A New Framework for Sharp and Efficient Resolution of NCSP with Manifolds of Solutions

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Numerical CSP

Traditional CSP definition

- $\langle V, D, C \rangle$ where
  - $V = \{x_1, \ldots, x_n\}$
  - $D = \{D_1, \ldots, D_n\}$
  - $C = \{c_1, \ldots, c_m\}$

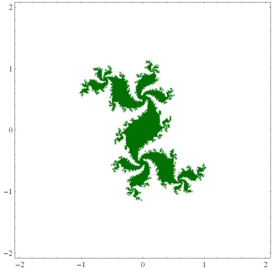
$\longrightarrow$ Variable assignments that satisfy constraints

Numerical CSP Definition

- Continuous domains (usually intervals)
- $\langle \mathbf{x}, [\mathbf{x}], C \rangle$ where
  - $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$
  - $[\mathbf{x}] = [x_1] \times \cdots \times [x_n] \subseteq \mathbb{R}^n$
  - $C = \{c_1, \ldots, c_m\}$ with $c_k : \mathbb{R}^n \longrightarrow \{0, 1\}$

$\longrightarrow$ Solution set:

$$\{\mathbf{x} \in [\mathbf{x}] : c_1(\mathbf{x}) \wedge \cdots \wedge c_m(\mathbf{x})\} \subseteq \mathbb{R}^n$$



$c(\mathbf{x}) \equiv (\forall n \in \mathbb{N}, f^{(n)}(\mathbf{x}) \leq 0)$

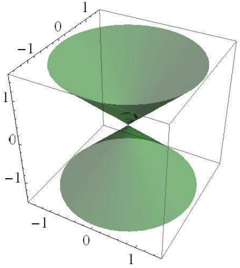
Infinitely complex solution sets

Equality Constraints

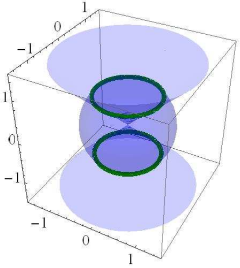
- $c(\mathbf{x}) \equiv f(\mathbf{x}) = 0$ with f compound of usual differentiable elementary functions
- Typically: $\{c_1, \dots, c_m\}$ with $m \leq n$

$\Longrightarrow$ solution set = manifold of dimension $n - m$

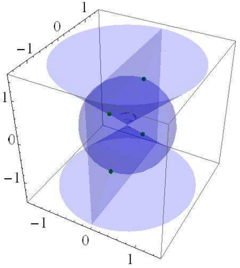
- Focus : $n - m \geq 1$



$f_1(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2$
(dim = 2)



$f_2(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 1$
(dim = 1)



$f_3(\mathbf{x}) = x_1$
(dim = 0)

The Branch and Prune Algorithm

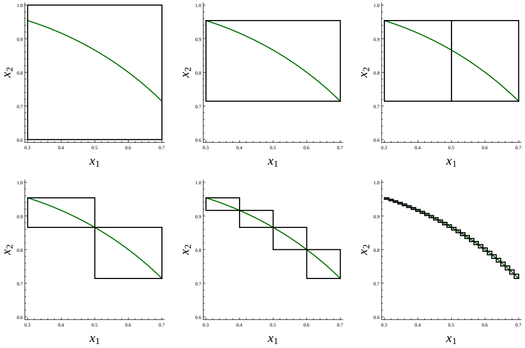
- Input: Set of constraints and one box domain
- Output: List of boxes which contains all solutions
- Algorithm: Succession of **filtering** and **branching**

Example

- $f(\mathbf{x}) = x_1^2 + x_2^2 - 1$
- $[x_1] = [0.3, 0.7]$ and $[x_2] = [0.6, 1]$

$\Longrightarrow$ $n = 2$ and $m = 1$

$\Longrightarrow$ $\dim = n - m = 1$



Locality of Filtering

- Several constraints $\Longrightarrow$ local filtering $\oplus$ propagation
  - Possible slow convergence to a fixed point
  - Fixed point possibly a poor quality enclosure

$\longrightarrow$ Global constraint through *preconditioning* <u>when dim $= 0$</u>

Issues in the Branch and Bound Algorithm

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$\longrightarrow$ Global constraint through *preconditioning* <u>when $\dim = 0$</u>

Existence of Solutions

- Non rational solutions
- Numerical evaluation (using floating points) of the constraints cannot prove existence

$\longrightarrow$ *Existence theorems* (e.g. interval Newton) <u>when $\dim = 0$</u>

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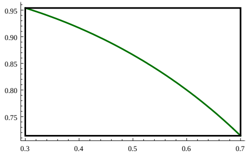
Wrapping Effect

- Solution set not parallel to some axis

$\Longrightarrow$ not efficiently enclosed inside a box

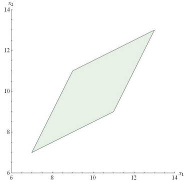
- Not an issue when $\dim = 0$



Parallelepiped domains

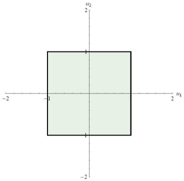
Parallelepipeds

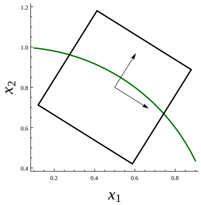
- Image of a box through an affine map
  - $A \in \mathbb{R}^{n \times n}$
  - $[\mathbf{u}] \in \mathbb{IR}^n$
  - $\tilde{\mathbf{x}} \in \mathbb{R}^n$
- $\big(A, [\mathbf{u}], \tilde{\mathbf{x}}\big) = \{A \cdot \mathbf{u} + \tilde{\mathbf{x}} : \mathbf{u} \in [\mathbf{u}]\}$
- Widely used in Interval Analysis:
  - Class of sets larger than boxes
  - Still tractable ($n^2 + 2n$ floating point numbers)



New NCSP Definition

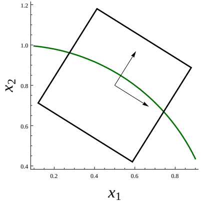
- $\langle \mathbf{x}, (A, [\mathbf{u}], \tilde{\mathbf{x}}), C \rangle$
- Tasks: Define
  - Filtering parallelepiped domains
  - Bisecting parallelepiped domains
  - Updating parallelepiped domains orientations





$$\mathbf{x} \in (A, [\mathbf{u}], \tilde{\mathbf{x}}), f(\mathbf{x}) = 0$$

Filtering Parallelepiped Domains: Decreasing the Wrapping Effect



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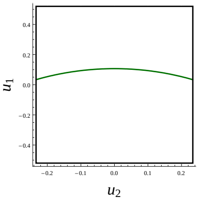
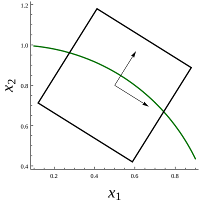


$\mathbf{u} \in [\mathbf{u}]\ ,\ f(A \cdot \mathbf{u} + \tilde{\mathbf{x}}) = 0$



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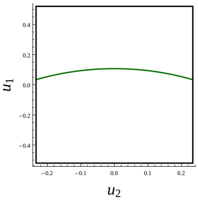

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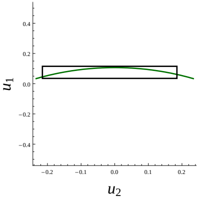
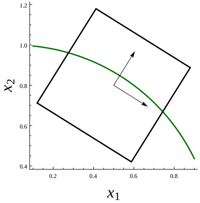


Filtering the Auxiliary NCSP

- Many occurrences of variables: $f(a_{11}u_1 + a_{12}u_2 + \tilde{x}_1, \ldots) = 0$

$\longrightarrow$ Derivative based filtering (interval Newton, or box-consistency)



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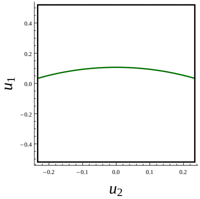

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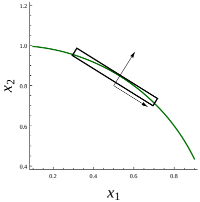

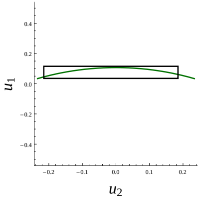

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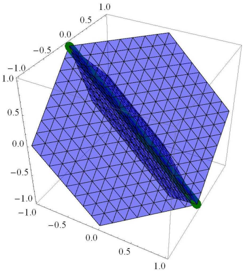
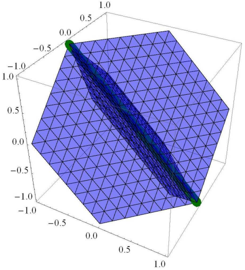
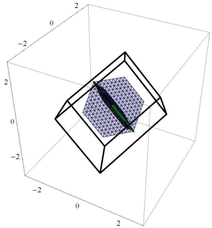


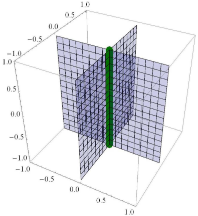
Box Domain

- Each constraint has solutions on every sides

$\Rightarrow$ No possible contraction







Box Domain

- Each constraint has solutions on every sides

⇒ No possible contraction

Parallelepiped Basis

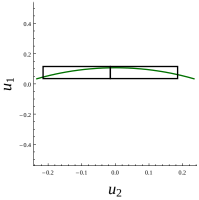
- Constraints are parallel to axes

⇒ Perfect contraction (up to rounding errors)

- **Global constraint** + **no wrapping effect** + **existence of solution proved**
- Nonlinear constraints: asymptotically same behavior

Bisecting Parallelepiped Domains

- $[\mathbf{u}] \rightarrow [\mathbf{u}'])$ and $[\mathbf{u}'']$
  with $[\mathbf{u}] = [\mathbf{u}'] \cup [\mathbf{u}'']$



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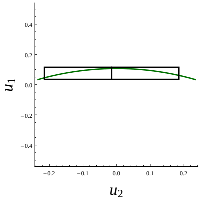

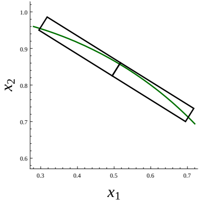

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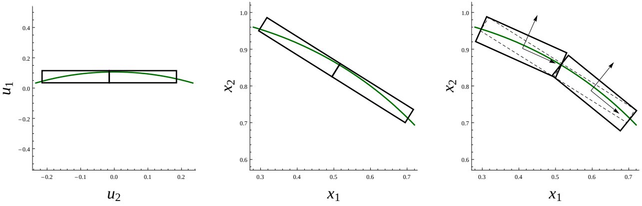
Updating Parallelepiped Domains

- Recenter : $\tilde{\mathbf{x}} = A \cdot \mathrm{mid}[\mathbf{u}]$ and $[\mathbf{u}] = [\mathbf{u}] - \mathrm{mid}[\mathbf{u}]$
- Update A using $D\mathbf{f}(\tilde{\mathbf{x}})$
- (Addition of linear inequalities to prevent overlapping)



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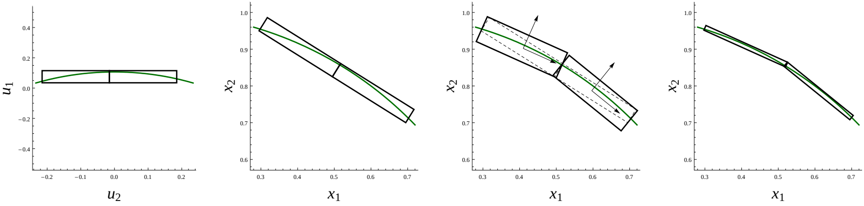

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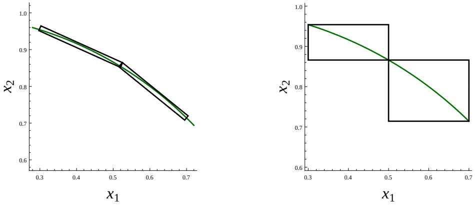
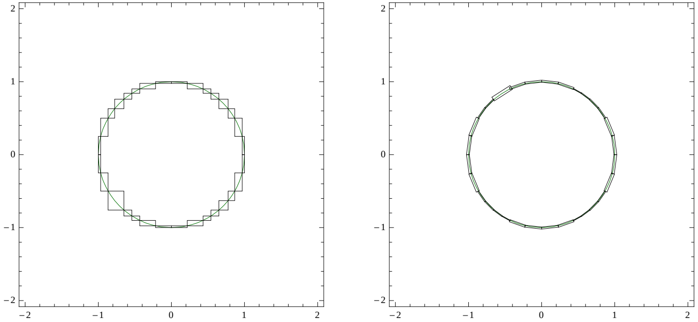


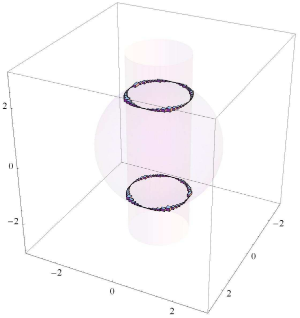
Outline

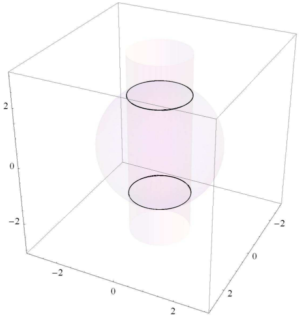
1. NCSPs with Parallelepiped domains
2. Experiments



Surfaces Intersection in 3D

- Intersection of a sphere and a cylinder
- After 100 bisections using box domains and parallelepiped domains
- Parallelepiped domains provide
  - Sharper enclosure
  - Existence proof along all parallelepipeds

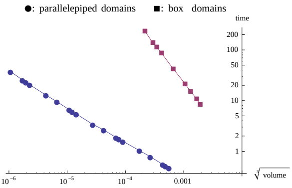


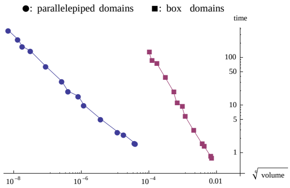


Test Problems from the Literature

- The Layne-Watson Exponential Cosine Curve ($n = 3$, $m = 2 \Rightarrow$ dim $= 1$)
- The Parametrized Broyden Tridiagonal ($n = 5$, $m = 4 \Rightarrow$ dim $= 1$)
- Comparison: Log-log plots of time vs reduced volume[a] of the enclosure

[a] Reduced volume= $\sqrt[n]{\text{volume}}$





Comments

- Parallelepiped domains: at least 100 times quicker to obtain the same volume
- Lines in log-log plots $\Rightarrow$ time increases polynomially w.r.t. (reduced volume)$^{-1}$
- **Parallelepiped domains improve the degree of the polynomial complexity**

Conclusion

Abstract

- Preconditioning: Key global constraint for NCSP *with discrete solution set*
- Contribution: New preconditioning process for under-constrained systems of equations

$\longrightarrow$ Global constraint for under-constrained systems of equations

  - Strong contraction
  - Strongly decreased wrapping effect
  - Proof of Solution Existence

Perspectives

- Current experiments: $dim = m - n = 1$
- Theory ok for $dim > 2$, experiments?
- Applications: Robotics, global optimization, 3D geometric modelers, etc...