Contractors in lattices for solving generalized constraint satisfaction problems

Journée *calcul ensembliste* et *interprétation abstraite*

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Presentation available at http://youtu.be/rRh8azmaWqc

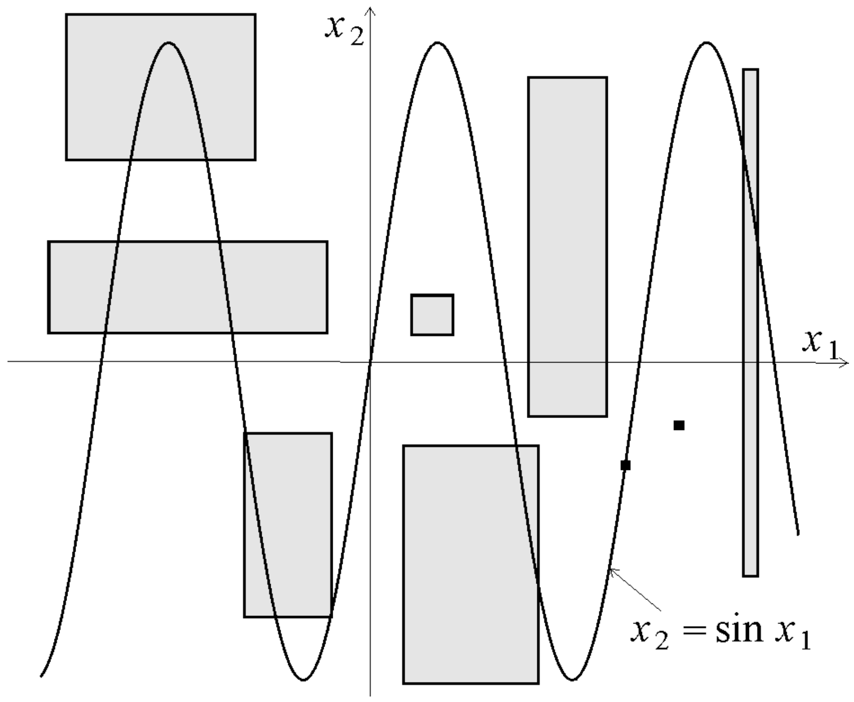
1 Contractors

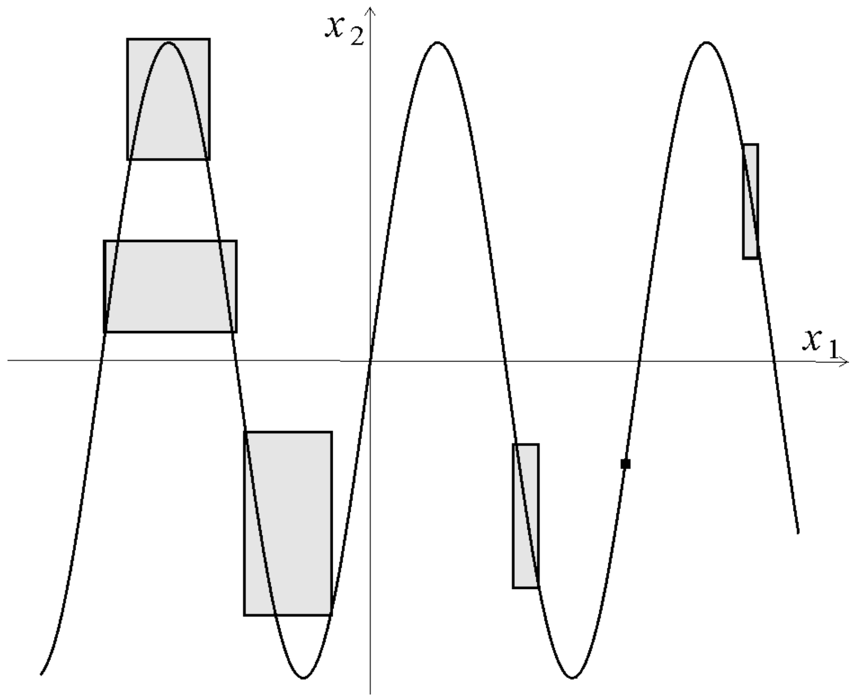
The operator $\mathcal{C} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

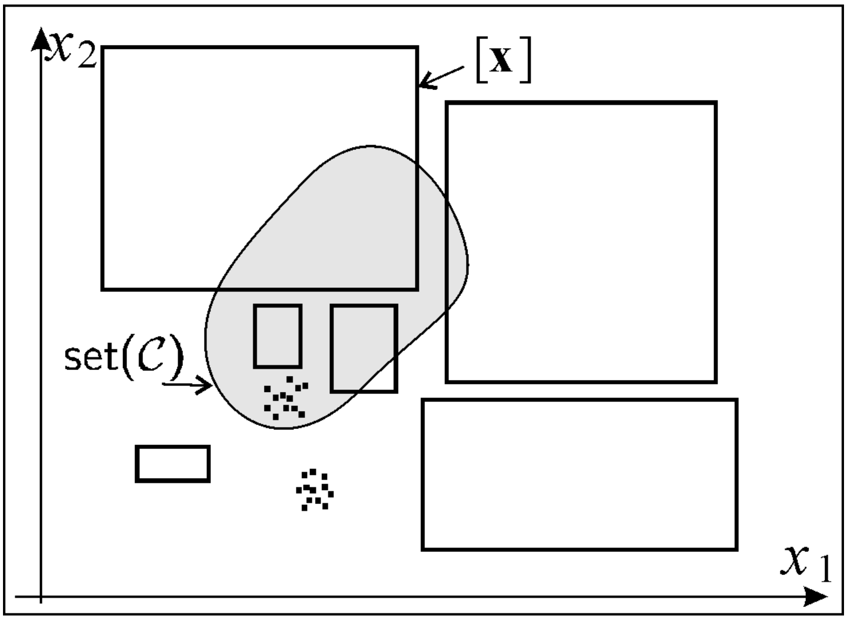
Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$

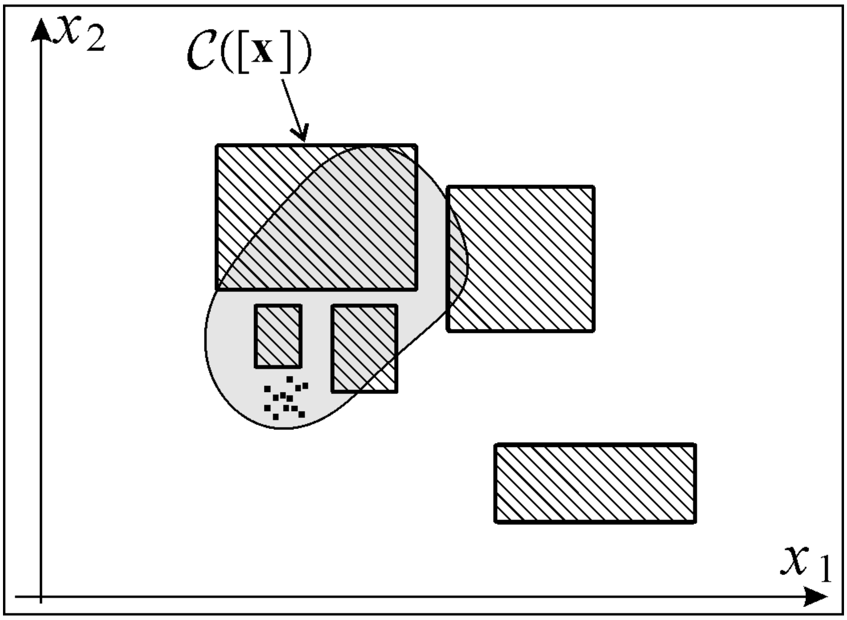

x_2
x_1
$x_2 = \sin x_1$



x_2
x_1



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

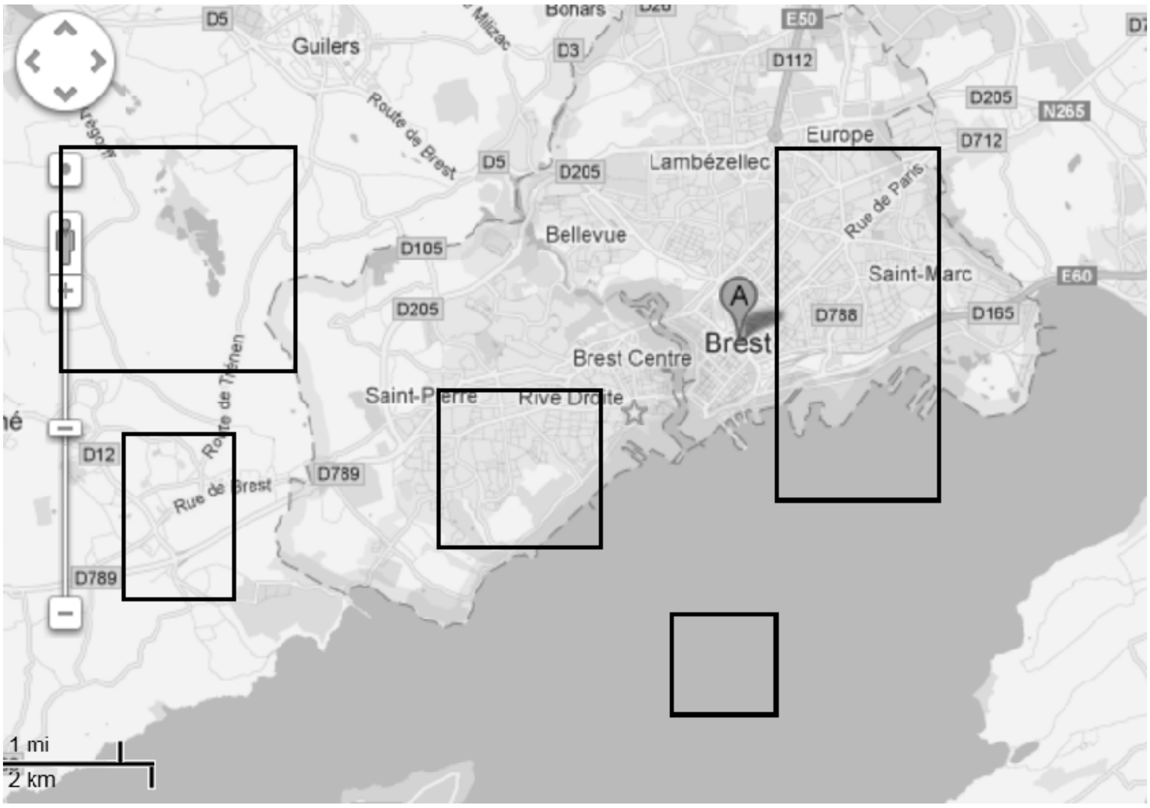
$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *idempotent* if	$\mathcal{C}\left(\mathcal{C}([\mathbf{x}])\right) = \mathcal{C}([\mathbf{x}])$

Contractor algebra

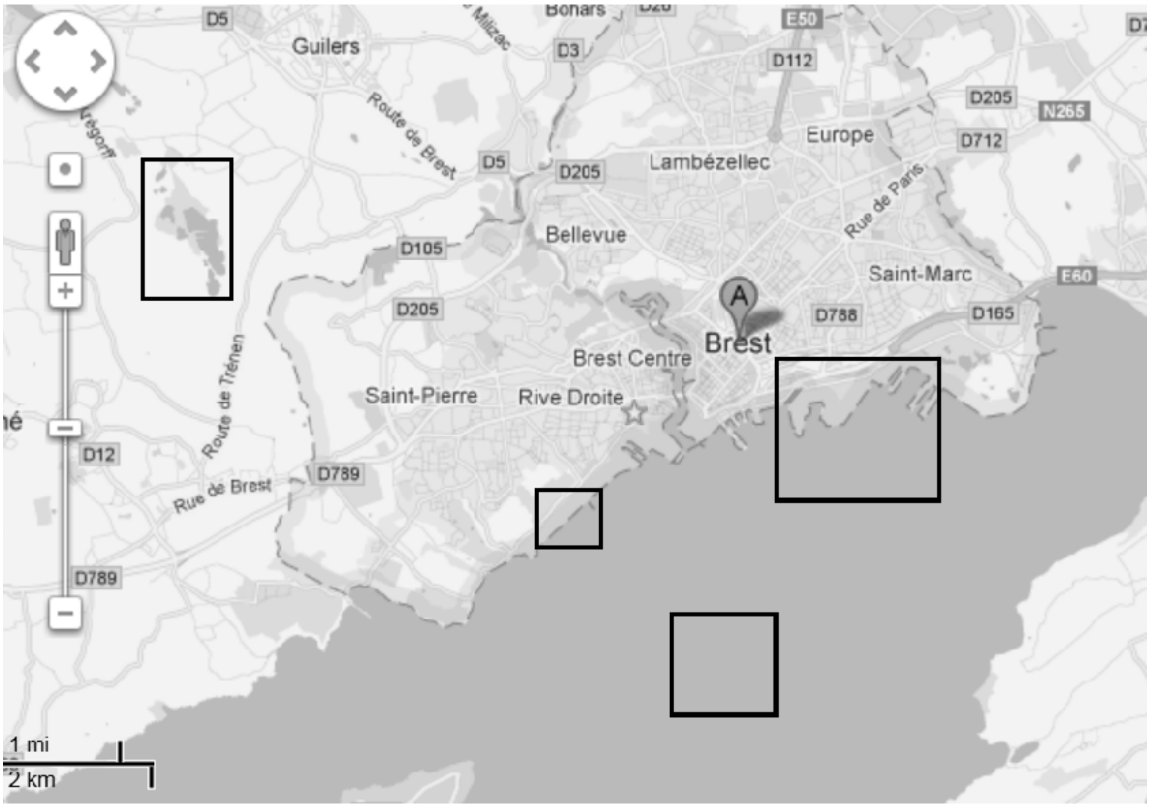
intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \overset{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \overset{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Contractor associated with a database

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Bohars
D3
E50
D112
D205
N265
D712
Europe
Lambézellec
Route de Brest
D5
D205
Bellevue
D105
Rue de Paris
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Route de Trénen
Saint-Pierre
Rive Droite
D12
Rue de Brest
D789
D789
1 mi
2 km



D5
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Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
D12
D789
Rue de Brest
D789
1 mi
2 km

2 Solver

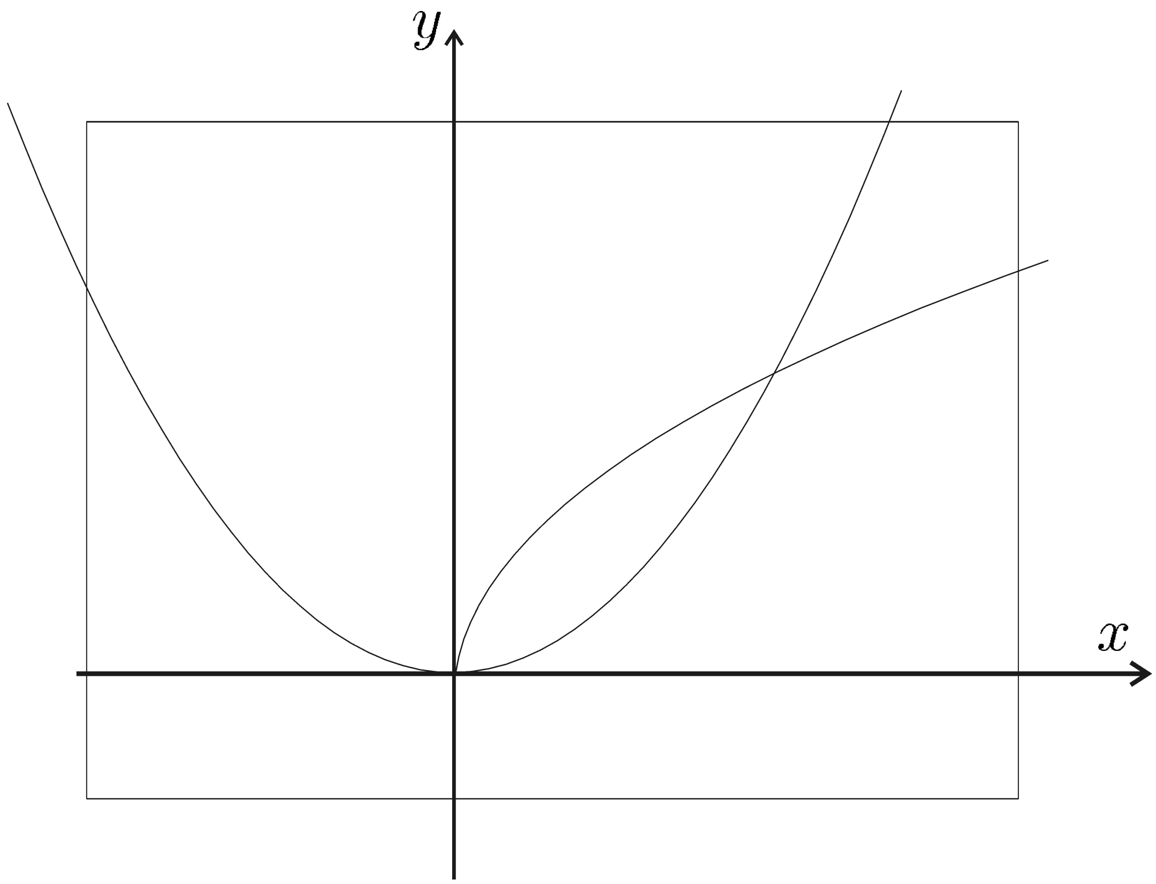
Example. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

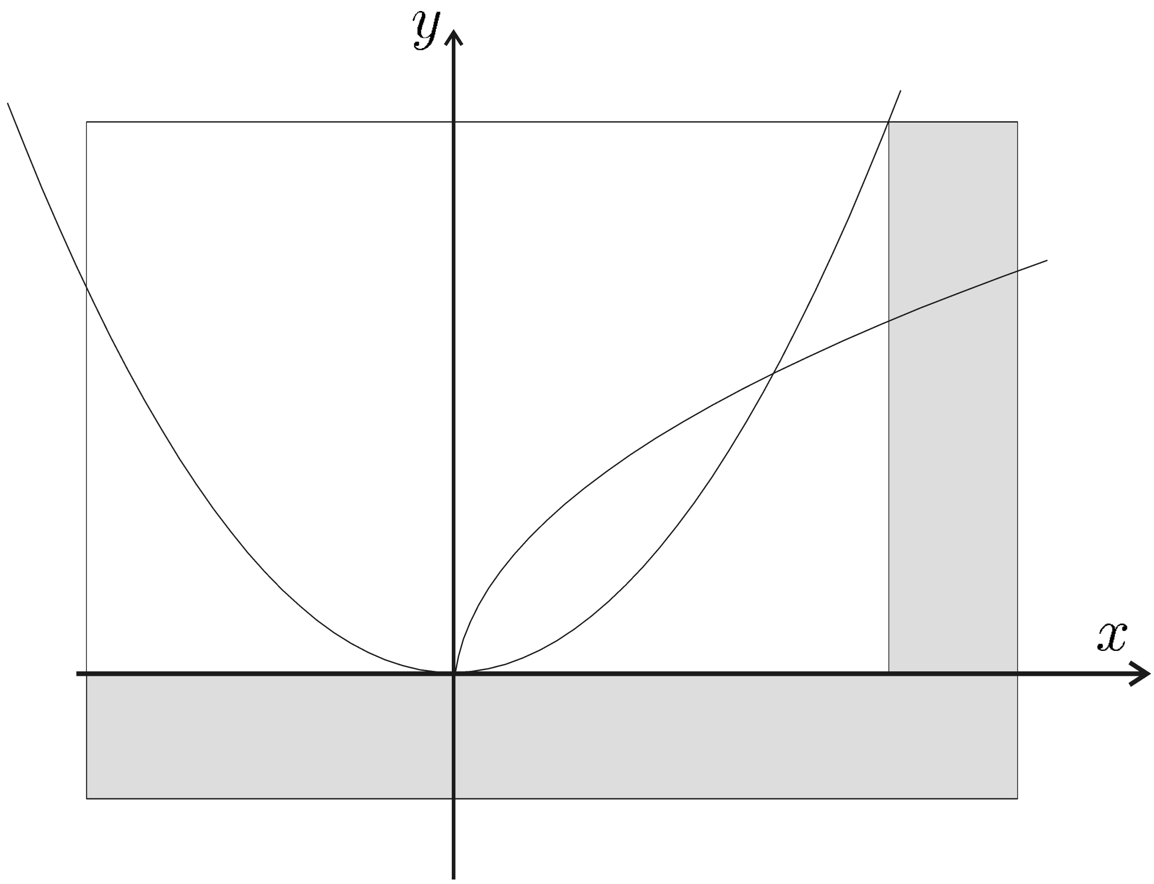
We build two contractors

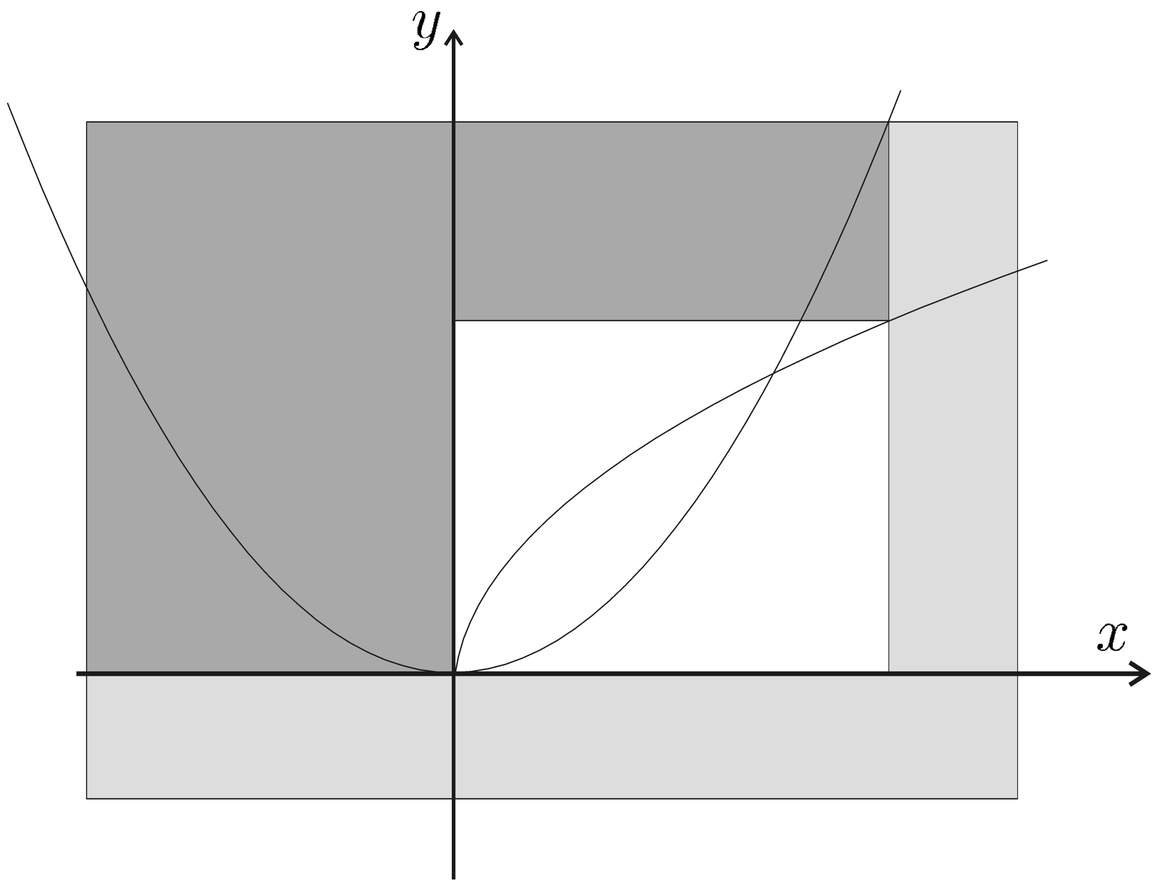
$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated with } y = x^2$$

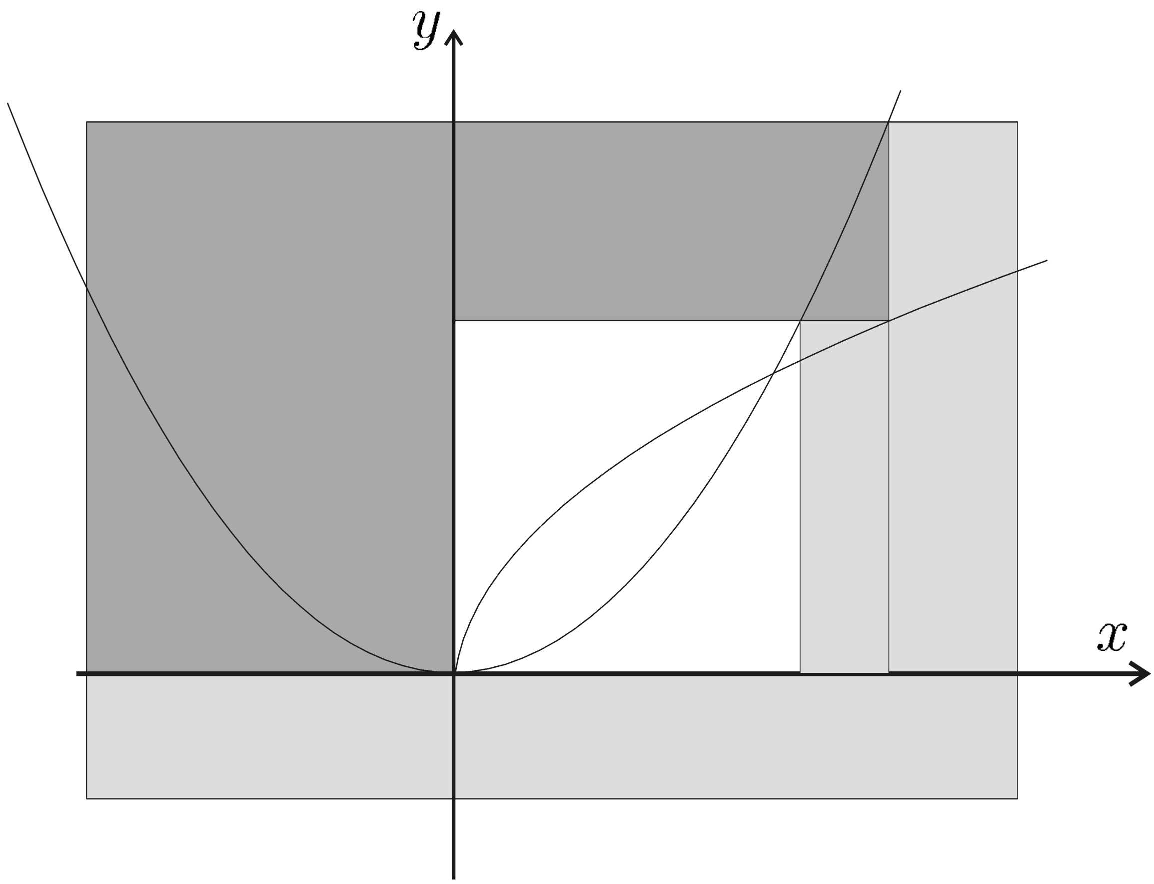
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated with } y = \sqrt{x}$$

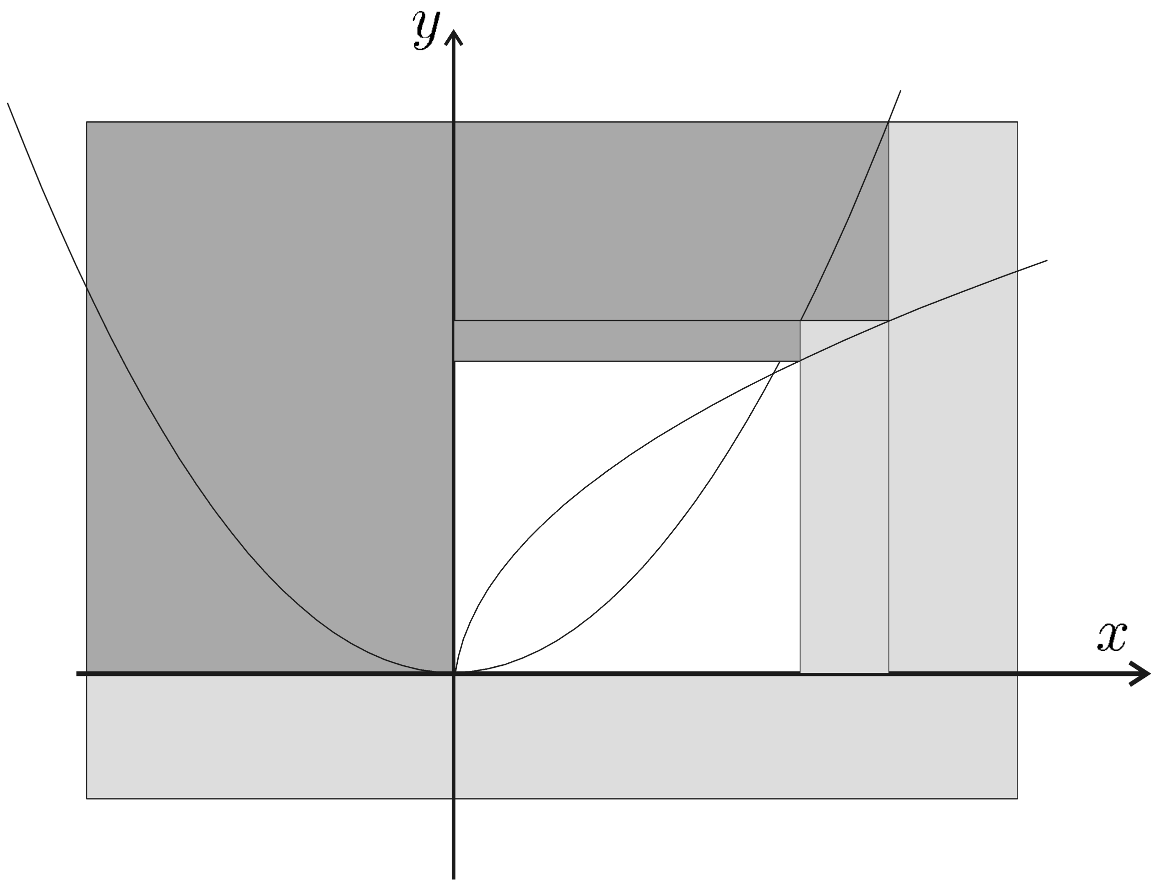


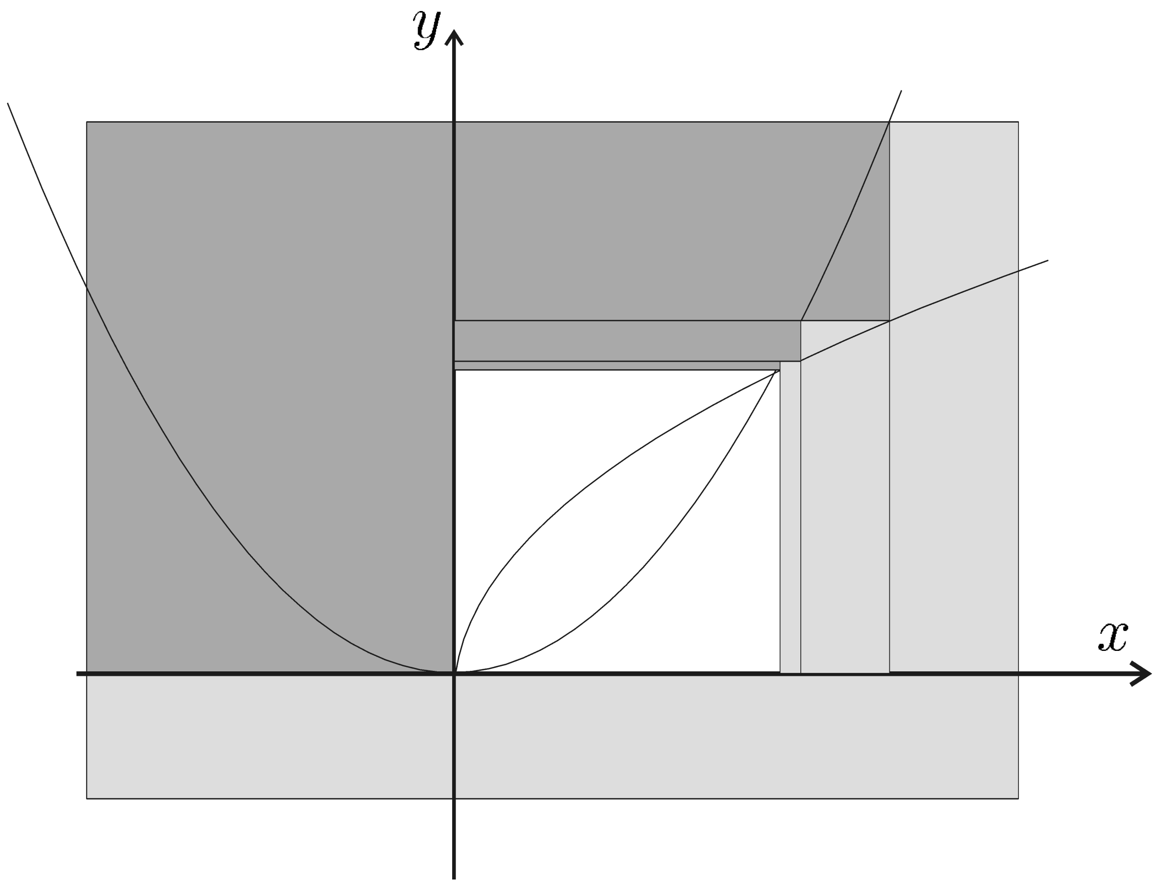
y
x


y
x

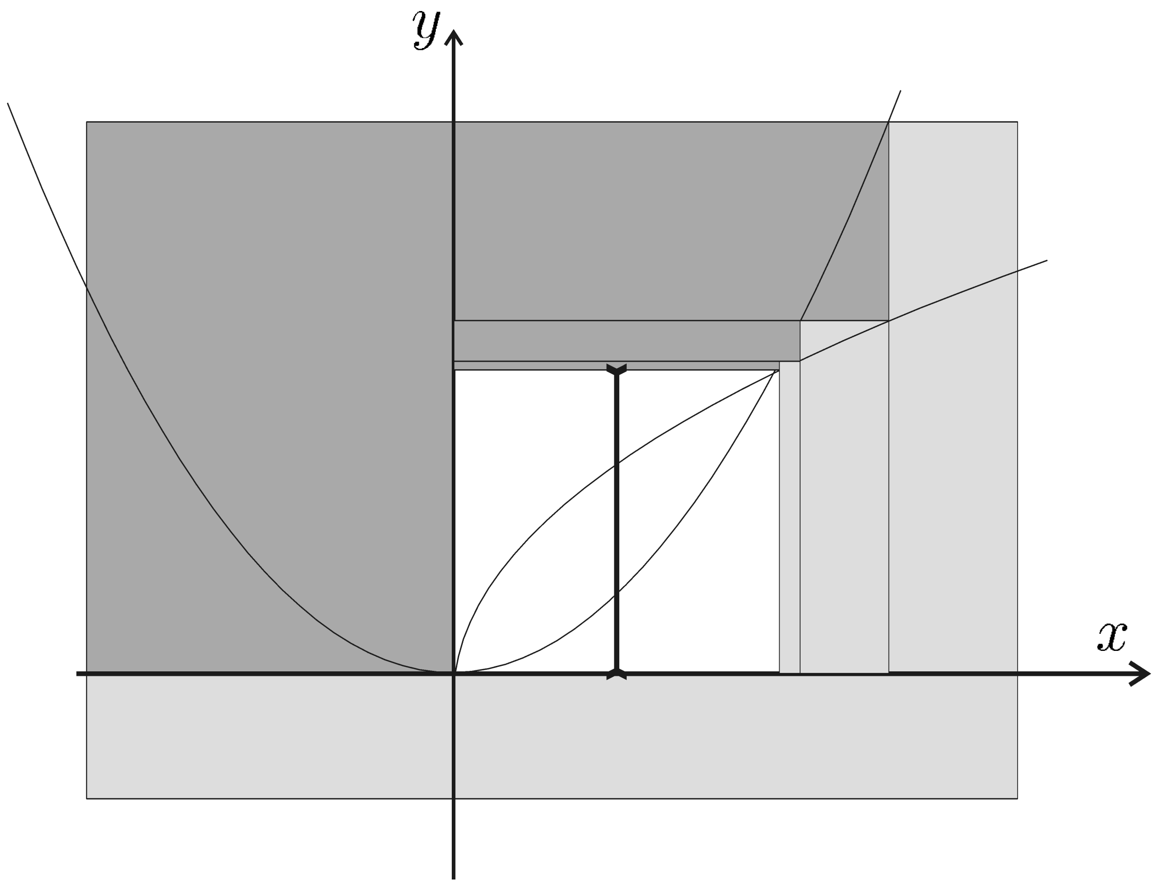

y
x


y
x

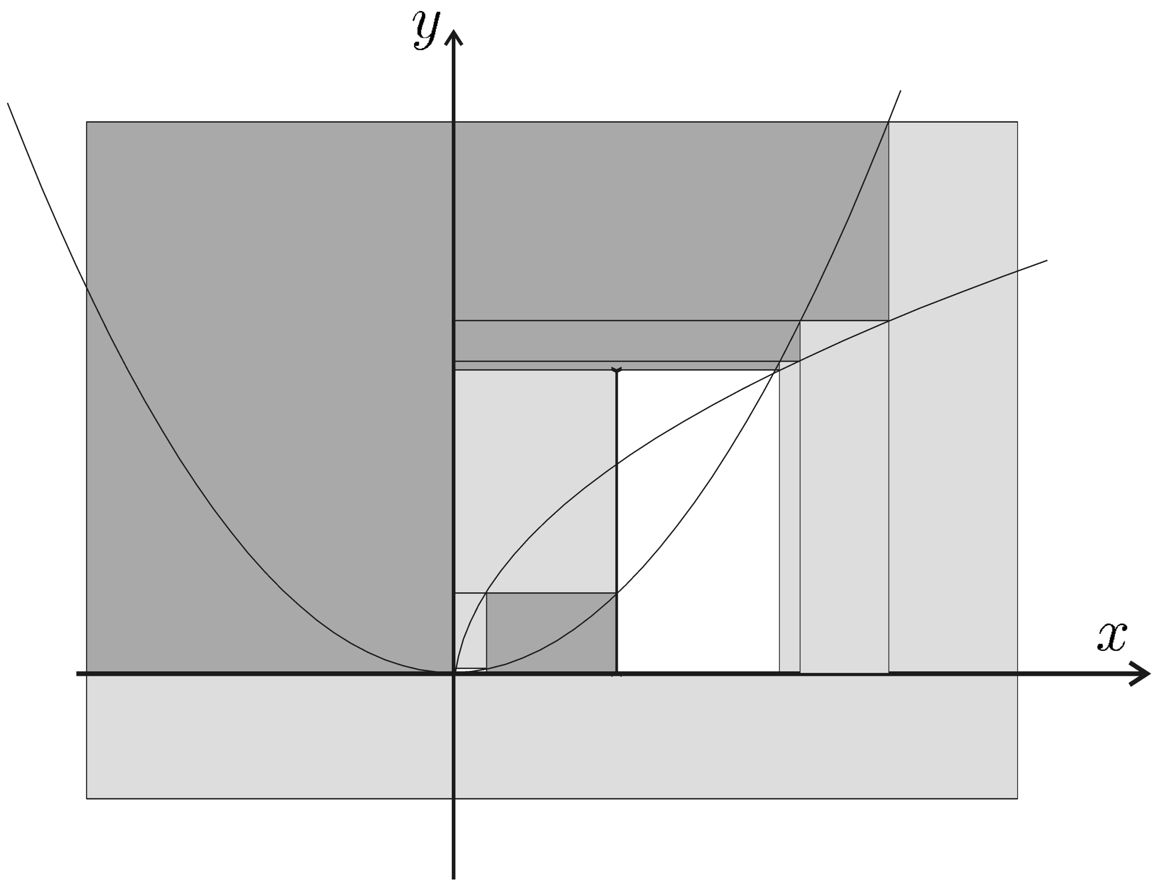

y
x

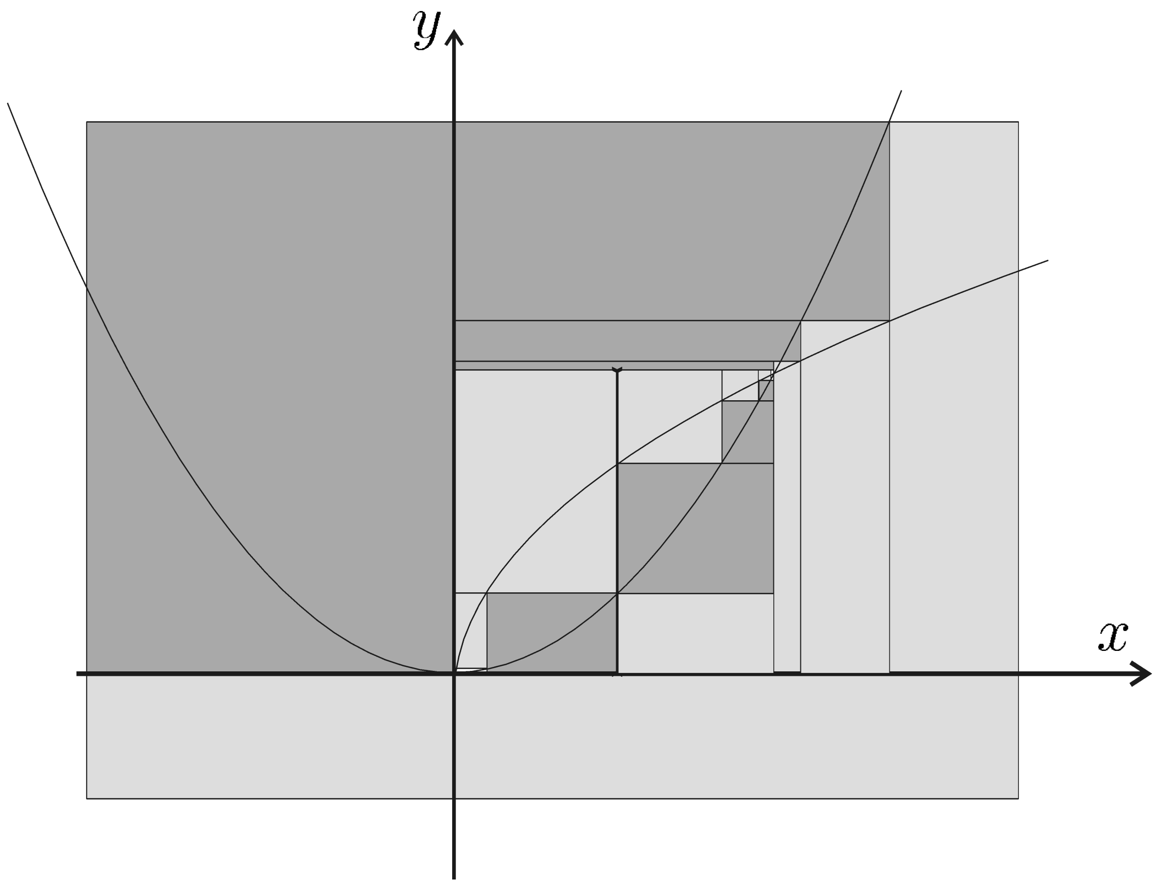


y
x



y
x


y
x



y
x

3 Generalized contractors

A *lattice* $(\mathcal{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds.

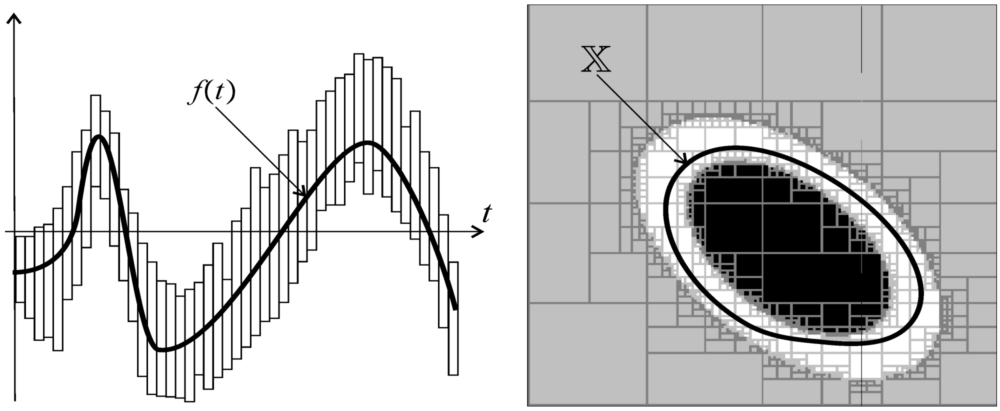
The *join*: $x \vee y$.

The *meet*: $x \wedge y$.

An *interval* $[x]$ of a complete lattice $\mathcal{E}$ is a subset of $\mathcal{E}$ which satisfies

$$[x] = \{x \in \mathcal{E} \mid \wedge [x] \leq x \leq \vee [x]\} .$$

Both $\emptyset$ and $\mathcal{E}$ are intervals of $\mathcal{E}$.



An interval function (or tube) and a set interval

A generalized CSP is composed of
variables $\{x_1, \ldots, x_n\}$,
constraints $\{c_1, \ldots, c_m\}$
domains $\{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$.

The domains $\mathbb{X}_i$ should belong to a lattice $(\mathcal{L}_i, \subset)$.

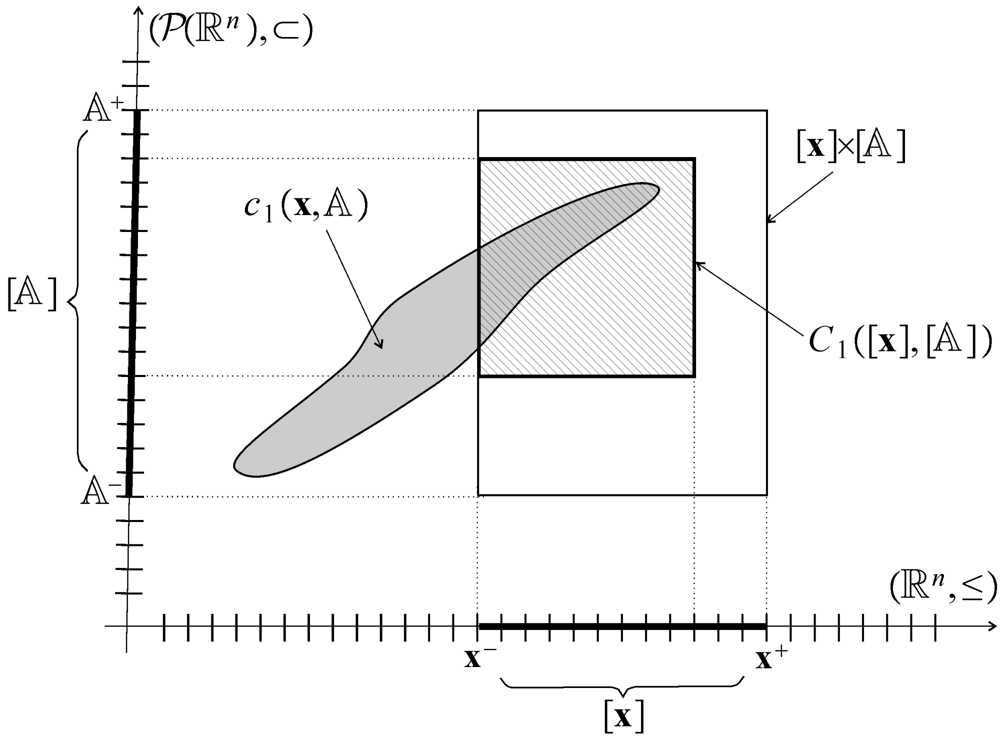
Define $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$.
An element $\mathbb{X}$ of $\mathcal{L}$ is the Cartesian product of n elements of $\mathcal{L}_i$: $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$.
The set $\mathbb{X}$ will be called *hyperdomain*.

A generalized *contractor* is an operator

$$\mathcal{C}: \begin{array}{ccc} \mathcal{L} & \rightarrow & \mathcal{L} \\ \mathbb{X} & \rightarrow & \mathcal{C}(\mathbb{X}) \end{array}$$

which satisfies

$$\begin{array}{ll} \mathbb{X} \subset \mathbb{Y} \Rightarrow \mathcal{C}(\mathbb{X}) \subset \mathcal{C}(\mathbb{Y}) & \text{(monotonicity)} \\ \mathcal{C}(\mathbb{X}) \subset \mathbb{X} & \text{(contractance)} \end{array}$$



$(\mathcal{P}(\mathbb{R}^n), \subset)$
$\mathbb{A}^+$
$[\mathbb{A}]$
$\mathbb{A}^-$
$c_1(\mathbf{x}, \mathbb{A})$
$[\mathbf{x}] \times [\mathbb{A}]$
$C_1([\mathbf{x}], [\mathbb{A}])$
$(\mathbb{R}^n, \leq)$
$\mathbf{x}^-$
$\mathbf{x}^+$
$[\mathbf{x}]$

4 Graph intervals

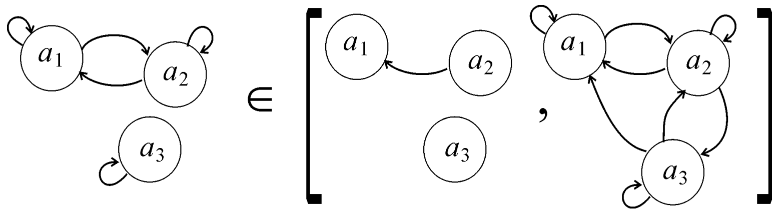
The set of graphs of $\mathcal{A}$ with the relation

$$\mathcal{G} \leq \mathcal{H} \Leftrightarrow \forall i, j \in \{1, \ldots, m\}, \; g_{ij} \leq h_{ij},$$

corresponds to a complete lattice. Intervals of graphs of $\mathcal{A}$ can thus be defined.

Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} [0,1] & [0,1] & 0 \\ 1 & [0,1] & [0,1] \\ [0,1] & [0,1] & [0,1] \end{pmatrix}$$



Define the minimal contractor $\mathcal{C}$ associated the constraint equivalence relation. We have

$$\mathcal{C}\left(\begin{array}{ccc} [0,1] & [0,1] & 0 \\ 1 & [0,1] & [0,1] \\ [0,1] & [0,1] & [0,1] \end{array}\right)=\left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & [0,1] \\ 0 & [0,1] & 1 \end{array}\right)$$

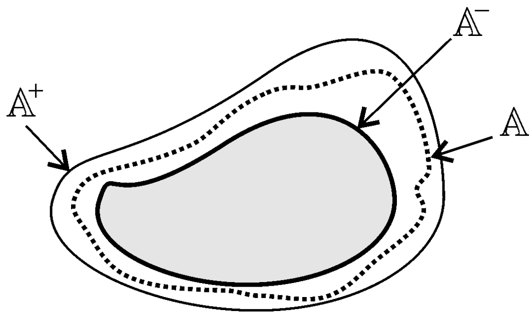
5 Set intervals

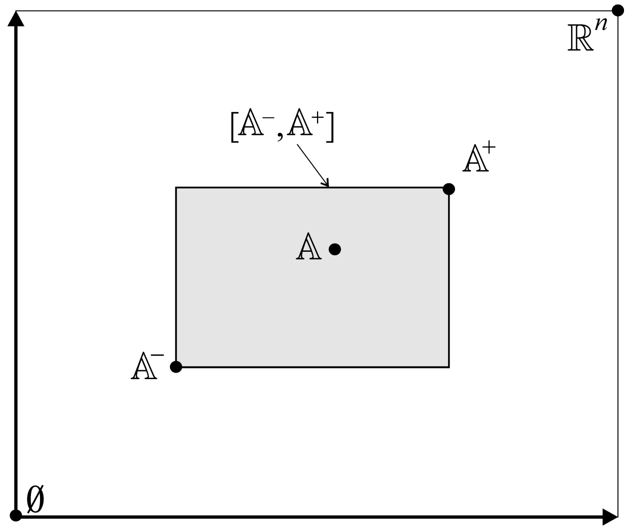
5.1 Definition

Given two sets $\mathbb{A}^-$ and $\mathbb{A}^+$ of $\mathbb{R}^n$, the pair $[\mathbb{A}] = \left[\mathbb{A}^-, \mathbb{A}^+\right]$ which encloses all sets $\mathbb{A}$ such that

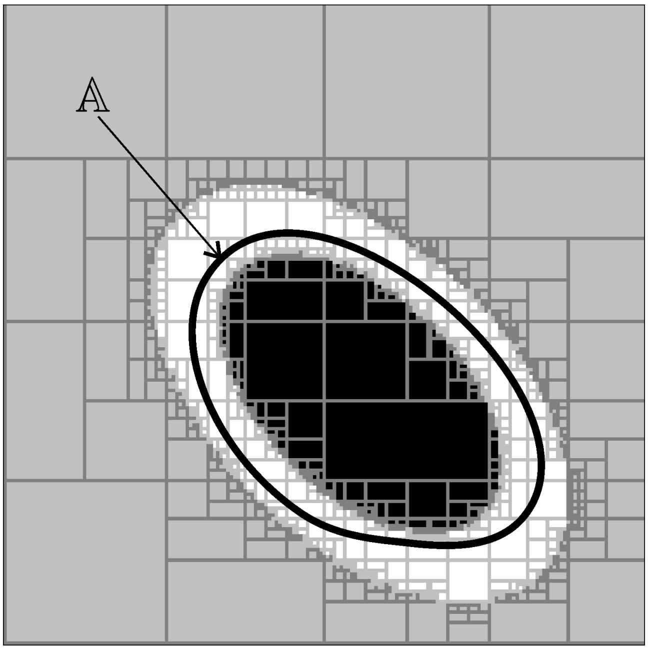
$$\mathbb{A}^- \subset \mathbb{A} \subset \mathbb{A}^+$$

is a *set interval*.





Lattice $(\mathcal{P}(\mathbb{R}^n), \subset)$

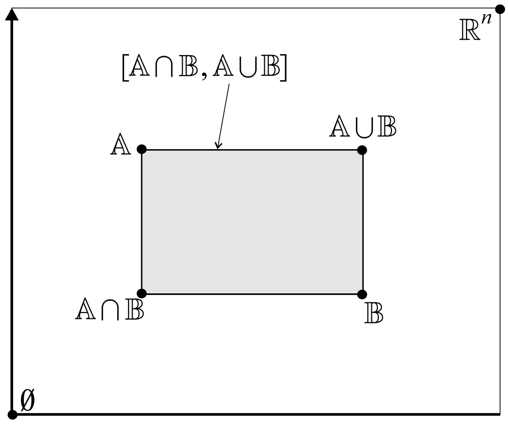


Machine representation of $\left[\mathbb{A}^{-}, \mathbb{A}^{+}\right]$

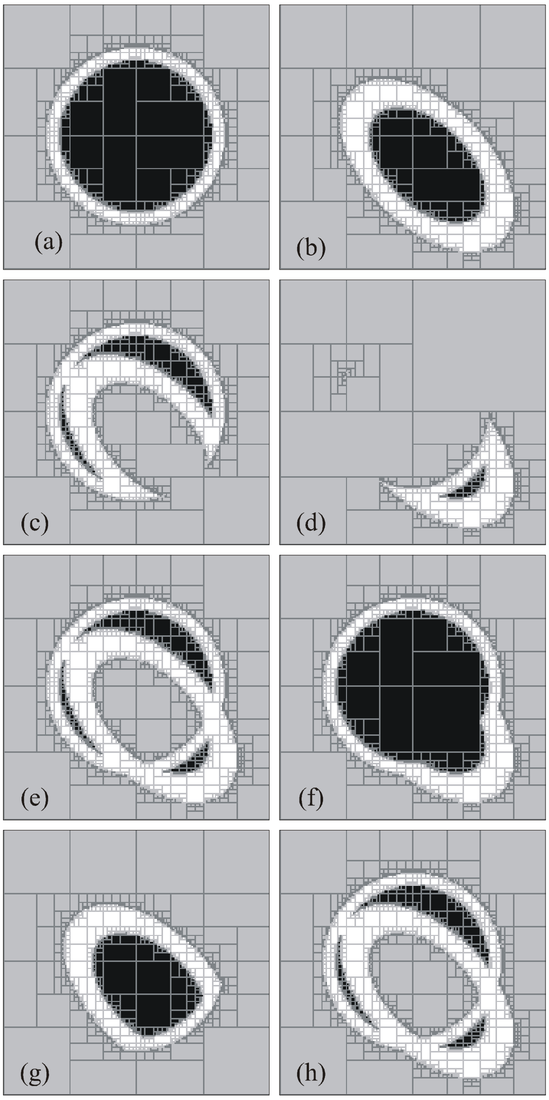
The set interval $[\emptyset, \emptyset]$ is a singleton : $\emptyset \in [\emptyset, \emptyset]$.
The set interval $[\emptyset, \mathbb{R}^n]$ encloses all sets of $\mathbb{R}^n$.
The empty set interval is denoted by $[\mathbb{R}^n, \emptyset]$.

Given two sets $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{R}^n$. The smallest set interval which contains $\mathbb{A}$ and $\mathbb{B}$ is

$$\square\{\mathbb{A},\mathbb{B}\} = [\mathbb{A}\cap\mathbb{B}, \mathbb{A}\cup\mathbb{B}]$$



5.2 Arithmetic



(a) $\mathbb{A} \in \left[\mathbb{A}^-, \mathbb{A}^+\right]$

(b) $\mathbb{B} \in \left[\mathbb{B}^-, \mathbb{B}^+\right]$

(c) $[\mathbb{A}] \setminus [\mathbb{B}]$

(d) $[\mathbb{B}] \setminus [\mathbb{A}]$

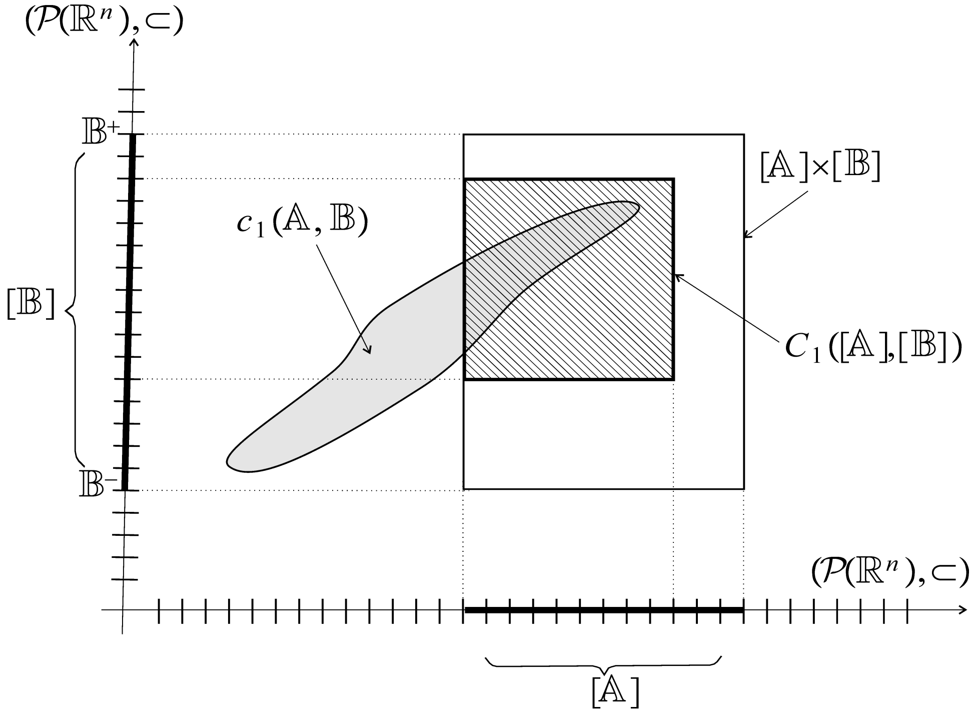
(e) $[\mathbb{A}] \setminus [\mathbb{B}] \cup [\mathbb{B}] \setminus [\mathbb{A}]$

(f) $[\mathbb{A}] \cup [\mathbb{B}]$

(g) $[\mathbb{A}] \cap [\mathbb{B}]$

(h) $([\mathbb{A}] \cup [\mathbb{B}]) \setminus ([\mathbb{A}] \cap [\mathbb{B}])$

5.3 Contractors



$(\mathcal{P}(\mathbb{R}^n),\subset)$
$\mathbb{B}^+$
$[\mathbb{A}]\times[\mathbb{B}]$
$c_1(\mathbb{A},\mathbb{B})$
$[\mathbb{B}]$
$C_1([\mathbb{A}],[\mathbb{B}])$
$\mathbb{B}^-$
$(\mathcal{P}(\mathbb{R}^n),\subset)$
$[\mathbb{A}]$

Consider the CSP

$$\left\{\begin{array}{c} \mathbb{A} \subset \mathbb{B} \\ \mathbb{A} \in [\mathbb{A}]\,, \mathbb{B} \in [\mathbb{B}]\,. \end{array}\right.$$

The optimal contractor is

$$\left\{\begin{array}{ll} \text{(i)} & [\mathbb{A}] := [\mathbb{A}] \sqcap ([\mathbb{A}] \cap [\mathbb{B}]) \\ \text{(ii)} & [\mathbb{B}] := [\mathbb{B}] \sqcap ([\mathbb{A}] \cup [\mathbb{B}]) \end{array}\right.$$

Consider the CSP

$$\begin{cases} \mathbb{A} \cap \mathbb{B} = \mathbb{C} \\ \mathbb{A} \in [\mathbb{A}] \, , \mathbb{B} \in [\mathbb{B}] \, , \mathbb{C} \in [\mathbb{C}] \, . \end{cases}$$

The optimal contractor is

$$\begin{cases} \text{(i)} & [\mathbb{C}] := [\mathbb{C}] \sqcap ([\mathbb{A}] \cap [\mathbb{B}]) \\ \text{(ii)} & [\mathbb{A}] := [\mathbb{A}] \sqcap ([\mathbb{C}] \cup ([\emptyset, \mathbb{R}^n] \setminus ([\mathbb{B}] \setminus [\mathbb{C}]))) \\ \text{(iii)} & [\mathbb{B}] := [\mathbb{B}] \sqcap ([\mathbb{C}] \cup ([\emptyset, \mathbb{R}^n] \setminus ([\mathbb{A}] \setminus [\mathbb{C}]))) \, . \end{cases}$$

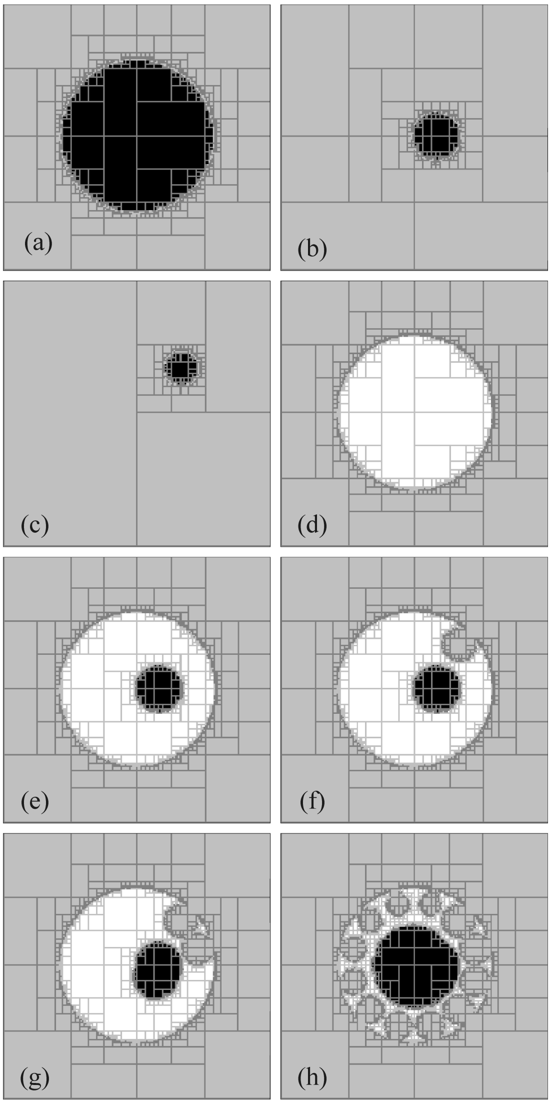
5.4 Application

Consider the following CSP

$$\begin{cases} \text{(i)} & \mathbb{X} \subset \mathbb{A} \\ \text{(ii)} & \mathbb{B} \subset \mathbb{X} \\ \text{(iii)} & \mathbb{X} \cap \mathbb{C} = \emptyset \\ \text{(iv)} & f(\mathbb{X}) = \mathbb{X}, \end{cases}$$

where $\mathbb{X}$ is an unknown subset of $\mathbb{R}^2$, f is a rotation with an angle of $-\frac{\pi}{6}$, and

$$\begin{cases} \mathbb{A} &= \left\{(x_1, x_2), x_1^2 + x_2^2 \leq 3\right\} \\ \mathbb{B} &= \left\{(x_1, x_2), (x_1 - 0.5)^2 + x_2^2 \leq 0.3\right\} \\ \mathbb{C} &= \left\{(x_1, x_2), (x_1 - 1)^2 + (x_2 - 1)^2 \leq 0.15\right\} \end{cases}$$



(a) $[\mathbb{A}]$

(b) $[\mathbb{B}]$

(c) $[\mathbb{C}]$

(d) $\mathbb{X} \subset \mathbb{A}$

(e) $\mathbb{B} \subset \mathbb{X}$

(f) $\mathbb{X} \cap \mathbb{C} = \emptyset$

(g) $f(\mathbb{X}) = \mathbb{X}$

(h) $(f(\mathbb{X}) = \mathbb{X})^{\infty}$

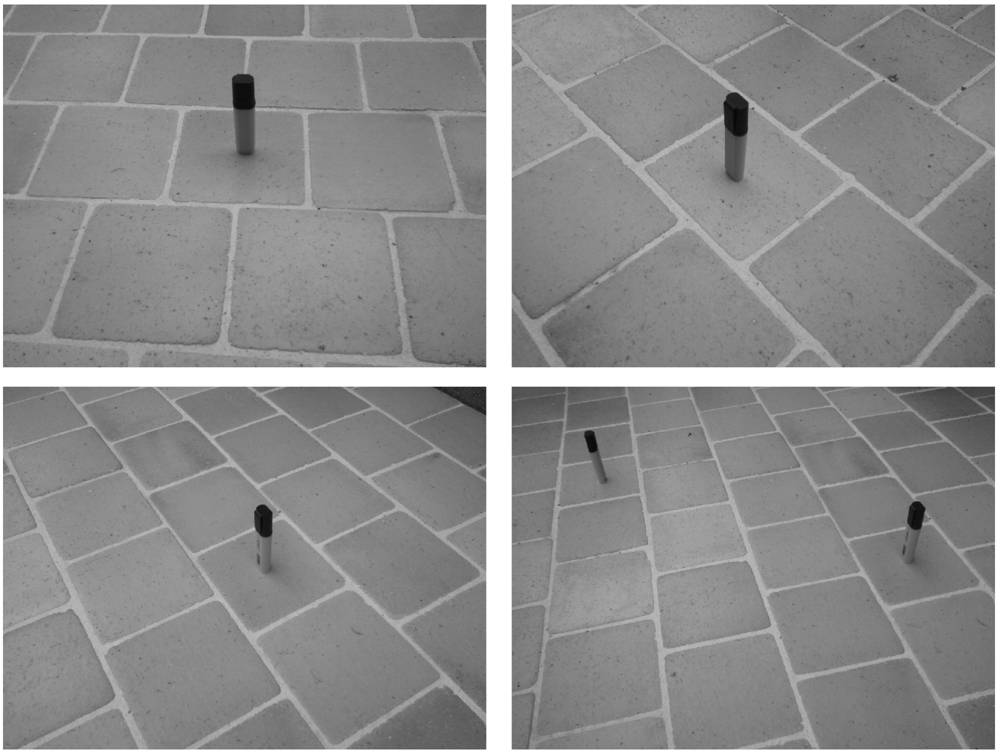
6 SLAM with indistinguishable marks

Robot: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}),\ \mathbf{x}(0) = \mathbf{0}$.
Marks $\mathcal{M} = \{\mathbf{m}(1), \mathbf{m}(2), \dots\} \subset \mathbb{R}^2$.

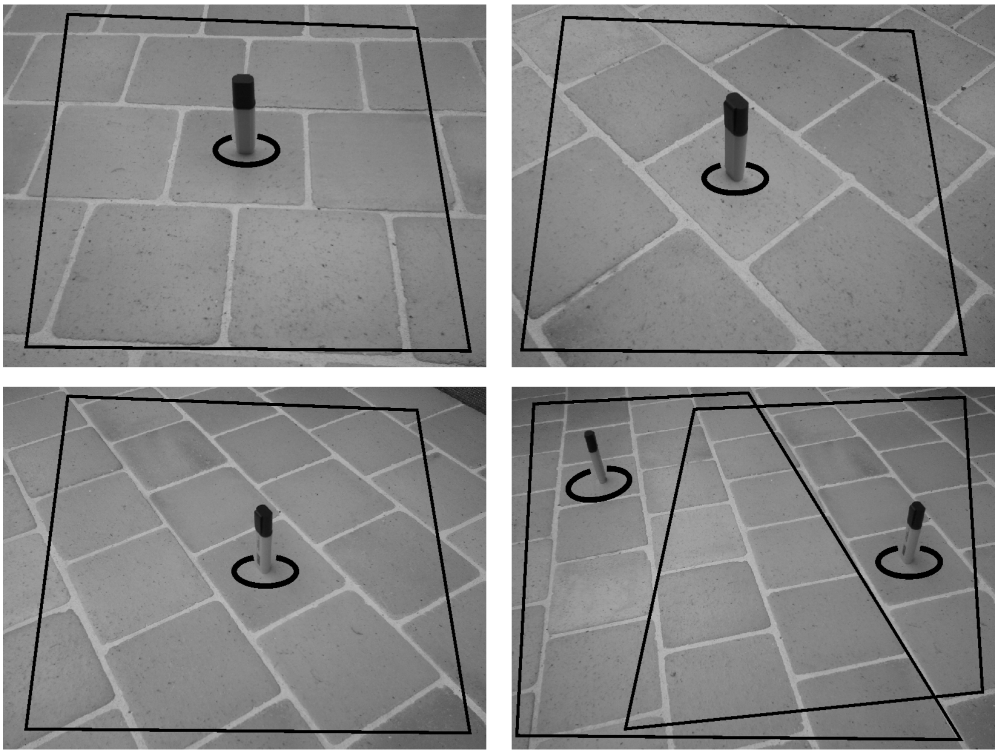
Context: indistinguishable point marks that are partially observable

Our SLAM is a *chicken and egg* problem of cardinality three:
(i) if the map and the associations are known, we have localization problem,
(ii) if the trajectory and the associations are known, we have a mapping problem
(iii) if the trajectory and the map are known we have an association problem.

The unknown variables have an heterogenous nature:
(i) marks $\mathbf{m}(j) \in \mathbb{R}^2$
(ii) trajectory $\mathbf{x}(t) : \mathbb{R} \longrightarrow \mathbb{R}^n$,
(iii) the free space $\mathbb{F} \in \mathcal{P}\left(\mathbb{R}^2\right)$
(iv) the data associations is a graph $\mathcal{G}$.



A *sector* $\mathbb{H}$ is a subset of $\mathbb{R}^2$ which contains a single mark.



Our SLAM problem:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) & \text{(evolution equation)} \\ (t_i, \mathcal{H}_i(\mathbf{x})) & \text{(sector list)} \end{cases}$$

where $t \in [0, t_{\mathsf{max}}]$, $\mathbf{u}(t) \in [\mathbf{u}](t)$.

Each set $\mathcal{H}_i(\mathbf{x}(t_i)) \subset \mathbb{R}^2$ contains a unique mark.

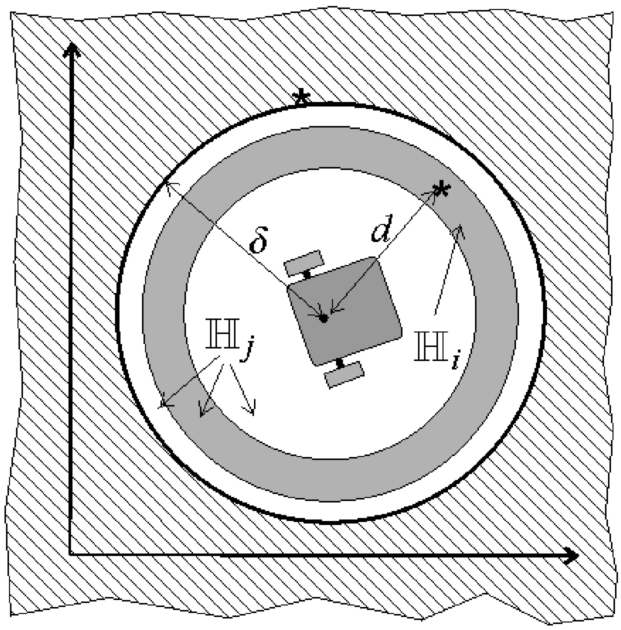
We have an egocentric representation.

We define $\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$.

Example 1. A robot moving in a plane and located at (x_1, x_2) . At t_3 the robot detects a unique mark at a distance $d \in [4, 5]$. We have

$$\mathcal{H}_3(\mathbf{x}) = \left\{ \mathbf{a} \in \mathbb{R}^2 \middle| (x_1 - a_1)^2 + (x_2 - a_2)^2 \in [16, 25] \right\}.$$

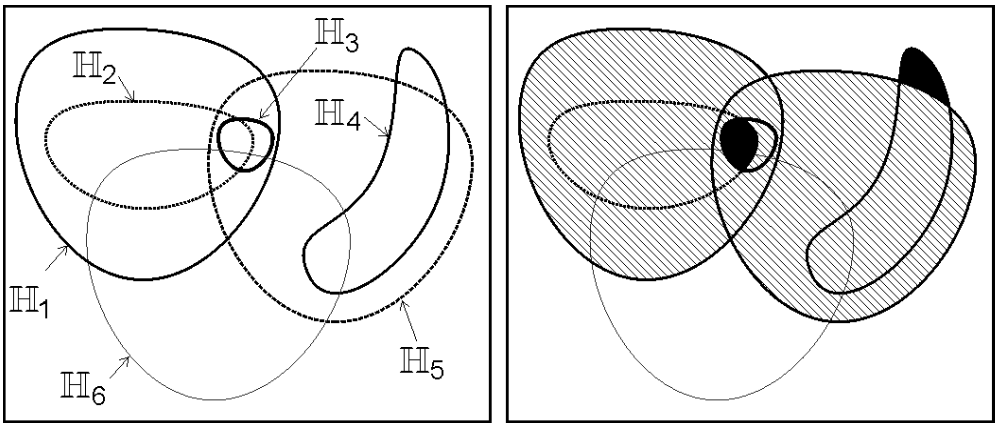
Example 2. We have two sectors $\mathbb{H}_i$ and $\mathbb{H}_j$.
Since $\mathbb{H}_i \subset \mathbb{H}_j$, $\mathbb{H}_j \backslash \mathbb{H}_i$ has no mark. Thus we can associate $\mathbb{H}_i$ with $\mathbb{H}_j$.



Theorem. Define the free space as $\mathbb{F} = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \mathbf{p} \notin \mathcal{M} \right\}$. Consider m sectors $\mathbb{H}_1, \ldots, \mathbb{H}_m$. Denote by $\mathbf{a}(i)$ the mark in $\mathbb{H}_i$. We have

$$\begin{array}{ll} \text{(i)} & \mathbb{H}_i \subset \mathbb{H}_j \Rightarrow \mathbf{a}(i) = \mathbf{a}(j) \\ \text{(ii)} & \mathbb{H}_i \cap \mathbb{H}_j = \emptyset \Rightarrow \mathbf{a}(i) \neq \mathbf{a}(j) \\ \text{(iii)} & \mathbb{H}_i \subset \mathbb{H}_j \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}. \end{array}$$

Example.



The two black zones contain a single mark and no mark exists in the hatched area.

Association graph. Consider m detections $\mathbf{a}(1), \ldots, \mathbf{a}(m)$. The *association graph* is the graph with nodes $\mathbf{a}(i)$ such that $\mathbf{a}(i) \to \mathbf{a}(j)$ means that $\mathbf{a}(i) = \mathbf{a}(j)$.

7 SLAM as a CSP

Variables

(i) the trajectory of the robot $\mathbf{x}$.

(ii) the sectors $\mathbb{H}_i$

(iii) the location of the mark $\mathbf{a}\,(i)$ detected at time t_i

(iv) the association graph $\mathcal{G}$

(v) the free space $\mathbb{F}$.

Domains

$$
\begin{aligned}
&\mathbf{x} \in [\mathbf{x}] = \left[\mathbf{x}^{-}, \mathbf{x}^{+}\right] \\
&\mathbf{a}\left(i\right) \in \mathbb{A}\left(i\right) \\
&\mathbb{H}_i \in [\mathbb{H}_i] = \left[\mathbb{H}_i^{-}, \mathbb{H}_i^{+}\right] \\
&\mathbb{F} \in [\mathbb{F}] = \left[\mathbb{F}^{-}, \mathbb{F}^{+}\right] \\
&\mathcal{G} \in [\mathcal{G}] = \left[\mathcal{G}^{-}, \mathcal{G}^{+}\right].
\end{aligned}
$$

Initialization

$[\mathbf{x}](t) = [-\infty, \infty]$ if $t > 0$ and $[\mathbf{x}](0) = \mathbf{0}$.

$\mathbb{A}(i) = \mathbb{R}^2$.

$\mathbb{H}_i \in \left[\emptyset, \mathbb{R}^2\right]$.

$\mathbb{F} \in \left[\emptyset, \mathbb{R}^2\right]$.

$\mathcal{G} \in [\emptyset, \top]$

Constraints

(i) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$

(ii) $\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$

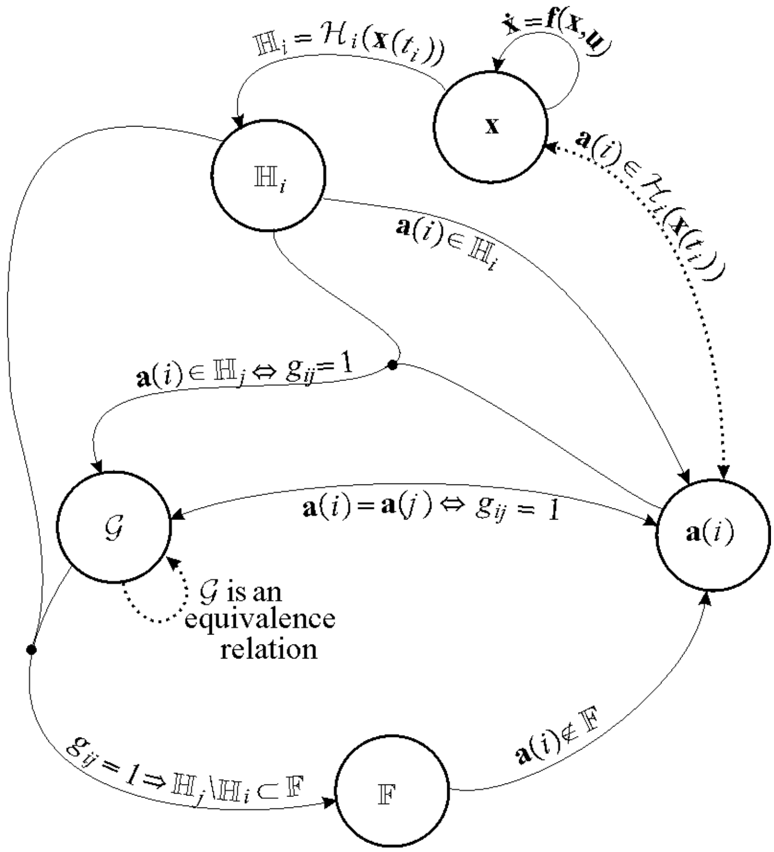
(iii) $\mathbf{a}(i) \in \mathbb{H}_i$

(iv) $\mathbf{a}(i) = \mathbf{a}(j) \Leftrightarrow g_{ij} = \mathbf{1}$

(v) $\mathbf{a}(i) \in \mathbb{H}_j \Leftrightarrow g_{ij} = \mathbf{1}$

(vi) $g_{ij} = \mathbf{1} \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}$

(vii) $\mathbf{a}(i) \notin \mathbb{F}$



$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})$
$\mathbb{H}_i = \mathcal{H}_i(\mathbf{x}(t_i))$
$\mathbf{x}$
$\mathbb{H}_i$
$\mathbf{a}(i) \in \mathcal{H}_i(\mathbf{x}(t_i))$
$\mathbf{a}(i) \in \mathbb{H}_i$
$\mathbf{a}(i) \in \mathbb{H}_j \Leftrightarrow g_{ij} = 1$
$\mathcal{G}$
$\mathbf{a}(i) = \mathbf{a}(j) \Leftrightarrow g_{ij} = 1$
$\mathbf{a}(i)$
$\mathcal{G}$ is an equivalence relation
$g_{ij} = 1 \Rightarrow \mathbb{H}_j \backslash \mathbb{H}_i \subset \mathbb{F}$
$\mathbb{F}$
$\mathbf{a}(i) \notin \mathbb{F}$

8 Test-case

Generation of the data.

A simulated robot follows a cycloid for 100sec.

10 marks inside $[-8,8] \times [-8,8]$.

A rangefinder collects the distance $\tilde{d}$ to the nearest mark.

Resolution. The robot is

$$\left\{\begin{array}{lll}\dot{x}_1 & = & u_1 \cos u_2 \\ \dot{x}_2 & = & u_1 \sin u_2.\end{array}\right.$$

The sector functions are

$$\begin{array}{lll}\mathcal{H}_i\left(\mathbf{x}\left(t_i\right)\right) & = & \left\{\mathbf{a} \mid \ \left\|\mathbf{a}-\mathbf{x}\left(t_i\right)\right\| \ \in \ \left[d_i\right]\right\} \\ \mathcal{H}_{i+1}\left(\mathbf{x}\left(t_{i+1}\right)\right) & = & \left\{\mathbf{a} \mid \ \left\|\mathbf{a}-\mathbf{x}\left(t_{i+1}\right)\right\| \ < \ \delta_{i+1}\right\}.\end{array}$$

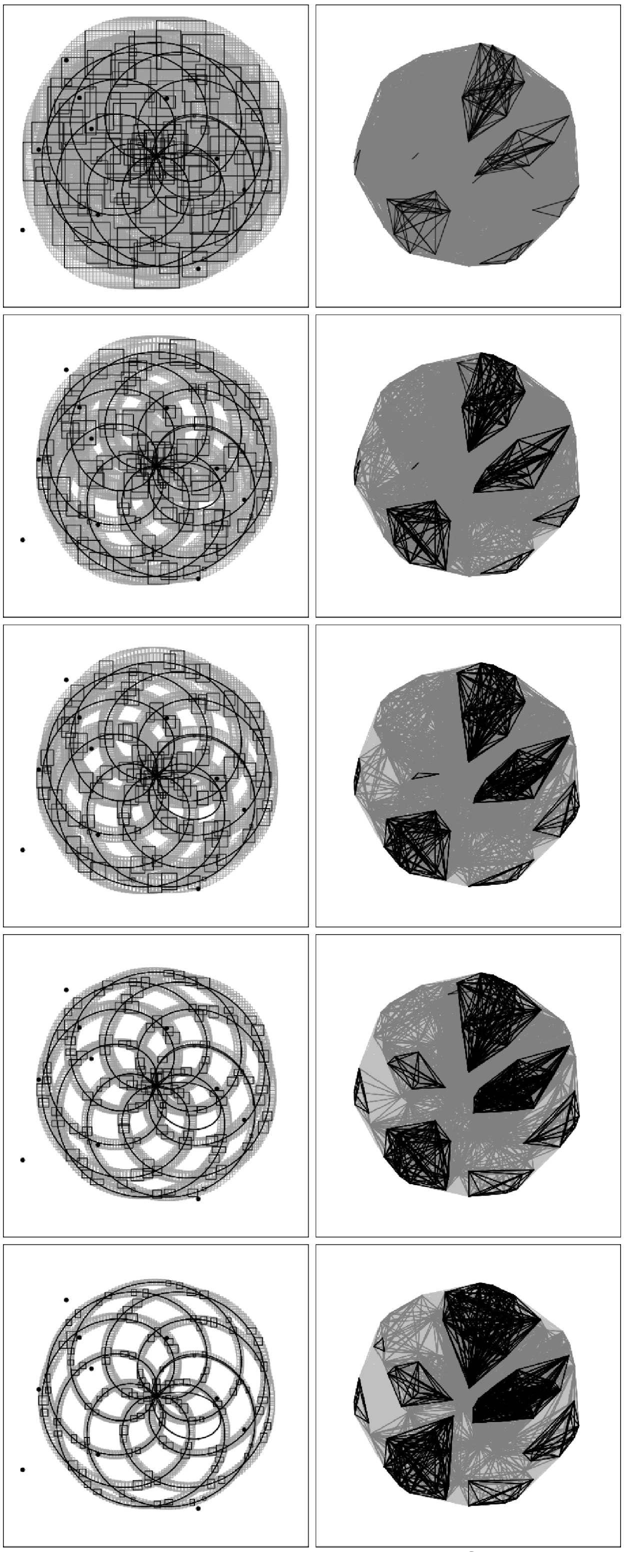
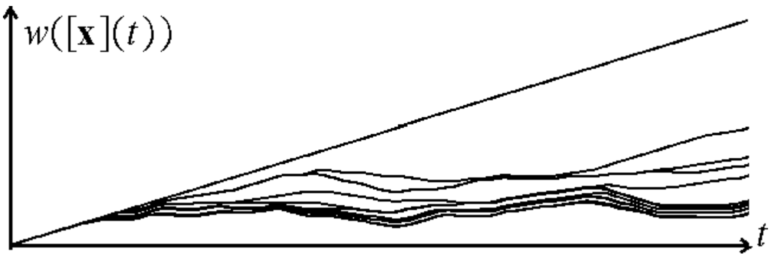
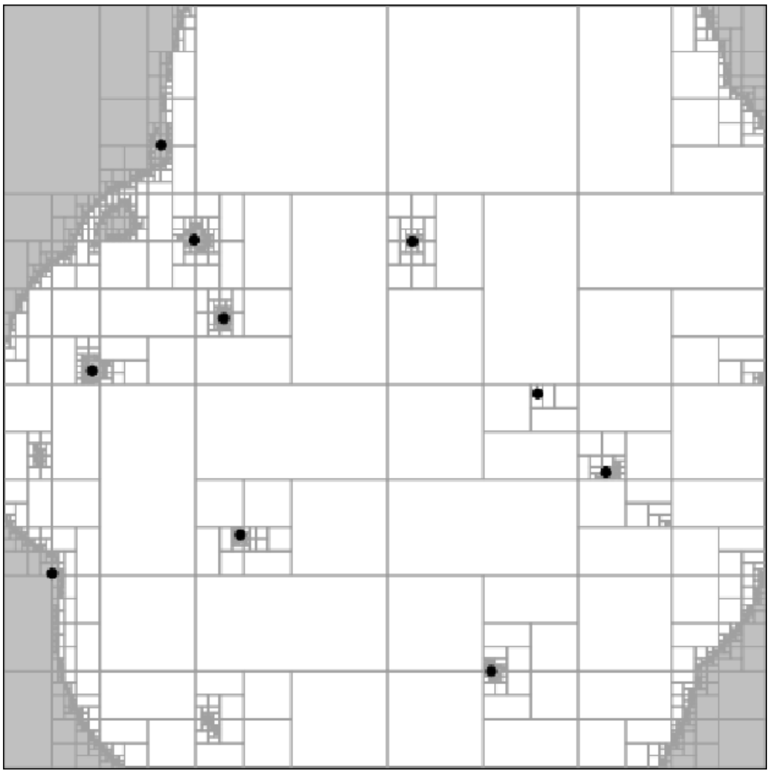


Illustration of the propagation. Left: the tube becomes more and more accurate. Right: The association graph has more and more arcs.



Superposition of the width of the tube $[\mathbf{x}](t)$

Associations. At the fixed point, 3888 associations have been found, 29128 pairs $(\mathbf{a}(i), \mathbf{a}(j))$ have been proven disjoint and 5400 pairs $(\mathbf{a}(i), \mathbf{a}(j))$ have not been classified.



Free space $\mathbb{F}$.