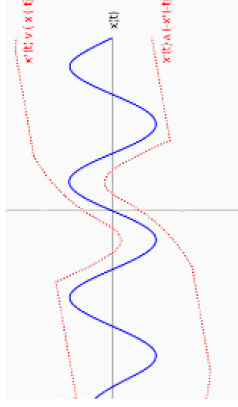
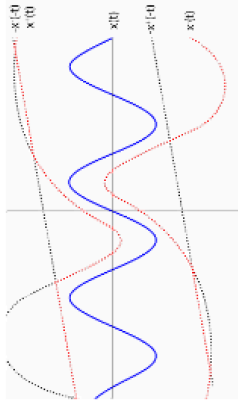
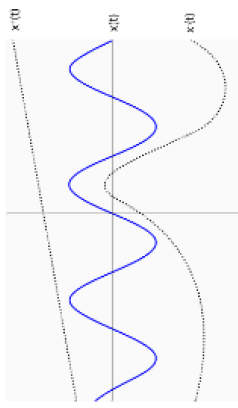
Proposition 4

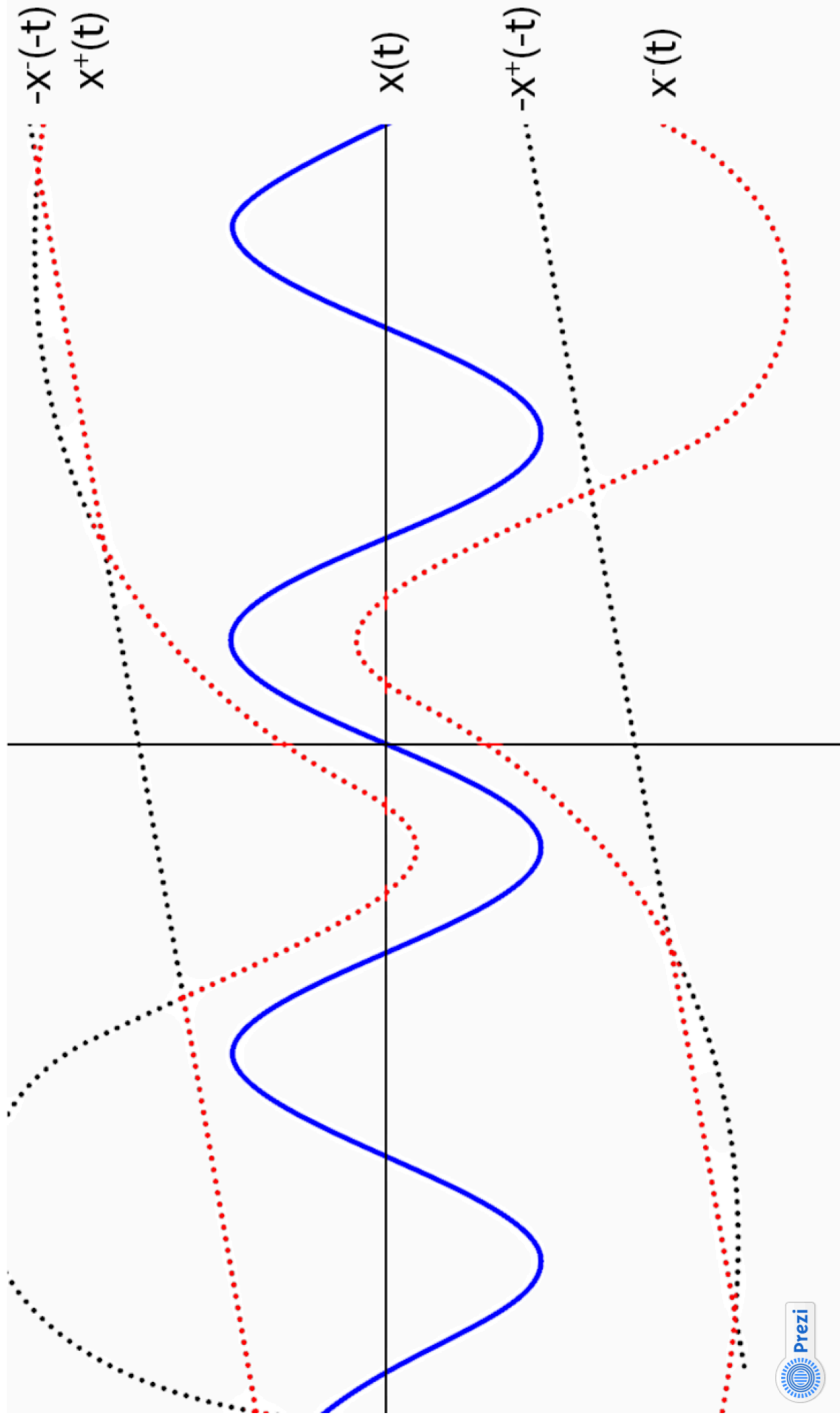
The minimal contractor associated with a central symmetry at $t=T$

$$C_{CSym}([x(T+t)]) =$$

$$[x^-(T+t) \vee (-x^+(T-t)); x^+(T+t) \wedge (-x^-(T-t))]$$

$$C_{odd}([x(t)]) = [x^-(t) \vee (-x^+(-t)); x^+(t) \wedge (-x^-(-t))]$$

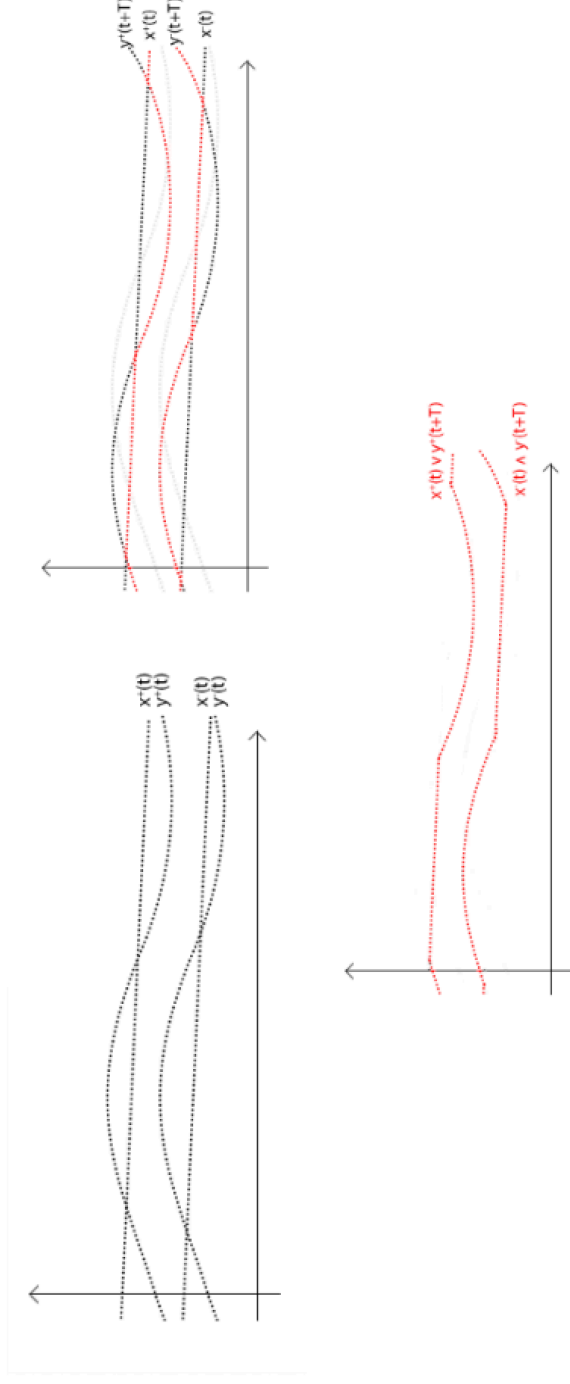




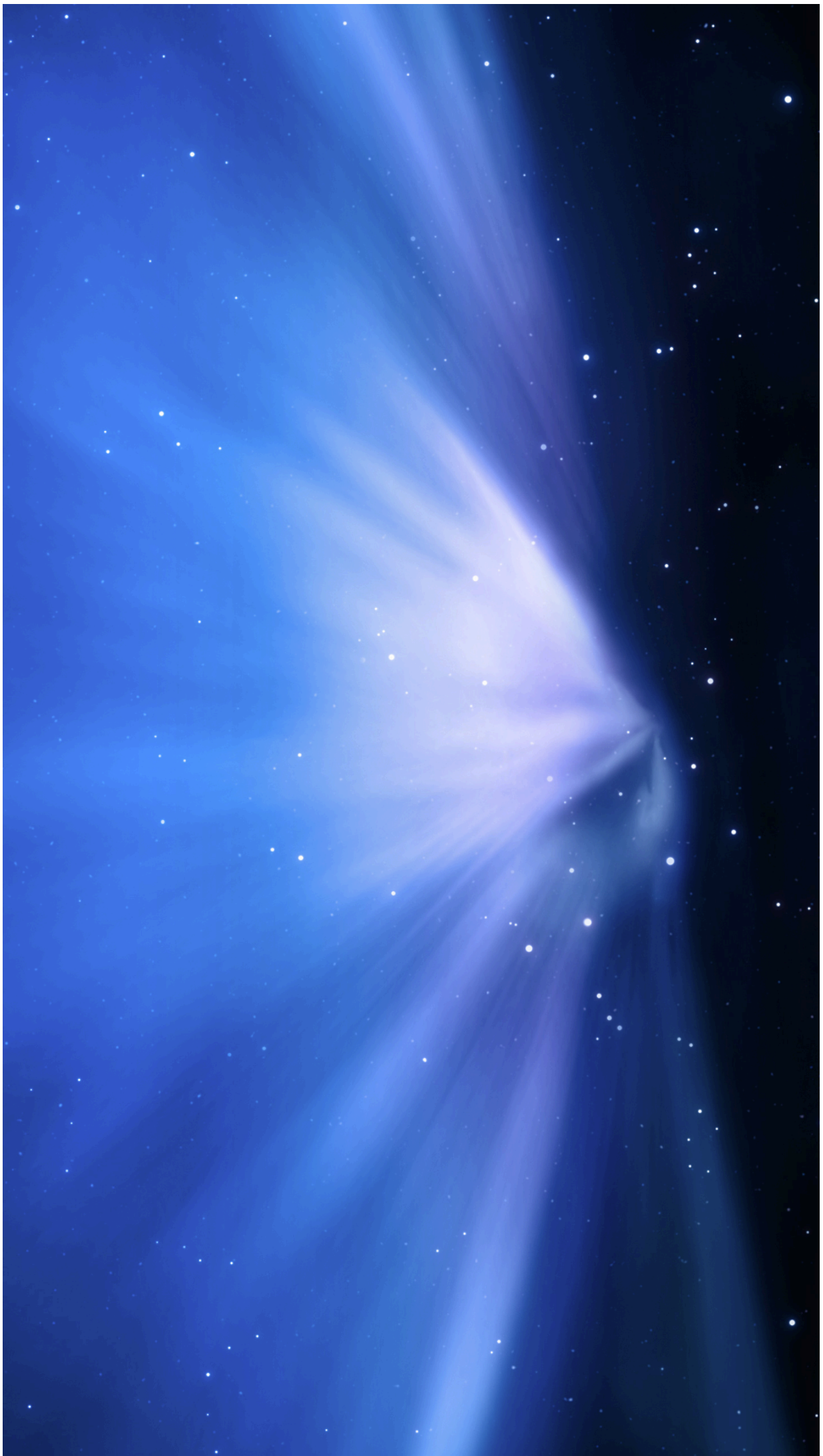
Proposition 5

The minimal contractor associated with a delay $x(t)=y(T+t)$

$$C_{delay} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \left(\begin{bmatrix} x^- \vee y^-(T+t), x^+ \wedge y^+(T+t) \\ x^- \vee y^-(T+t), x^+ \wedge y^+(T+t) \end{bmatrix} \right)$$



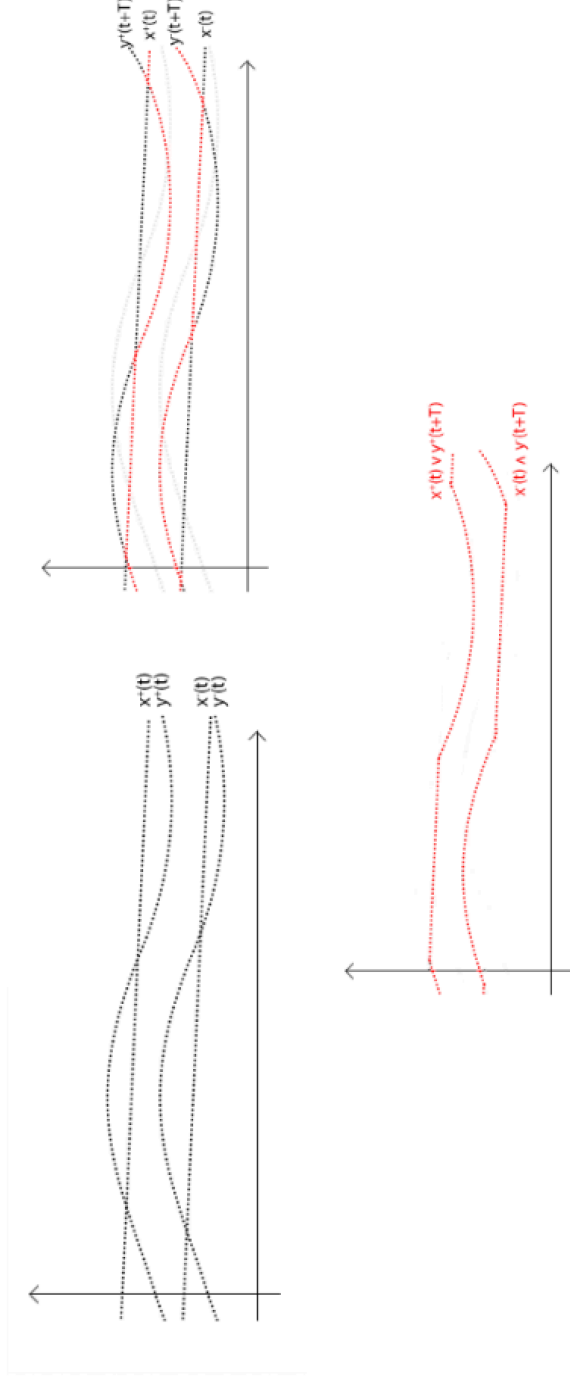
Non-synchronised constraint



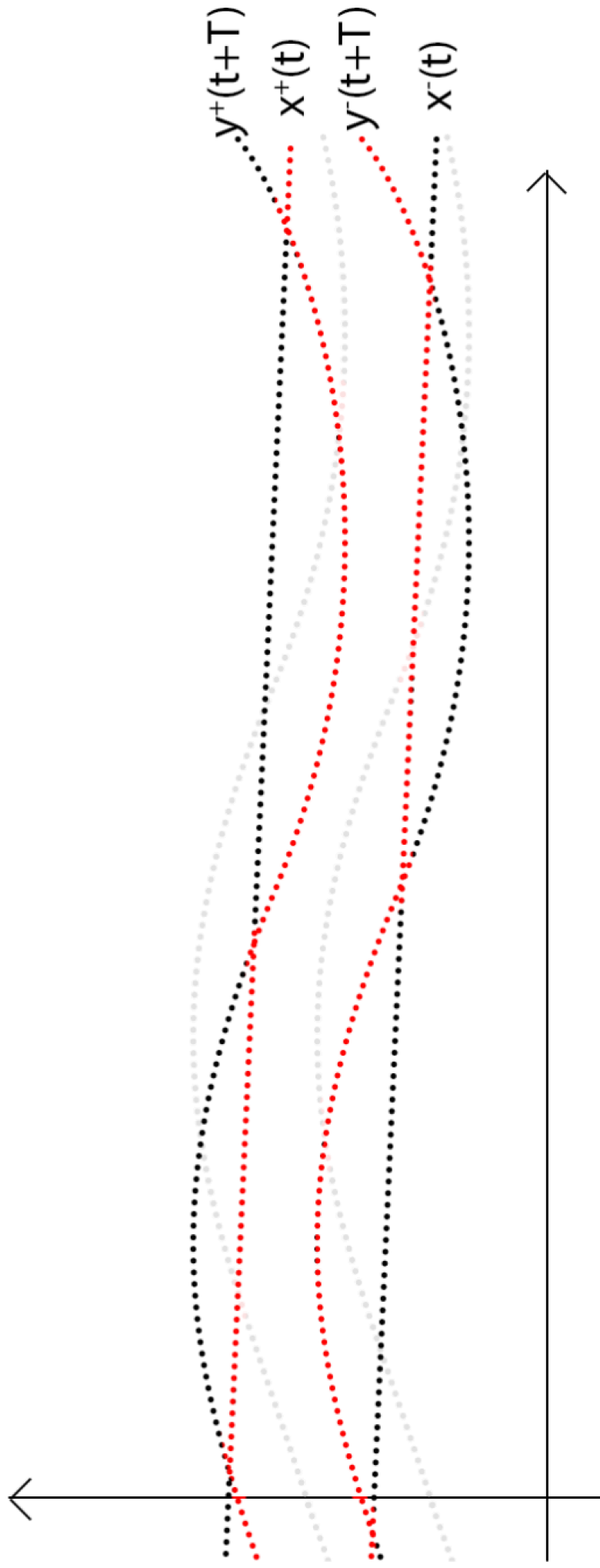
Proposition 5

The minimal contractor associated with a delay $x(t)=y(T+t)$

$$C_{delay} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \left(\begin{bmatrix} x^- \vee y^-(T+t), x^+ \wedge y^+(T+t) \\ x^- \vee y^-(T+t), x^+ \wedge y^+(T+t) \end{bmatrix} \right)$$



Non-synchronised constraint



3 - Contractors on tubes

Definitions

Contractor: $\gamma: \mathcal{P}(E) \rightarrow \mathcal{P}(E)$

Standard contractor: $\gamma: \mathcal{P}(E) \rightarrow \mathcal{P}(E)$

Contractor: $\gamma: \mathcal{P}(E) \rightarrow \mathcal{P}(E)$

Comprehension: $\gamma: \mathcal{P}(E) \rightarrow \mathcal{P}(E)$

Proposition 1

The standard contractor associated with γ is γ^* .

Proposition 2

The standard contractor associated with γ is γ^* .

Proposition 3

The standard contractor associated with γ is γ^* .

Proposition 4

The standard contractor associated with γ is γ^* .

Properties

$$a([-\infty; \infty]) \subset [-1; 1]$$

$$a([\frac{\pi}{2}, \pi]) \subset [-0.7(t - \frac{\pi}{2}) + 0.99,$$

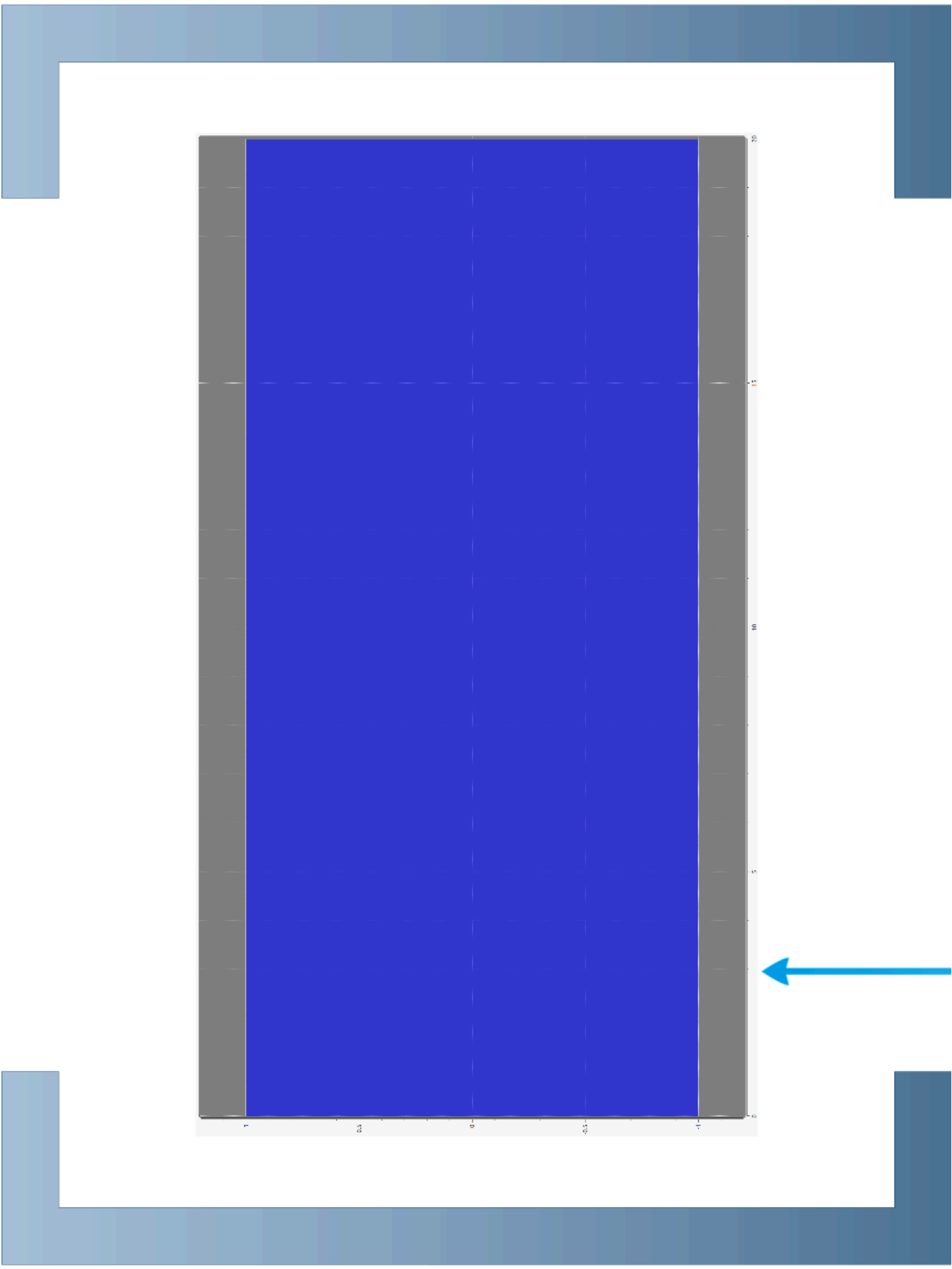
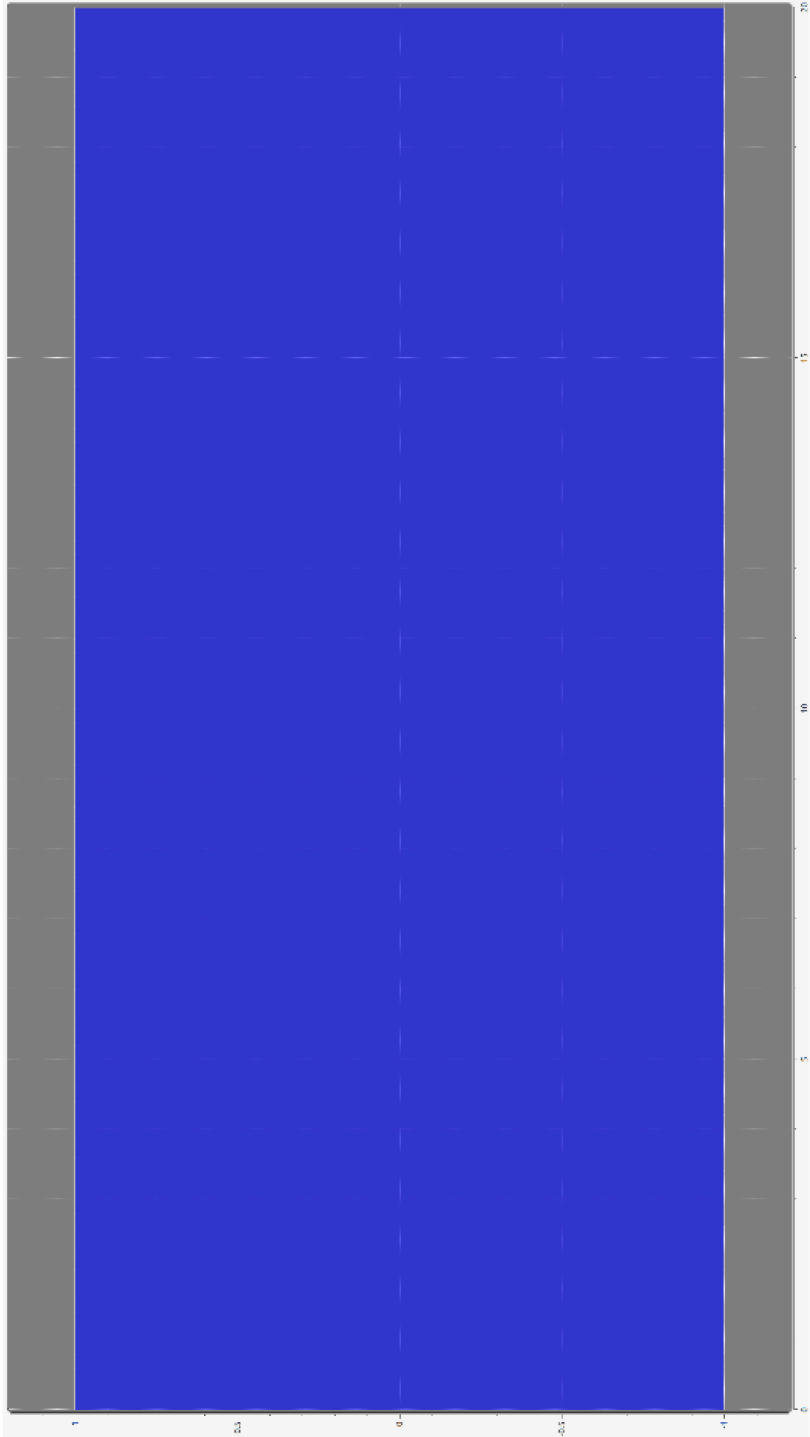
$$-0.1(t - \frac{\pi}{2}) + 1.01]$$

$$a(t + \pi) = -a(t)$$

$$a(t + 2\pi) = a(t)$$

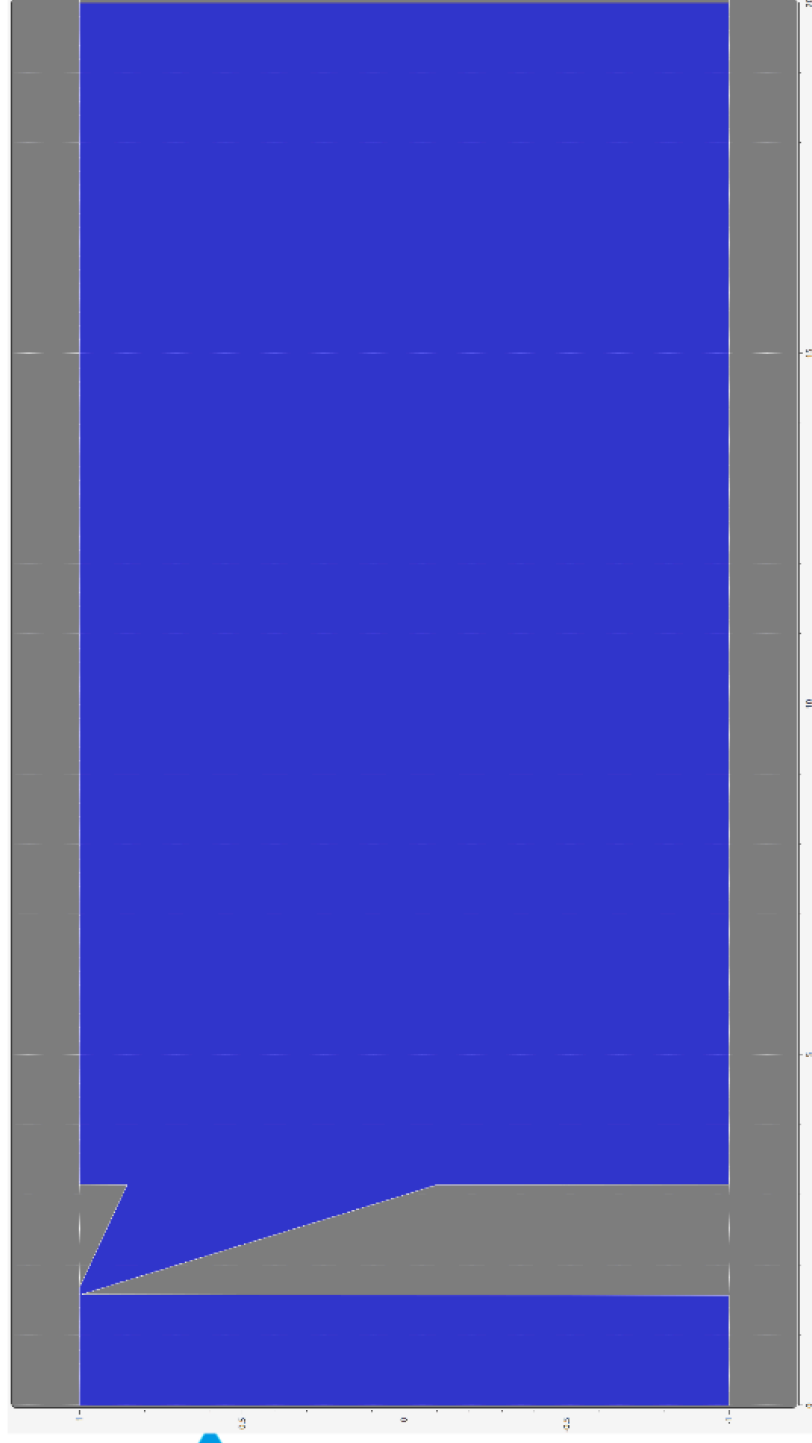
$$b(t - \frac{\pi}{2}) = a(t)$$

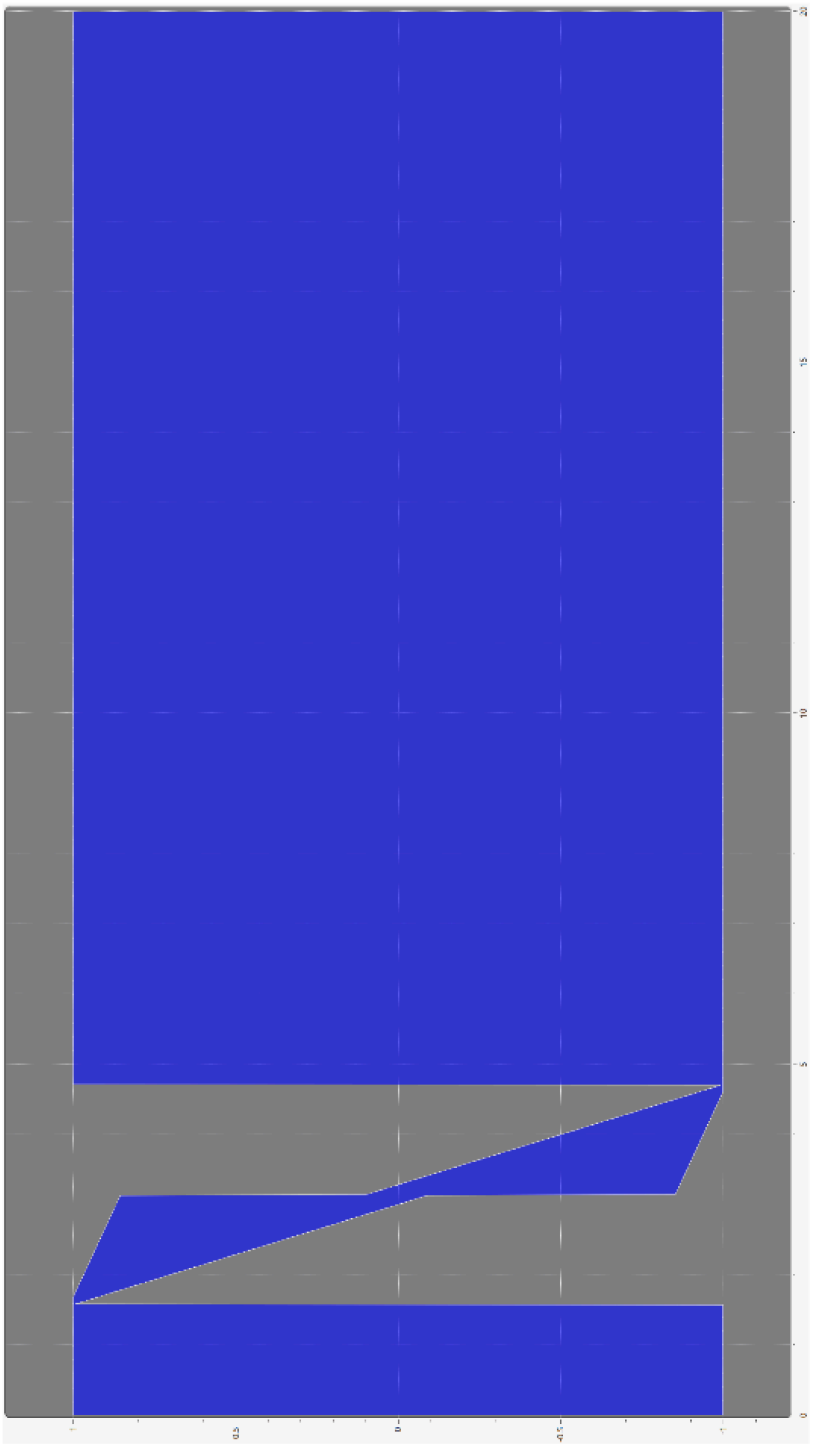
$$b(t) = \dot{a}(t)$$

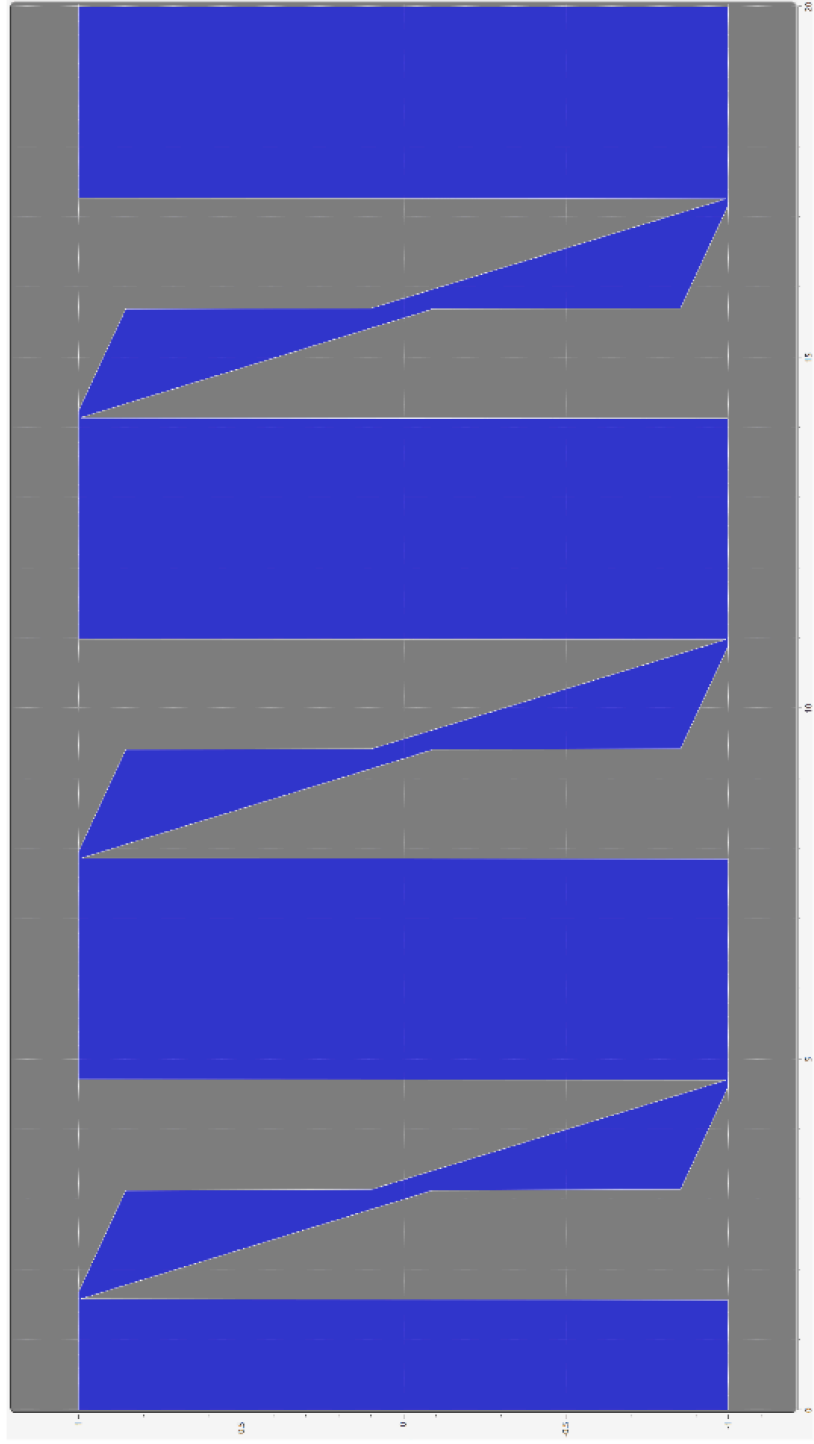


$$\forall t \in [\frac{\pi}{2}, \pi], \quad a(t) \leq -0.1(t - \frac{\pi}{2}) + 1.01$$

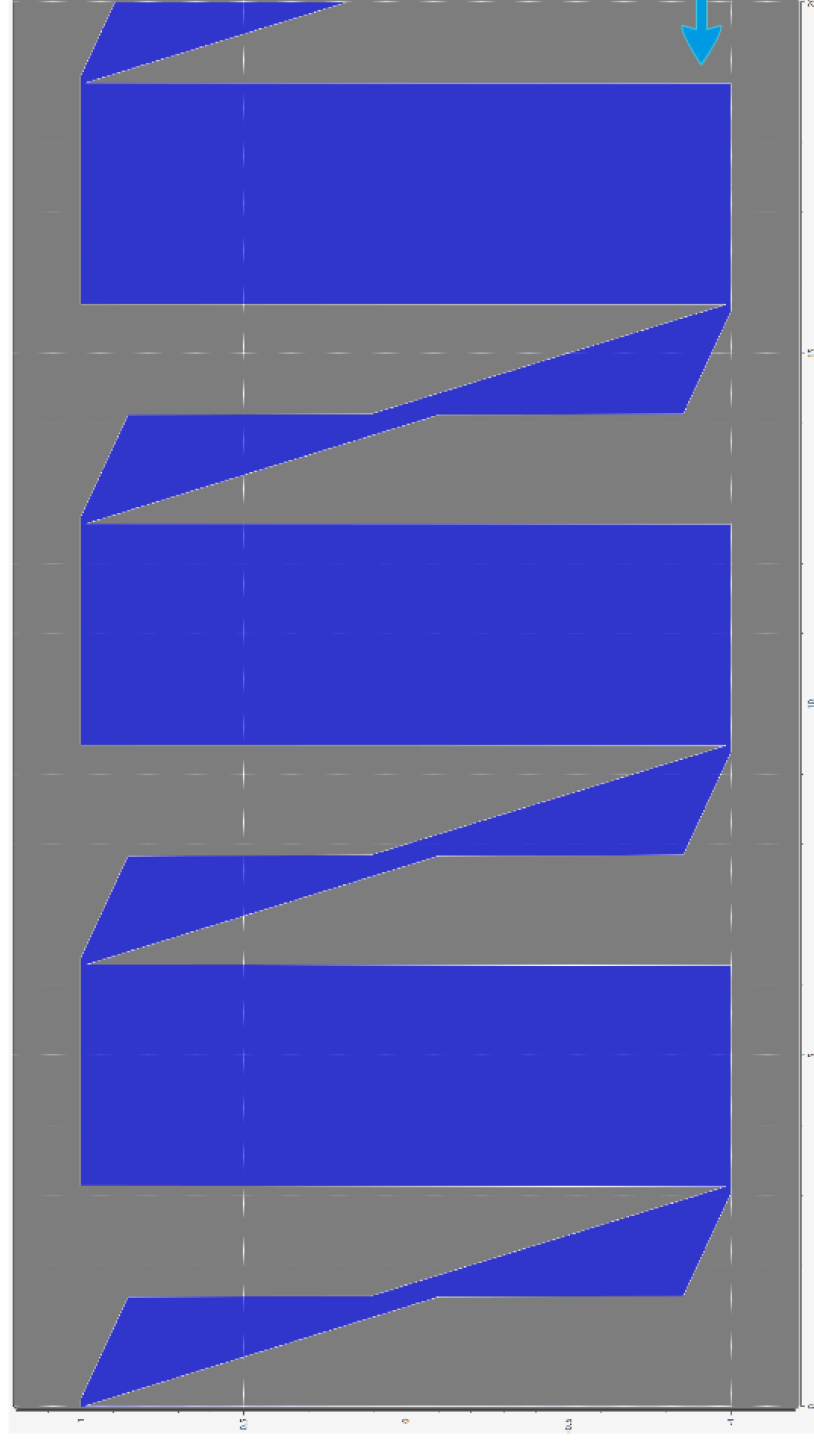
$$a(t) \geq -0.7(t - \frac{\pi}{2}) + 0.99$$



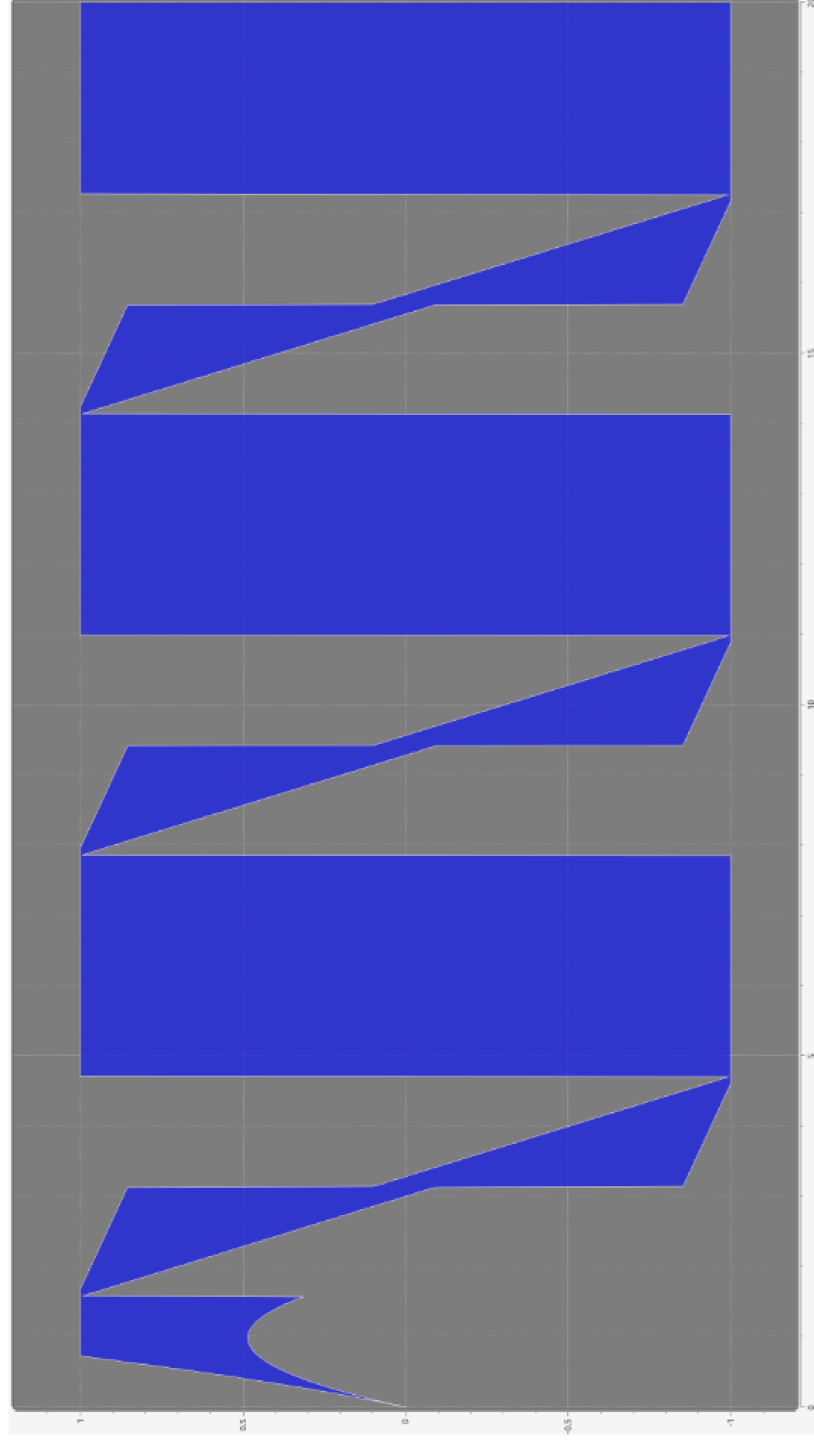


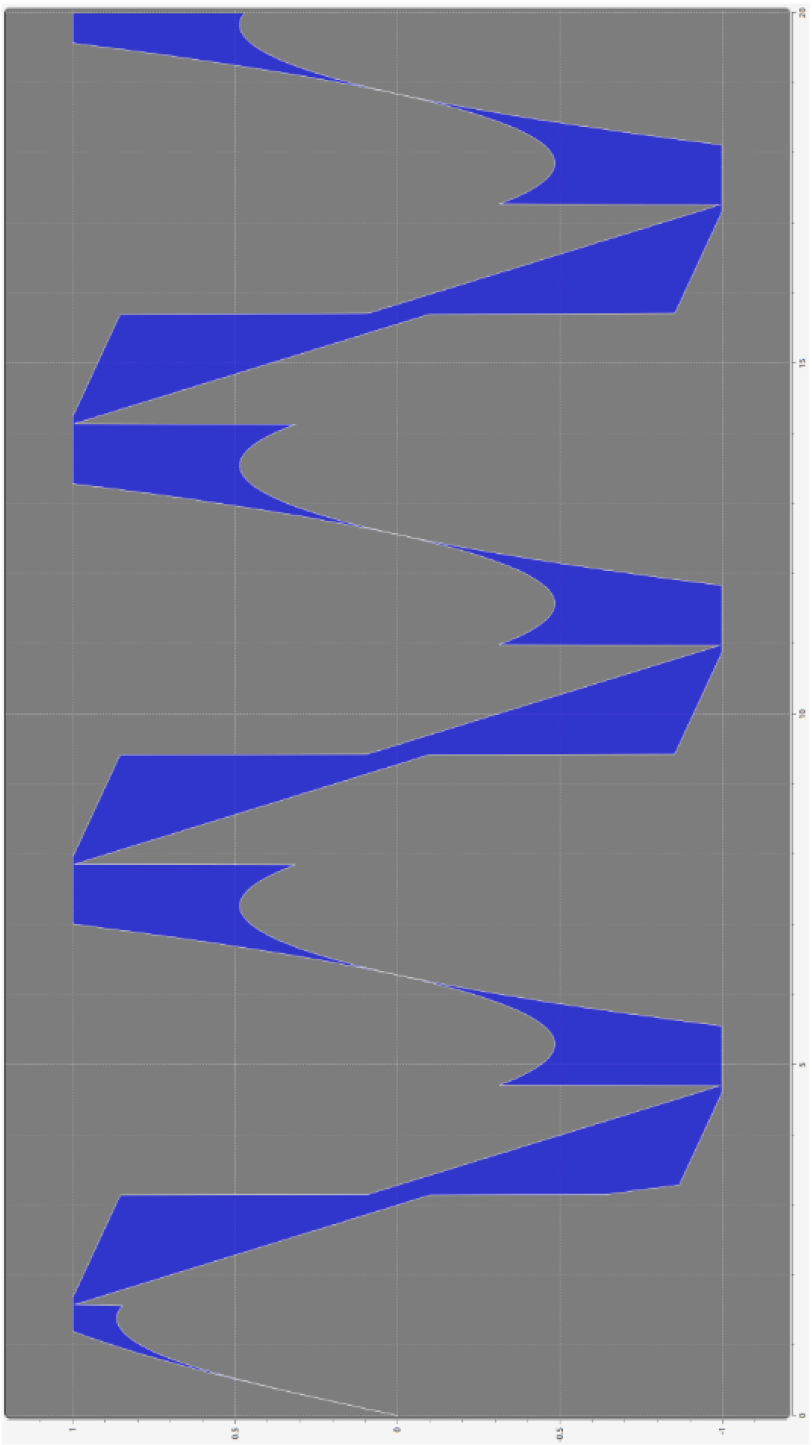


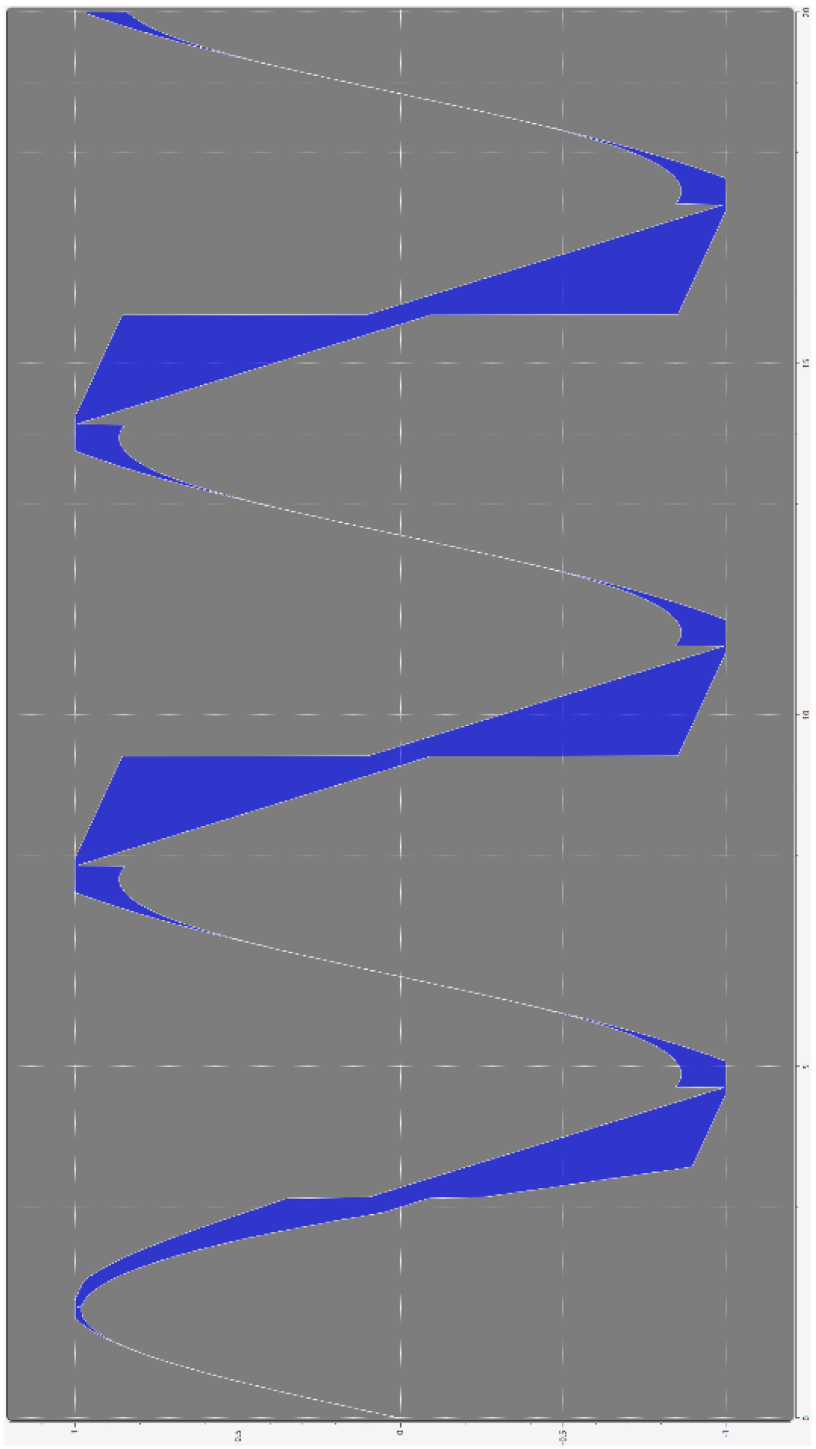
$$[b(t)] = [b(t)] \cap [a(t + \frac{\pi}{2})]$$

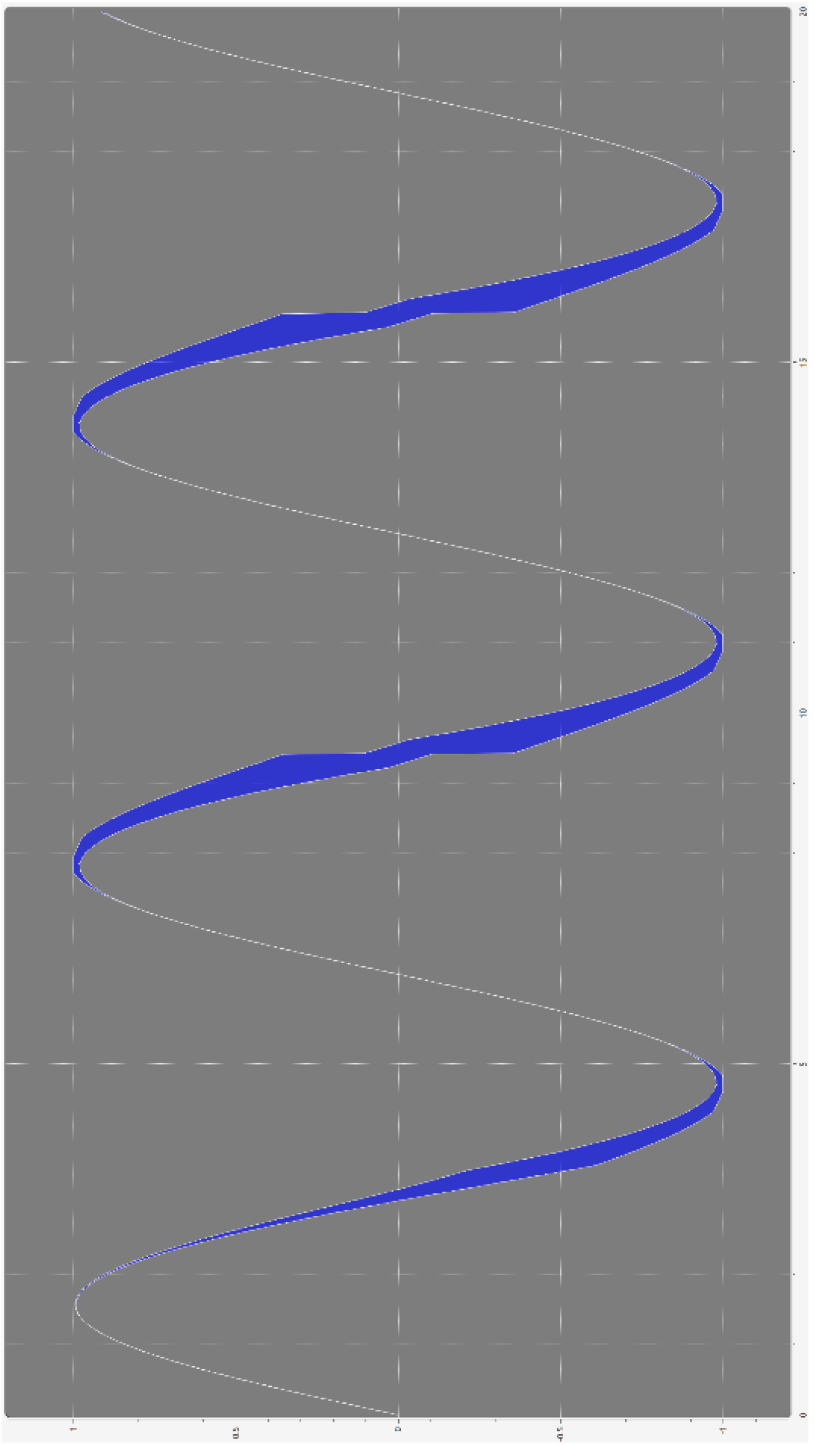


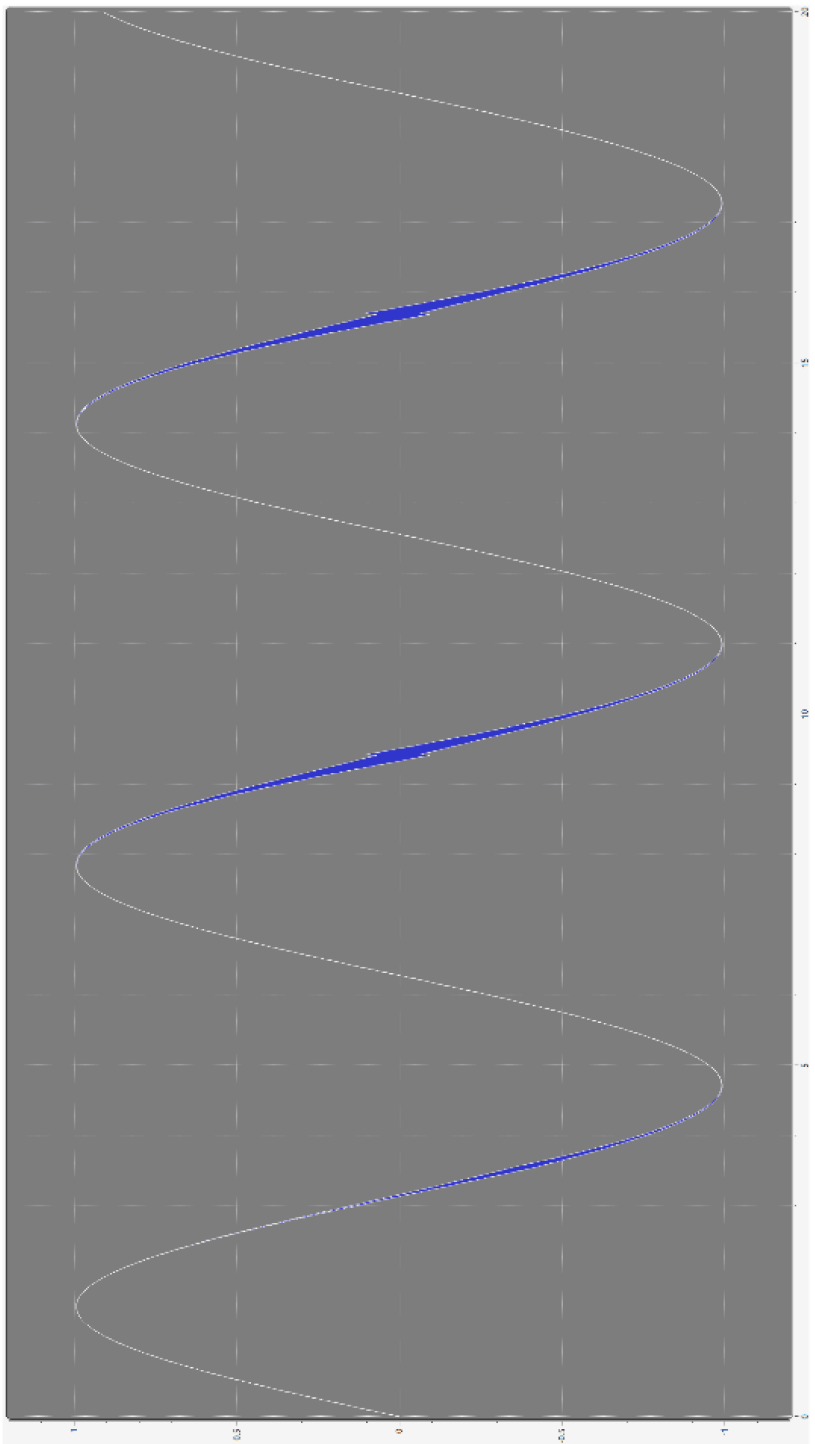
$$[a(t)] = [a(t)] \cap \left[\int_{\tau=0}^t \frac{db^+(\tau)}{dt} d\tau, \int_{\tau=0}^t \frac{db^-(\tau)}{dt} d\tau \right]$$

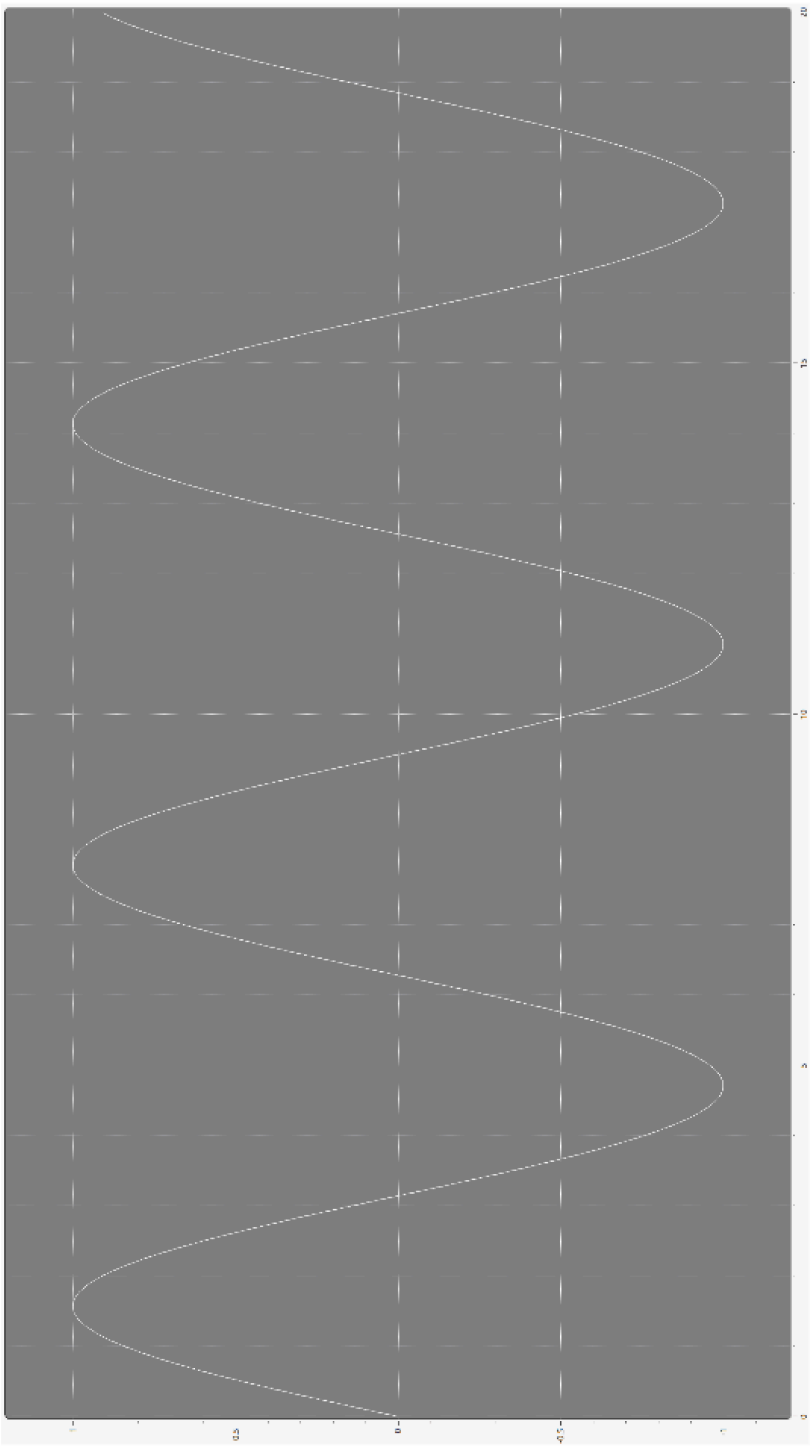




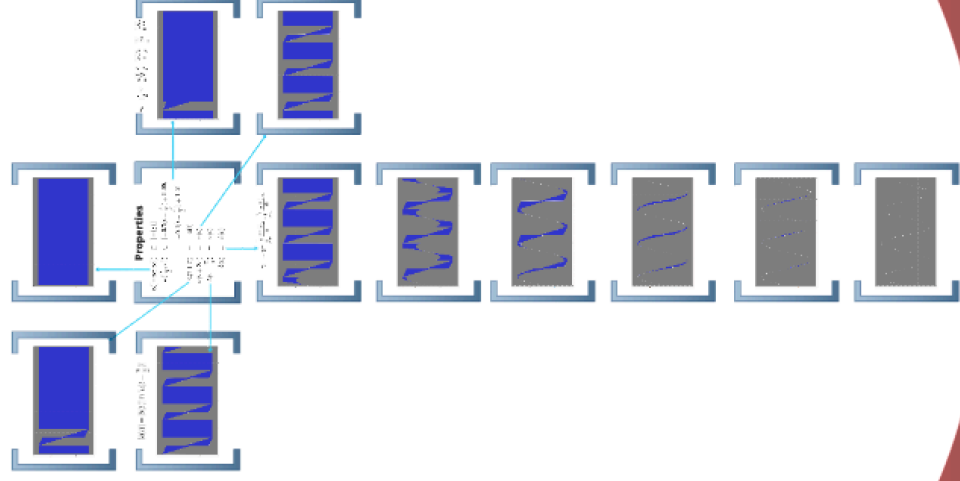








4 - Simple example



5 - Spring-Mass example

Continuous

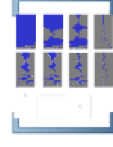
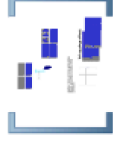
$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$



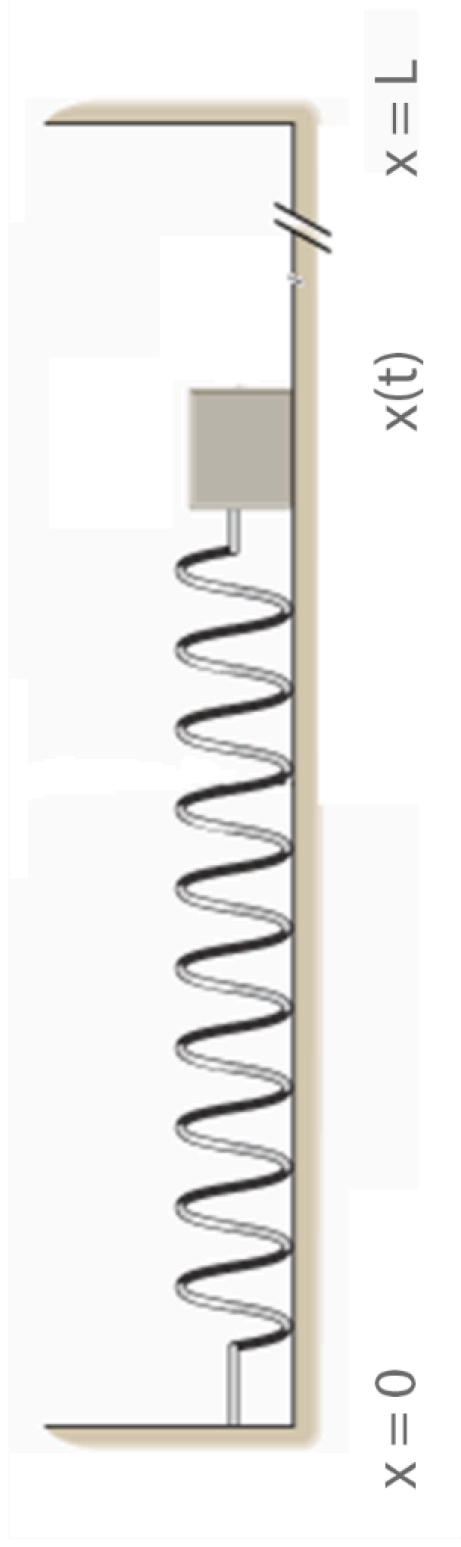
Simulation

Time (s)	0.0	0.5	1.0
Displacement (m)	0.0	0.1	0.2
Velocity (m/s)	0.0	0.1	0.2
Acceleration (m/s²)	0.0	0.1	0.2

Matlab Simulation



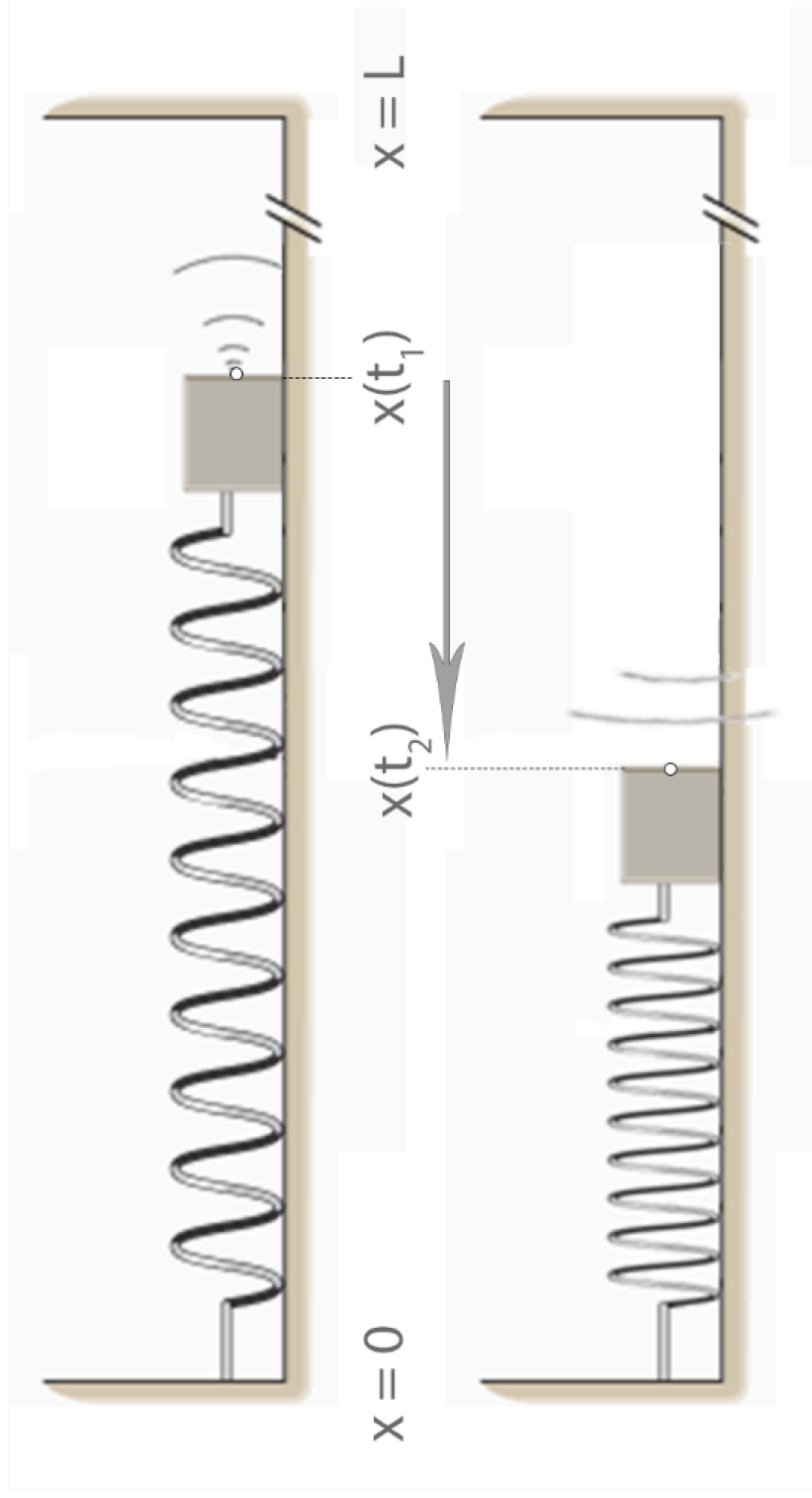
Spring-Mass system



$$m.\ddot{x} + \gamma.\dot{x} + \kappa.x - \beta x^3 = 0$$

Non-linear spring
No info on initial conditions

Spring-Mass-Sonar system



$$\begin{aligned} (L - x(t_1)) + (L - x(t_2)) &= c(t_2 - t_1) \\ \Leftrightarrow x(t_2) + x(t_1) &= 2L - c(t_2 - t_1) \end{aligned}$$

Contraintes

$$\frac{d}{dt} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \frac{(\beta x^2 - \kappa) \cdot x - \gamma \cdot \dot{x}}{m} \end{pmatrix}$$
$$x(t_i) + x(t_j) = 2L - c \cdot (t_i - t_j)$$

Simulation

$$x(3.00) + x(3.48) = 34.33$$

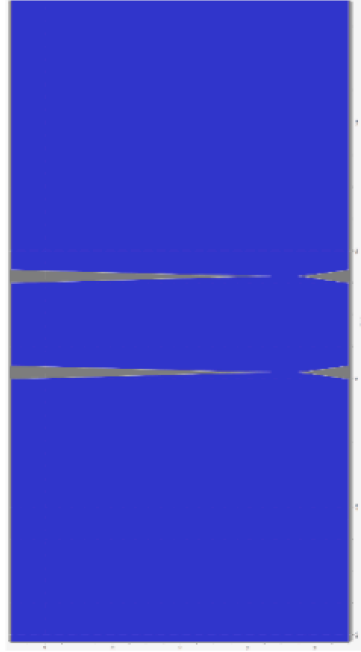
$$x(9.00) + x(9.54) = 15.46$$

$$x(12.00) + x(12.55) = 13.06$$

$$x(20.00) + x(20.51) = 24.38$$

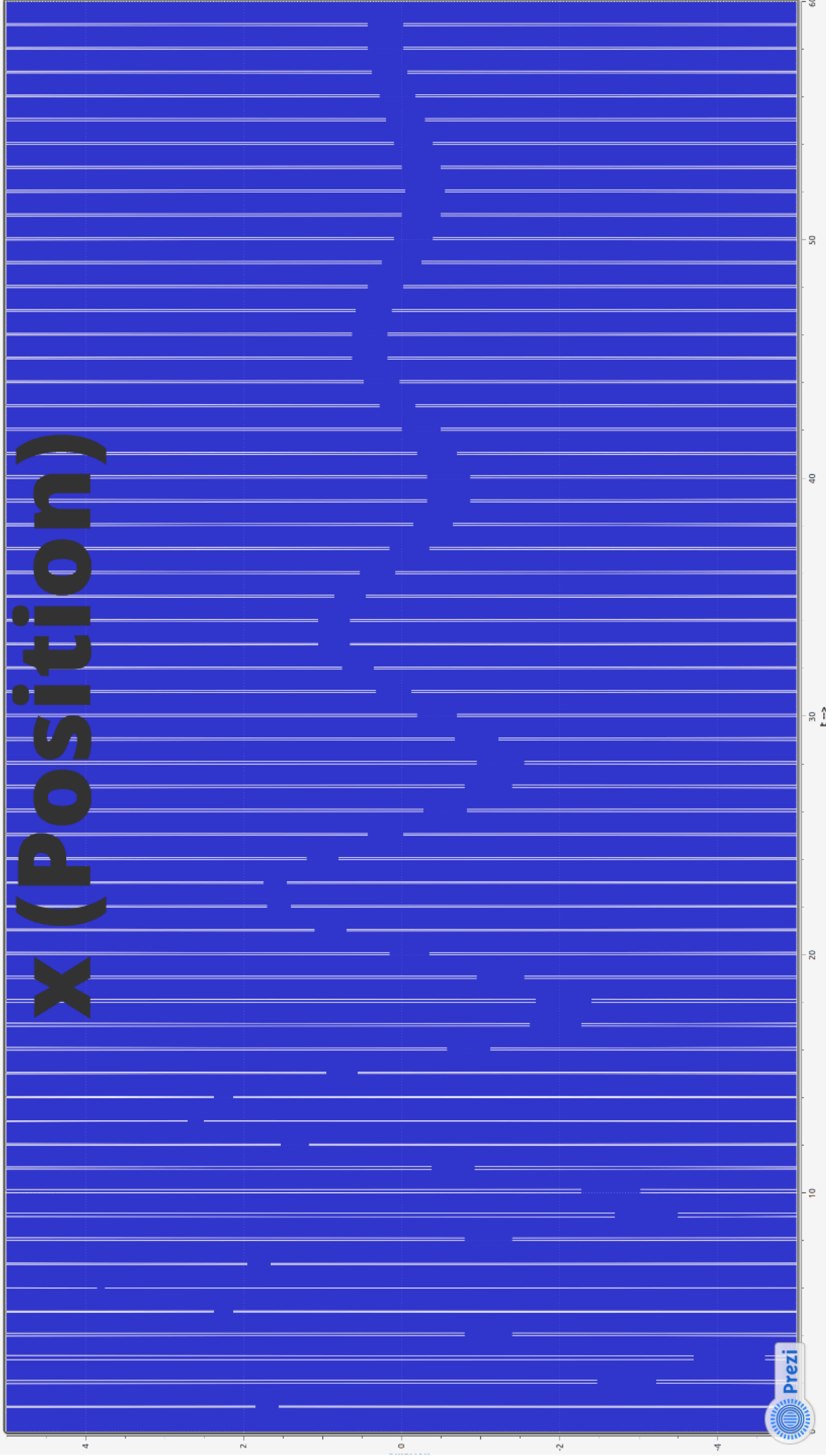
etc...

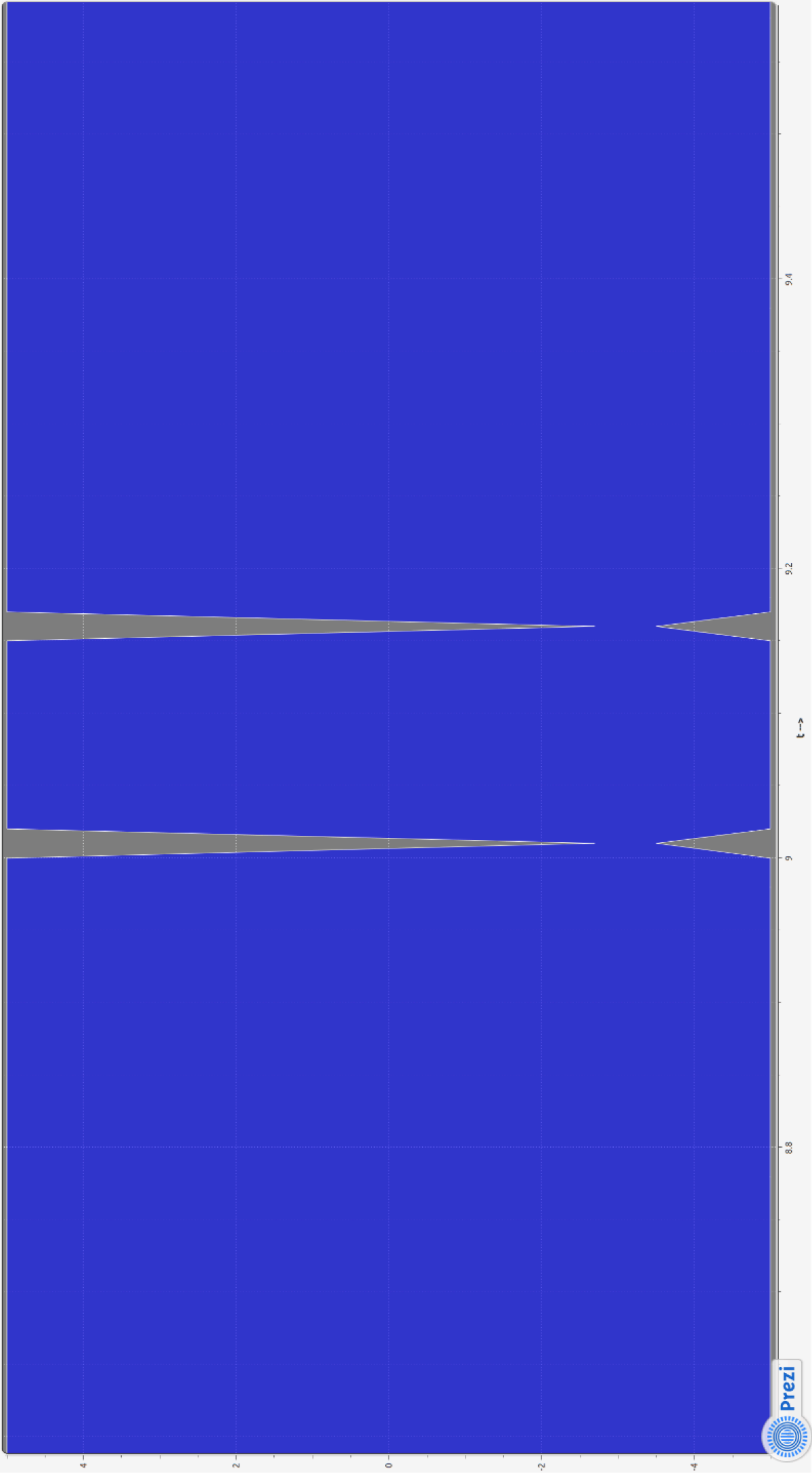
Matlab Simulation

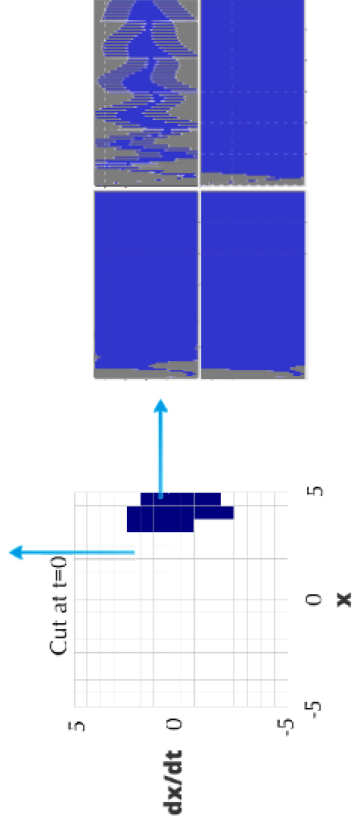
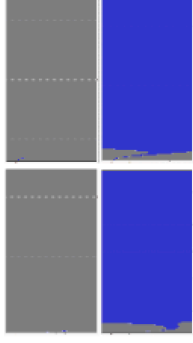


dx/dt (Speed)

x (Position)

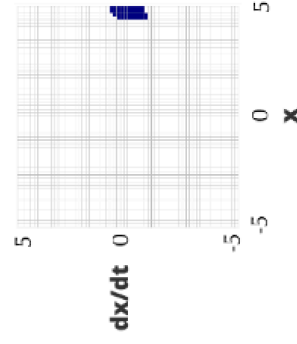


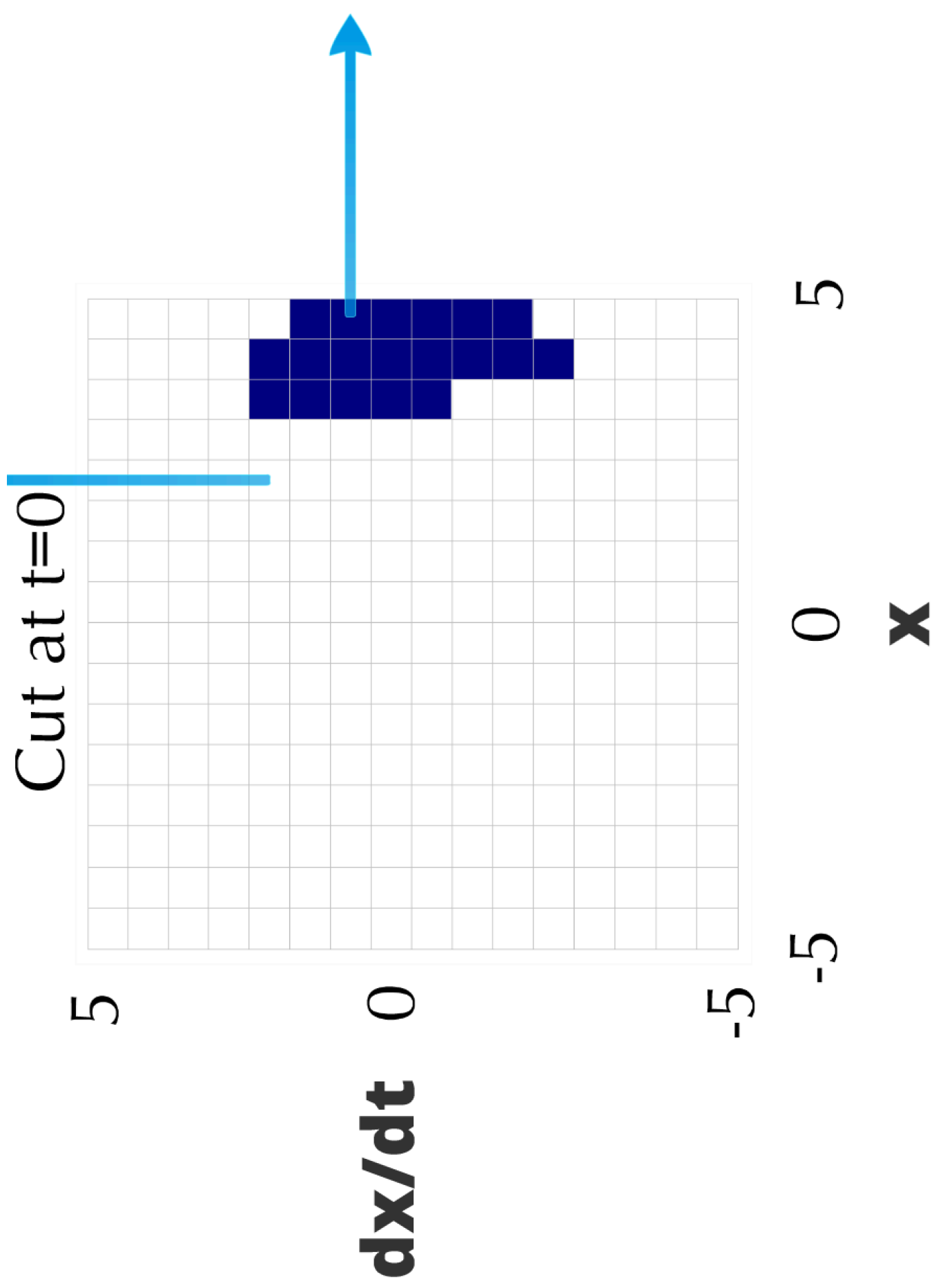


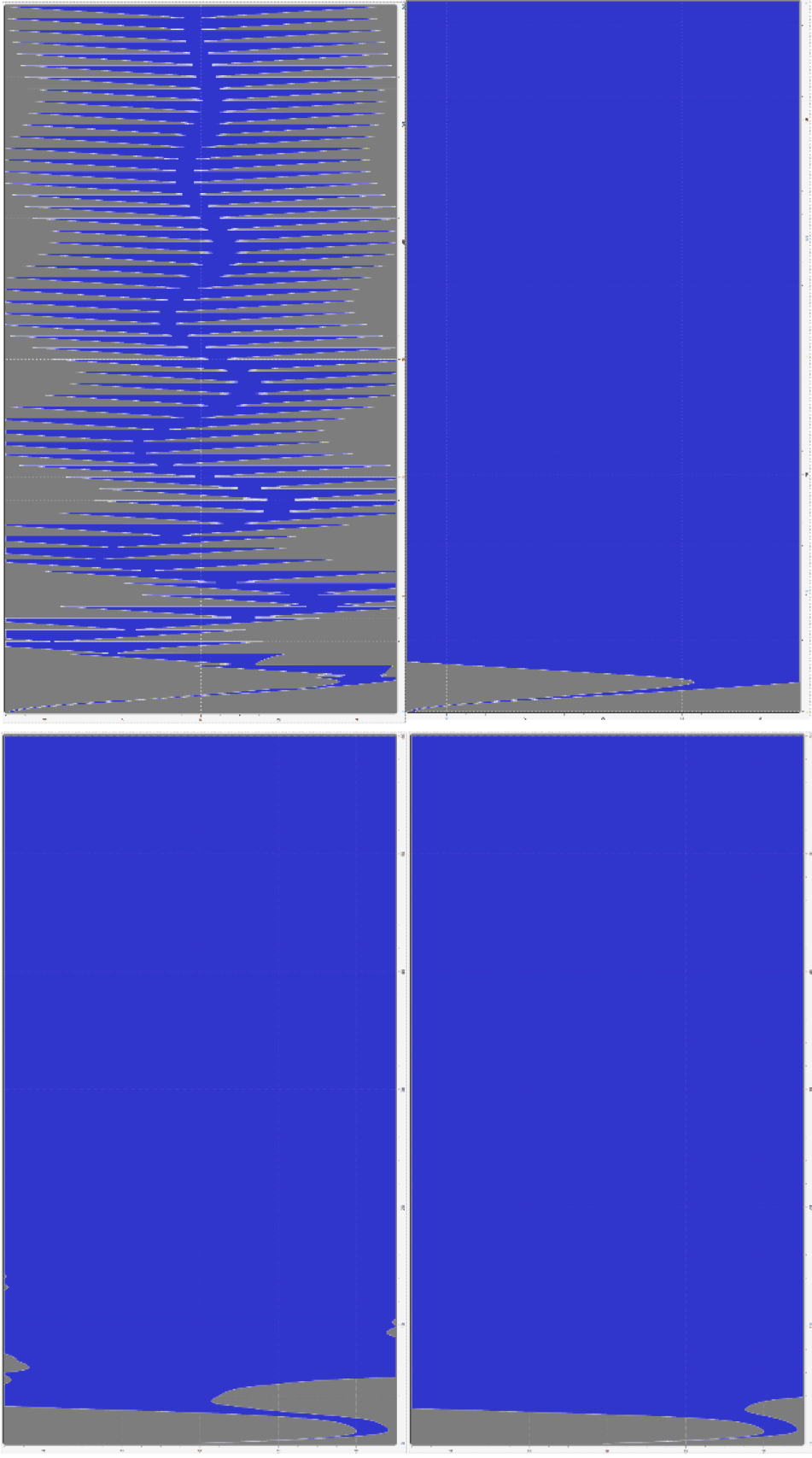


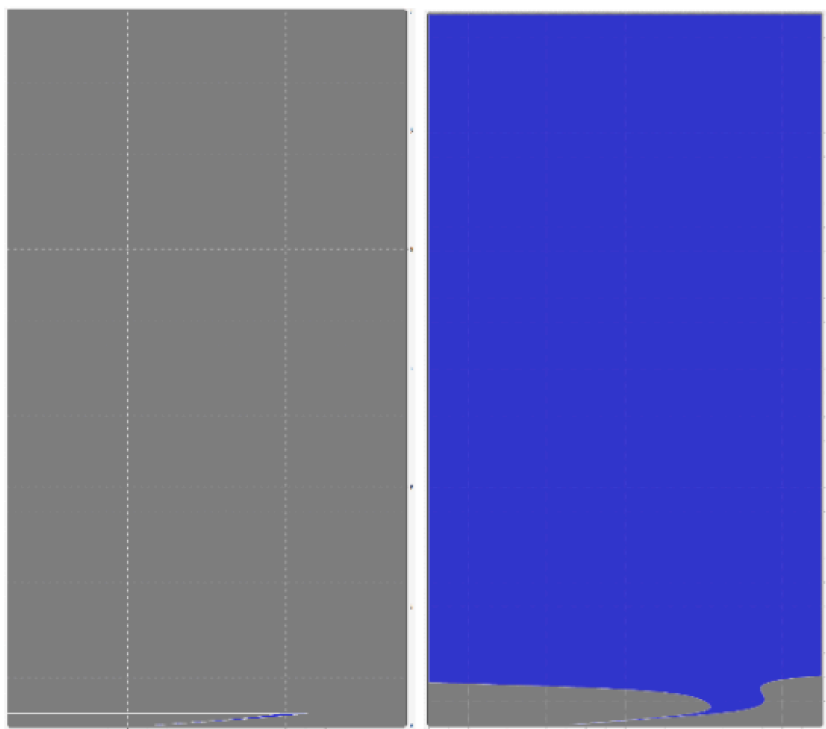
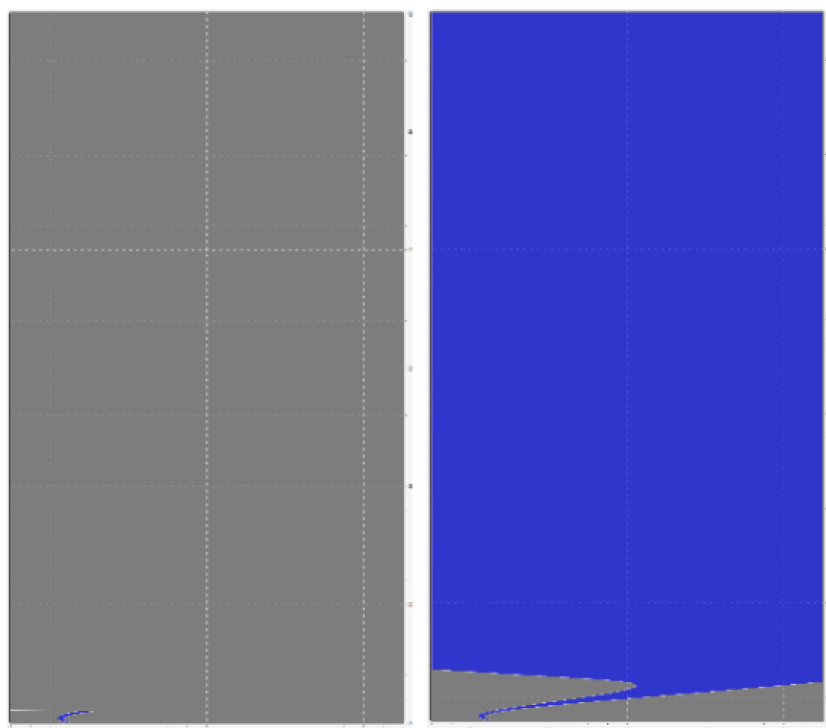
Solution : Union of all non-empty tubes
 We get an interval around the initial conditions

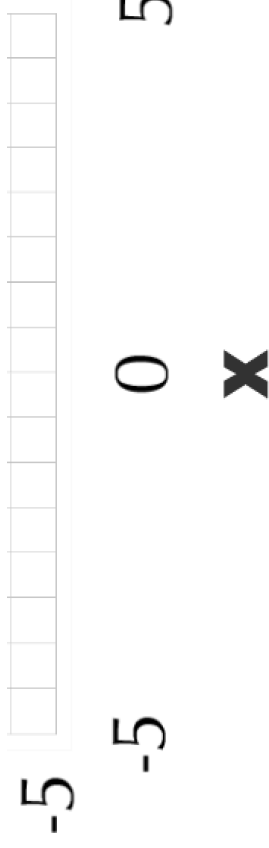
Wrapping effect



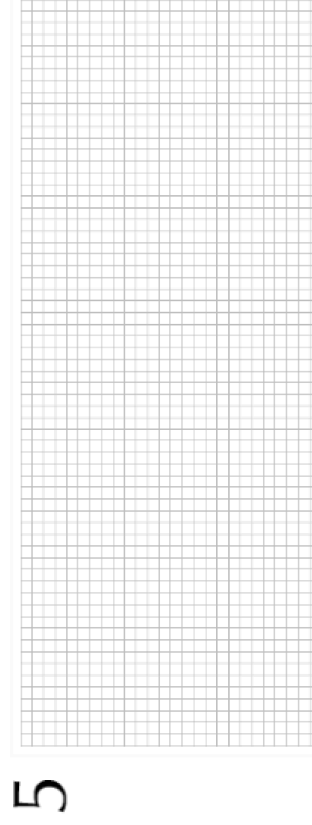




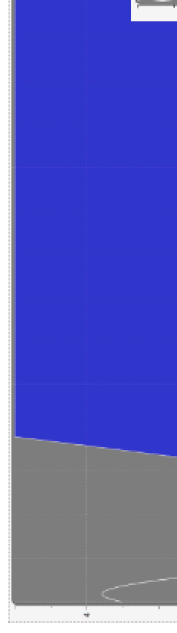


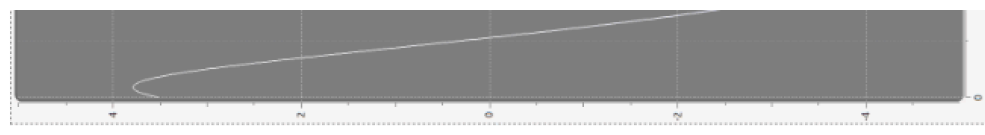
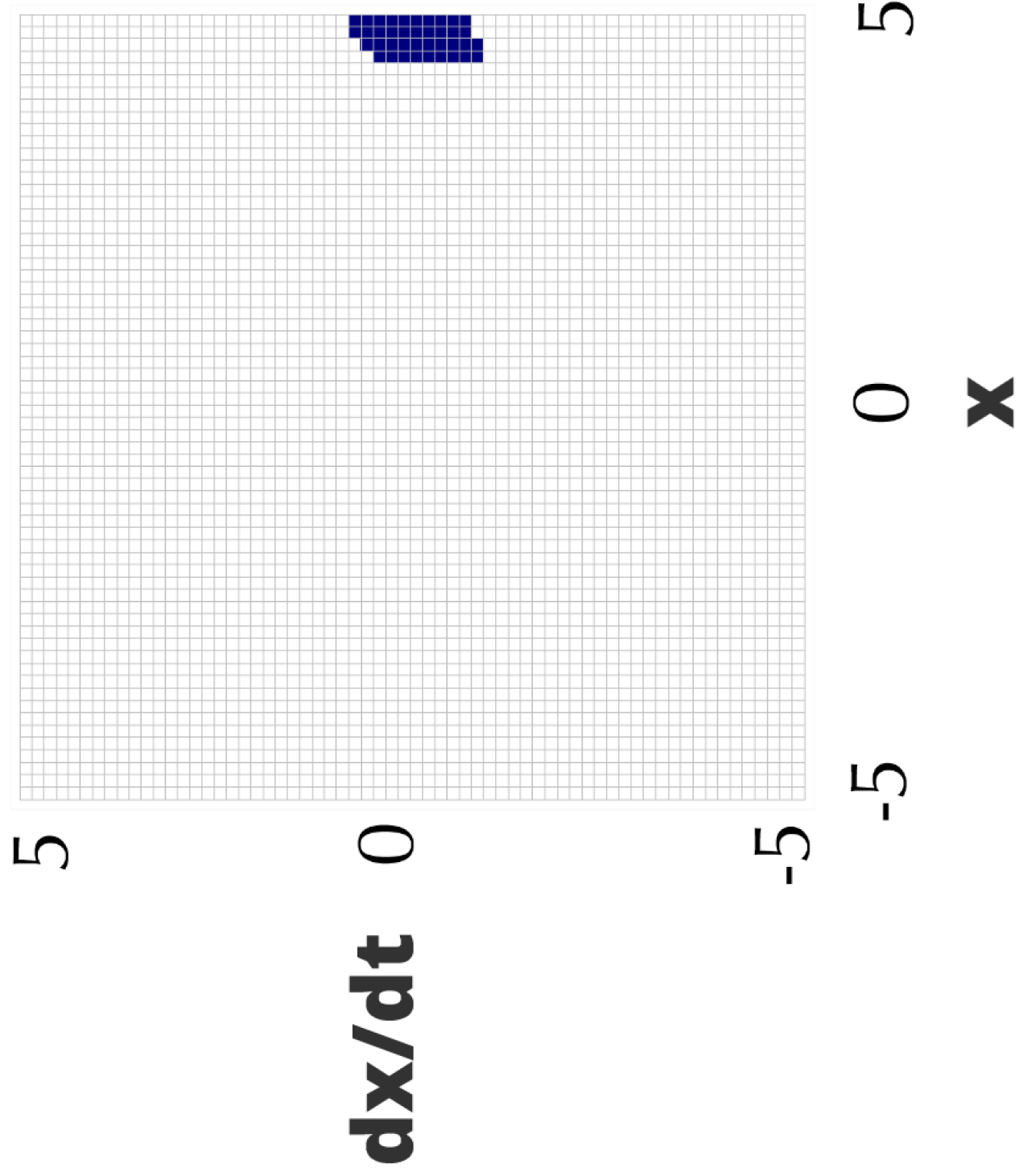


Solution : Union of all non-empty tubes
We get an interval around the initial
conditions

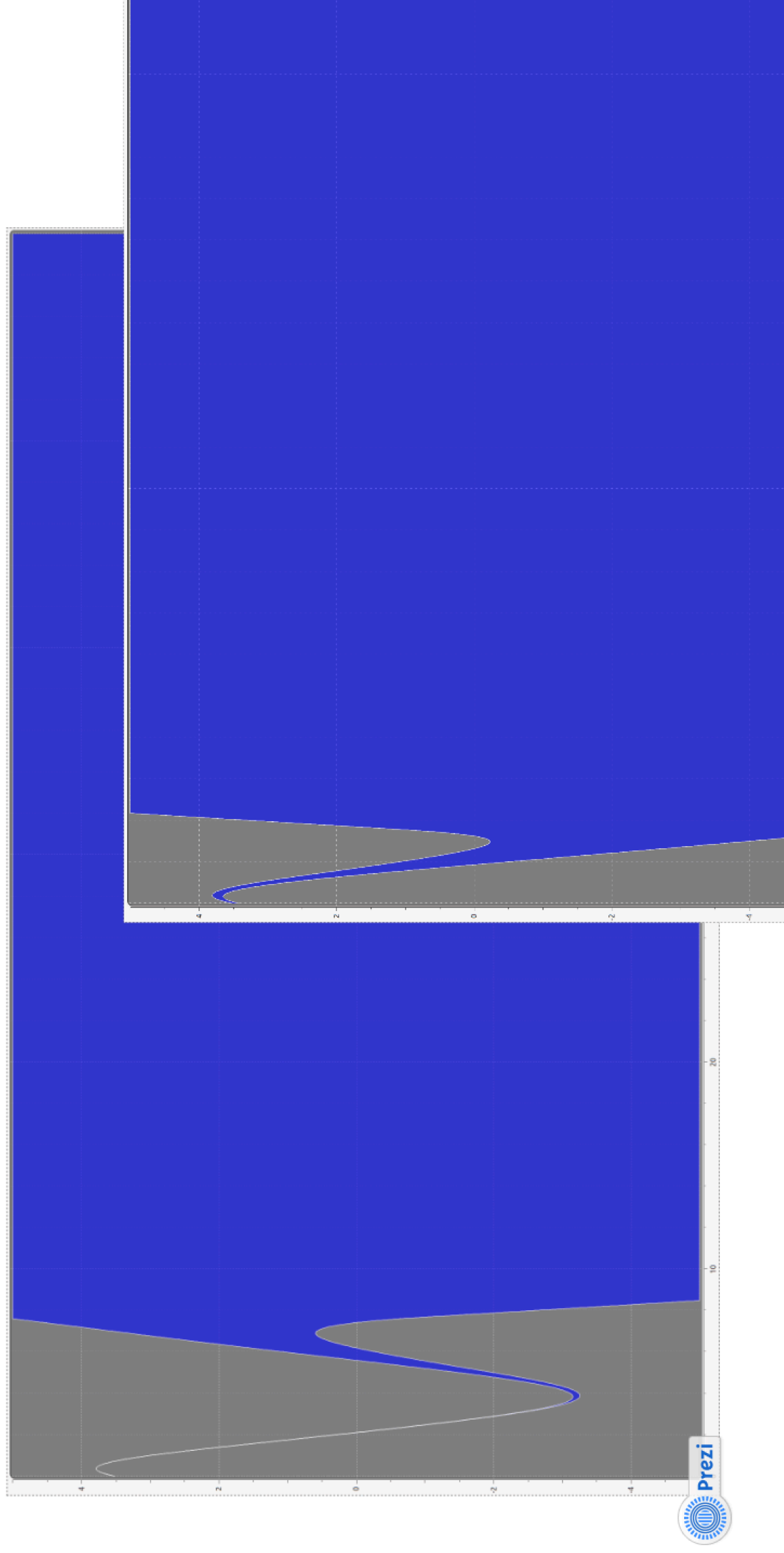


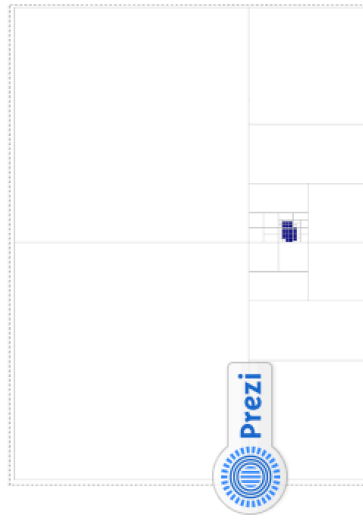
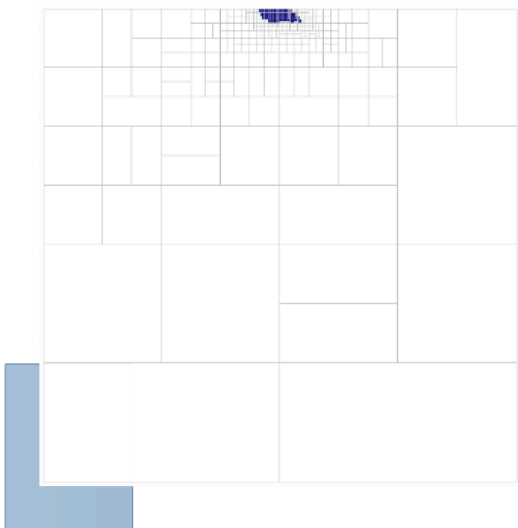
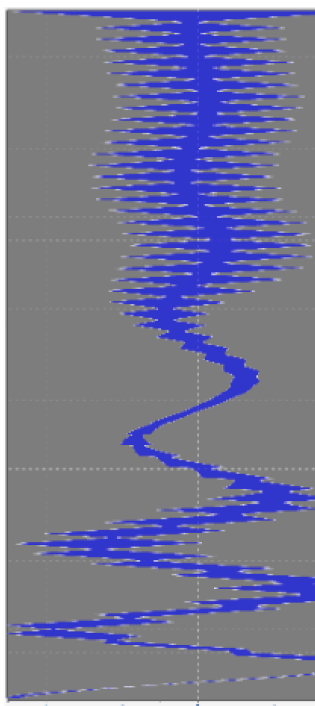
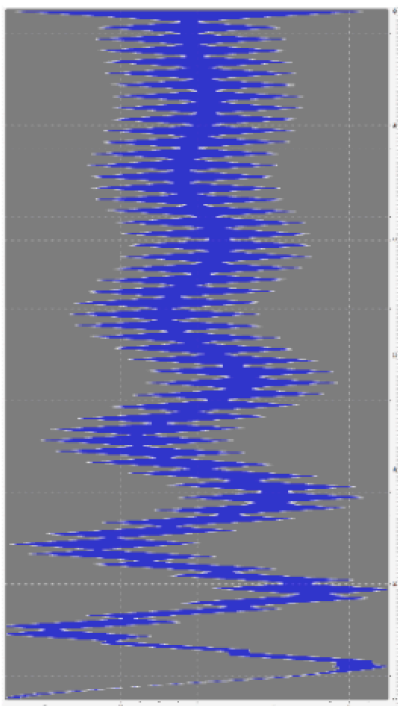
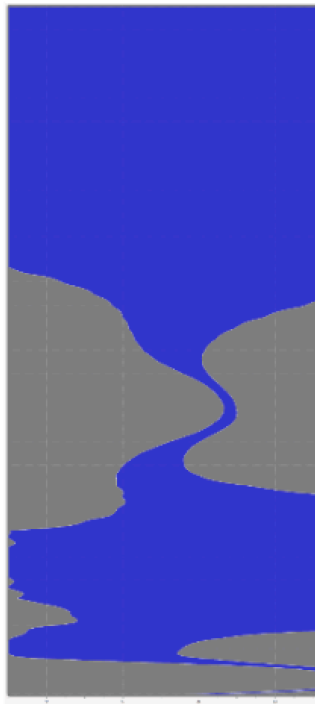
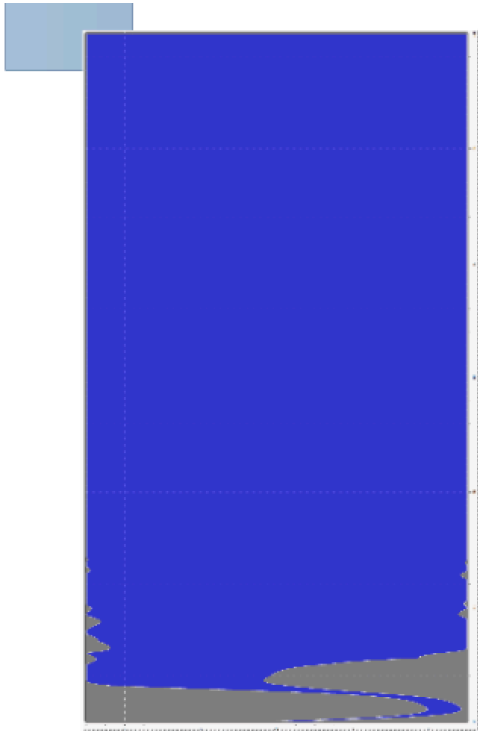
Wrapp

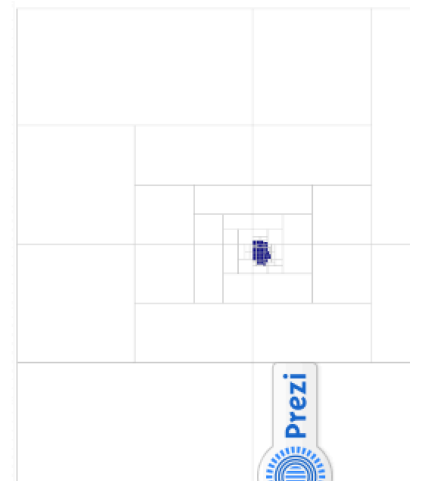
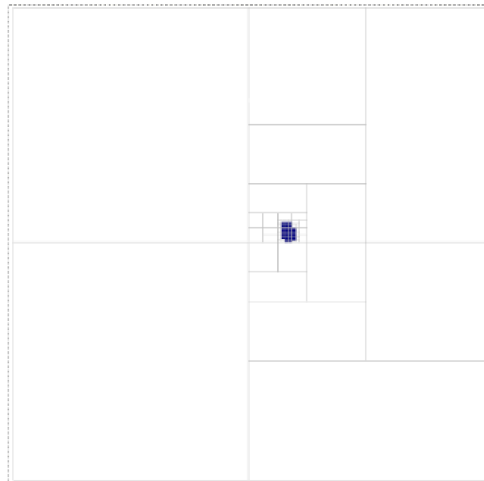
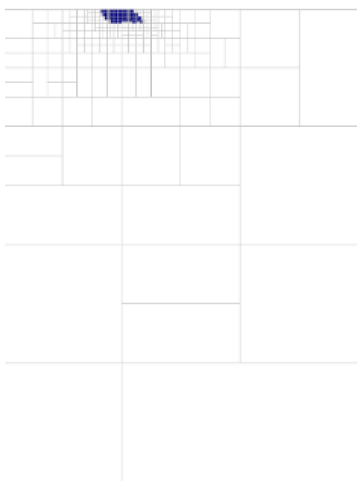
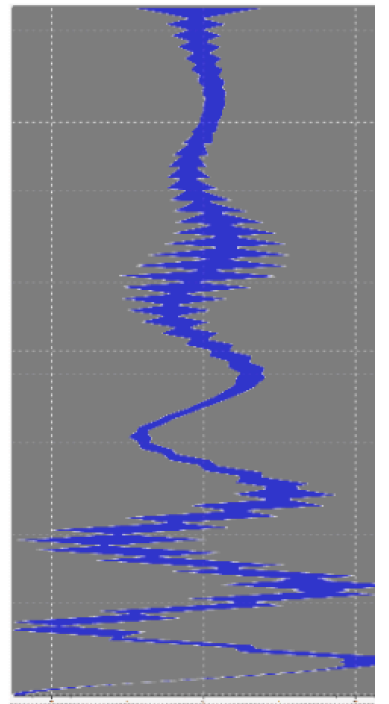
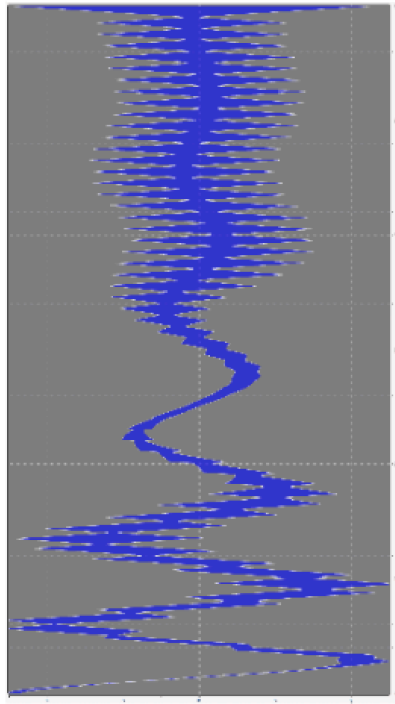
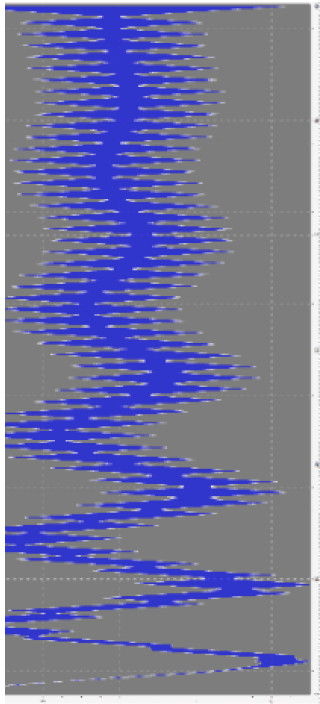
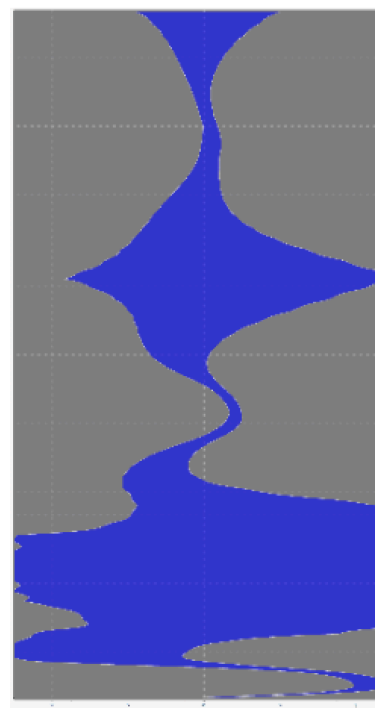
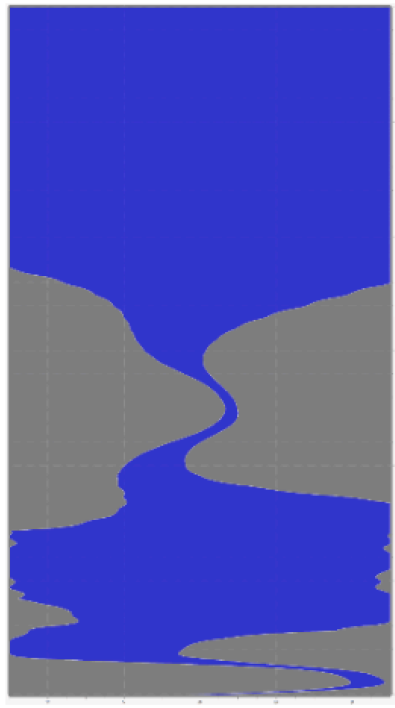
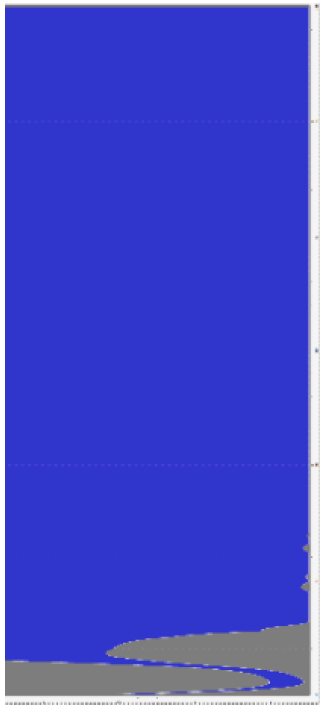


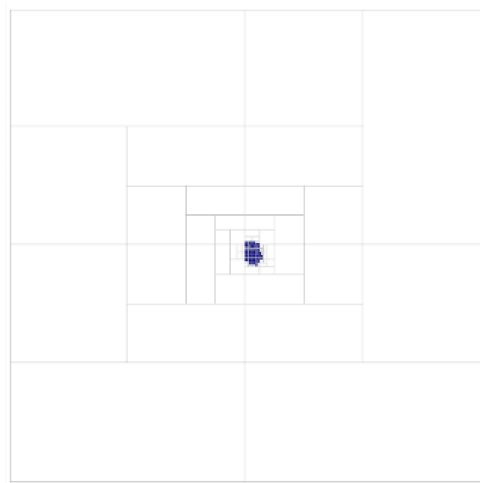
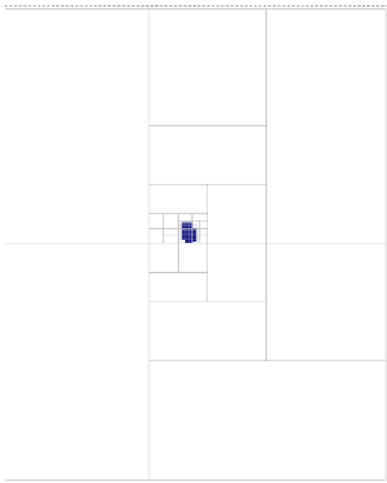
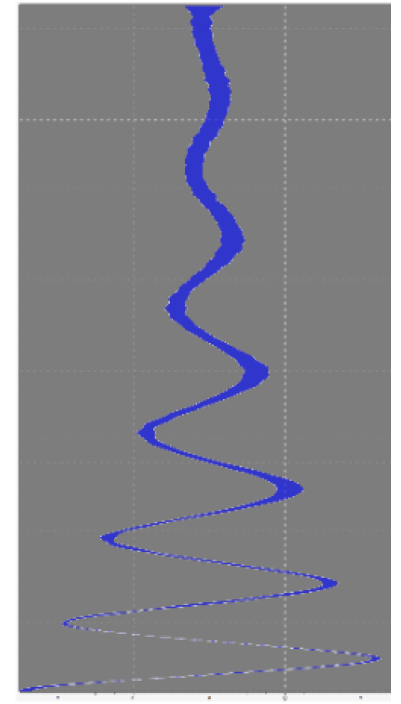
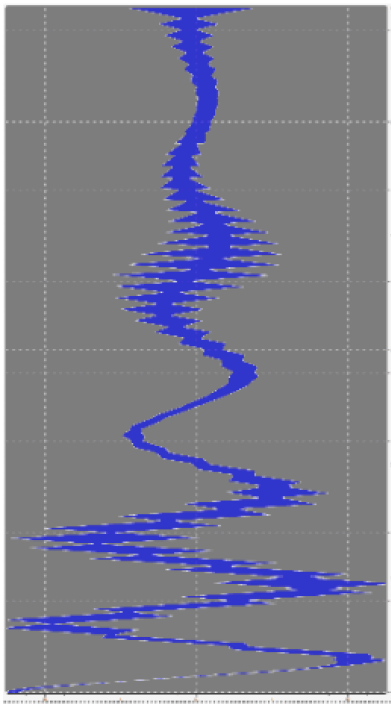
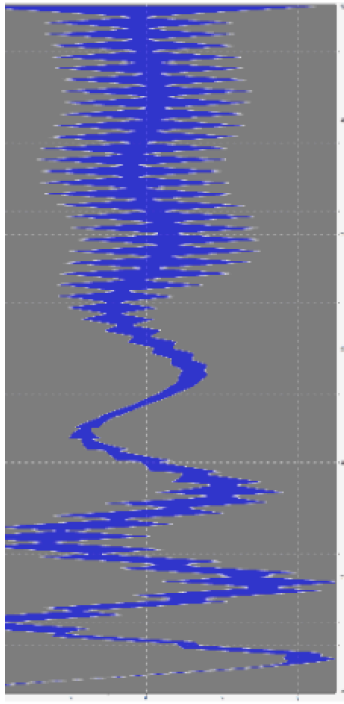
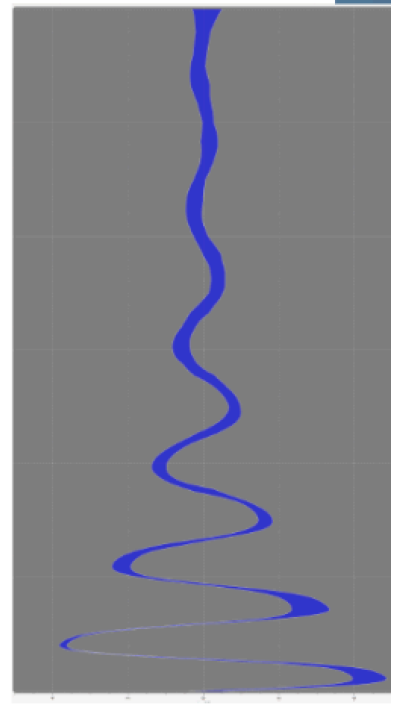
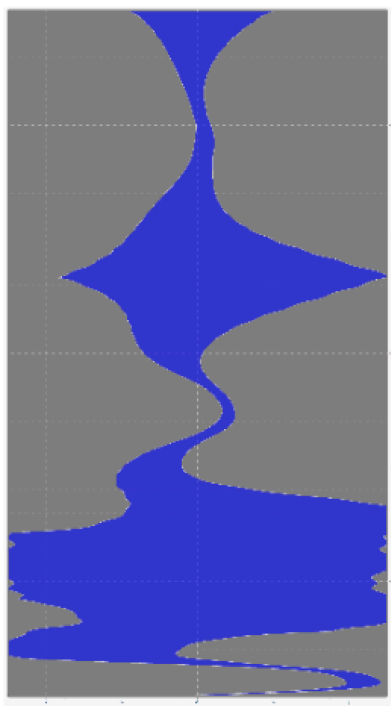
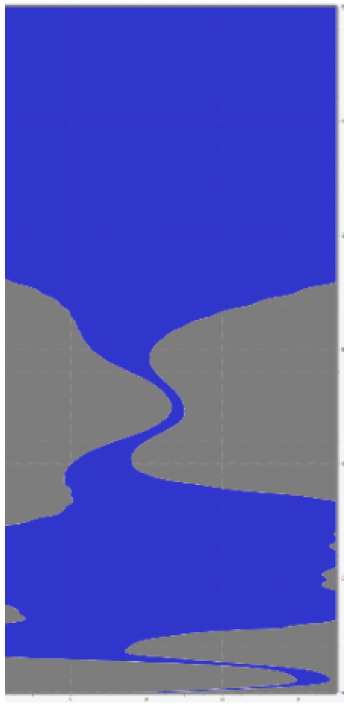


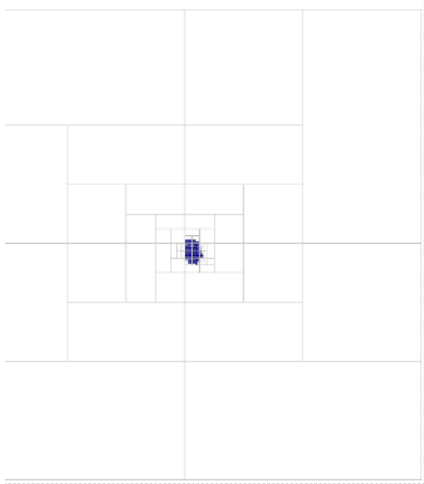
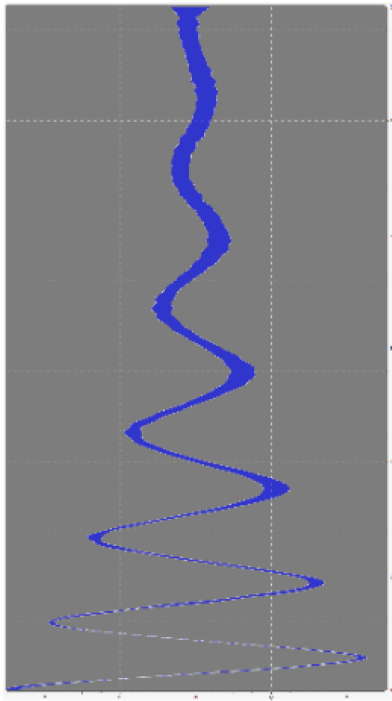
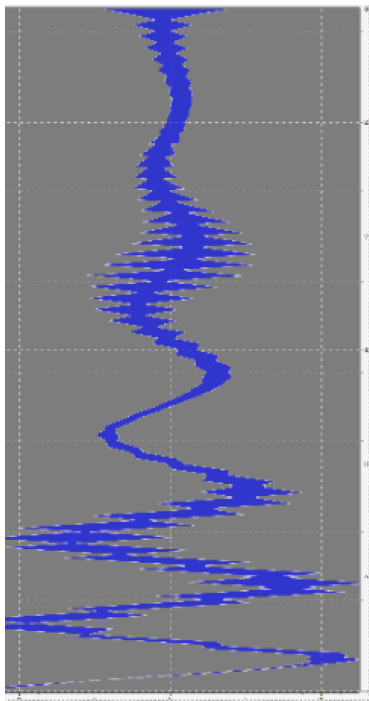
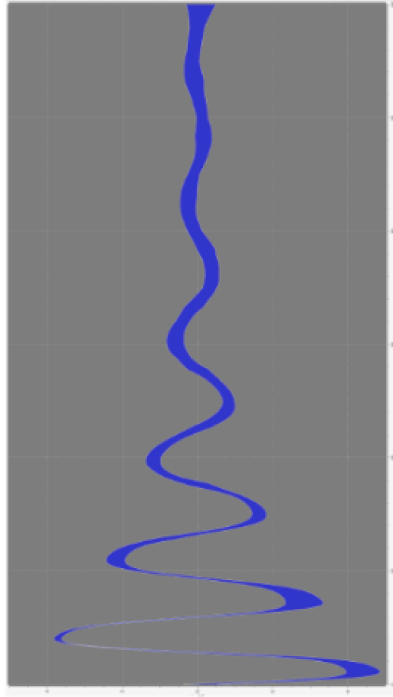
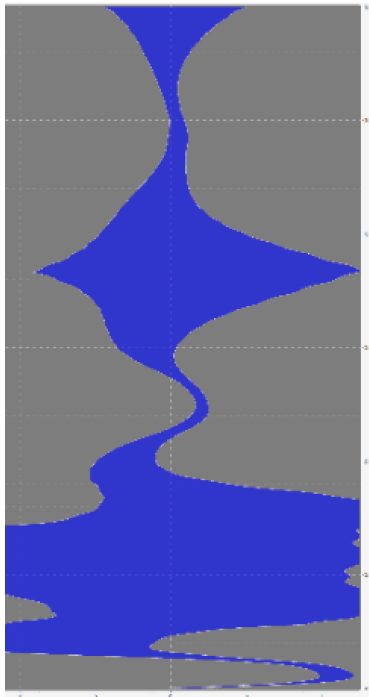
Wrapping effect











5 - Spring-Mass example

Continuous

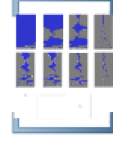
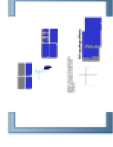
$$\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\frac{k}{m}x - \frac{b}{m}v \end{bmatrix}$$

Initial conditions: $x(0) = 1, v(0) = 0$



Simulation

Time (s)	Displacement (m)	Velocity (m/s)
0.0	1.0000	0.0000
1.0	0.7071	-0.7071
2.0	0.3679	-0.6320
3.0	0.2019	-0.5940
4.0	0.1193	-0.5647
5.0	0.0707	-0.5413
6.0	0.0432	-0.5230
7.0	0.0263	-0.5134
8.0	0.0161	-0.5089
9.0	0.0098	-0.5067
10.0	0.0061	-0.5060



6 - Conclusion

- Solve constraint satisfaction problem using tubes.
- Tubes are now implemented in IBEX (yesterday)
- Many applications involving constraints on trajectories.
- Trajectories of swarms of robot (Paladyn)