Identification of fuzzy regression models

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LISTIC

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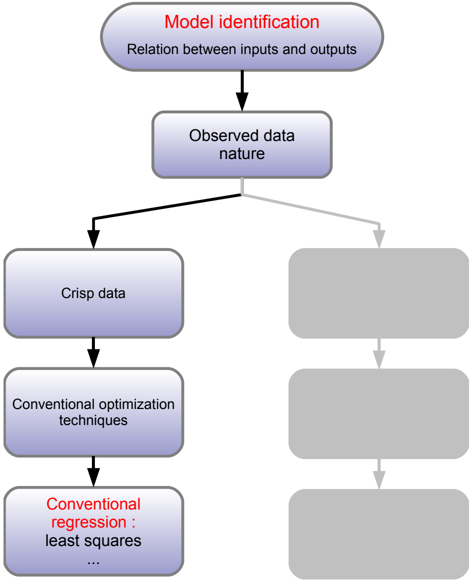
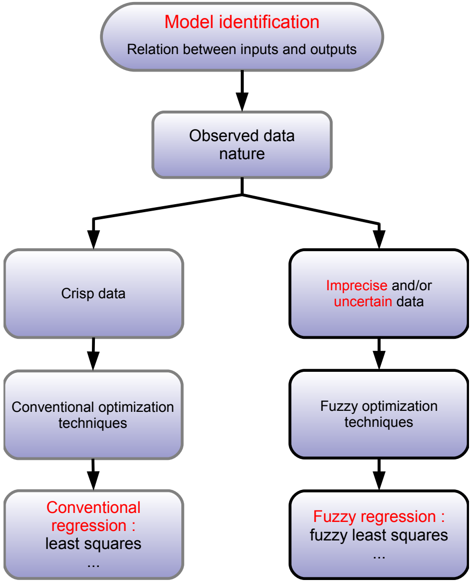
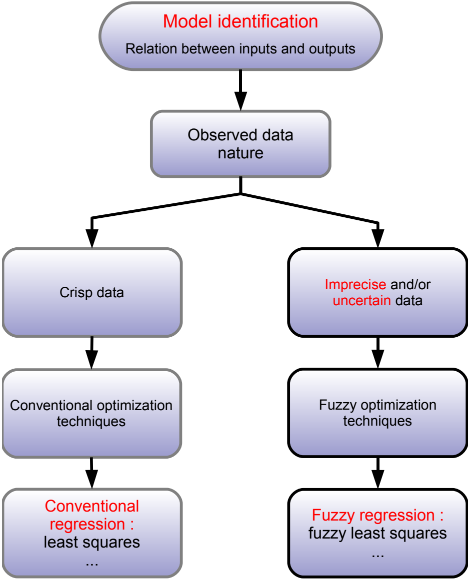
Introduction



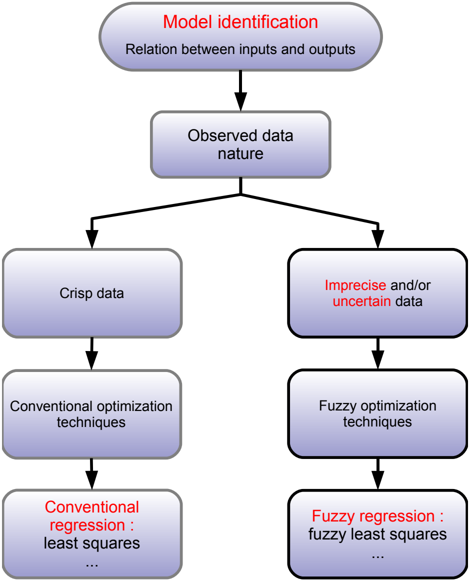

Model identification
Relation between inputs and outputs
Observed data nature
Crisp data
Imprecise and/or uncertain data
Conventional optimization techniques
Fuzzy optimization techniques
Conventional regression : least squares ...
Fuzzy regression : fuzzy least squares ...



General problem

- What kind of fuzzy models have to be used ?
- What kind of data have to be manipulated for the identification process ?

Introduction

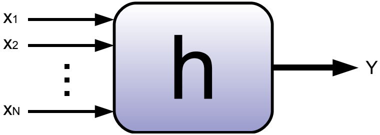


General problem

- What kind of fuzzy models have to be used ?
- What kind of data have to be manipulated for the identification process ?

⇒ How can we identify a fuzzy model on this kind of data ?

Notations and concepts



Considered model : multi-inputs, single-ouput

- Objective : to determine a relationship $Y = h(\mathbf{x})$ between inputs and output
- h considered as linear $\longrightarrow$ model of the form :

$$Y = \sum_{i=1}^{N} A_i \cdot x_i$$

- fuzzy model : fuzzy coefficients A_i

Notations and concepts

Observed data for the identification

Set of M observed data :

The j^{th} data :

- a crisp inputs vector $x_j = (x_{0j}, x_{1j}, ..., x_{Nj})$
- the corresponding fuzzy output Y_j

  $\longrightarrow$ a symmetrical triangular fuzzy number

Remark : choice made for the sake of simplicity

Notations and concepts

Conventional intervals

- Set of elements in $\mathbb{R}$ between a lower and an upper bound

$$a = \{x | a^- \leq x \leq a^+, x \in \mathbb{R}\}$$

- Midpoint and Radius

$$M(a) = M_a = (a^- + a^+)/2$$

$$R(a) = R_a = (a^+ - a^-)/2$$

- calculus on sets

Notations and concepts

Conventional intervals

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- calculus on sets

Conventional intervals : inclusion

- general relationship :

$$a \subseteq b \Leftrightarrow |M(b) - M(a)| \leq R(b) - R(a)$$

Notations and concepts

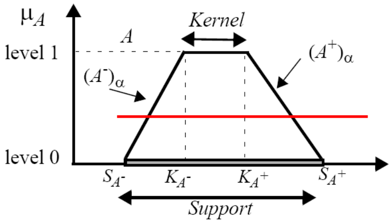
Fuzzy intervals

- Horizontal AND vertical dimension

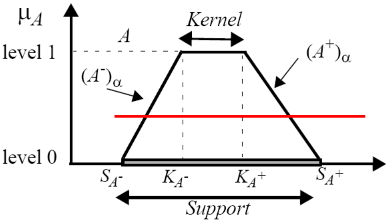
$\Rightarrow$ intervals at two levels :

  - kernel : $K_A = [K_A^-, K_A^+]$
  - support : $S_A = [S_A^-, S_A^+]$

- Kernel included in the Support
- function linking the two levels : profiles
- α-cut $\Rightarrow$ conventional interval



Notations and concepts



Fuzzy intervals : linear profiles

- general case : trapezoidal fuzzy interval

$$A = (K_A, S_A) = ([K_A^-, K_A^+], [S_A^-, S_A^+])$$

- a particular case : symmetrical triangular fuzzy interval

$$A = (K_A, R_A)$$

Notations and concepts

Notations and concepts

- inclusion of two fuzzy intervals : general definition

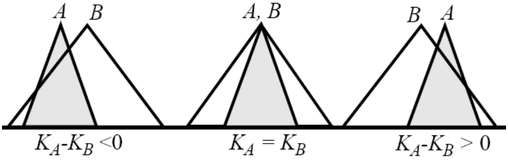
$$A \subseteq B \Leftrightarrow \forall x, \mu_A(x) \leq \mu_B(x)$$

- in the case of trapezoidal fuzzy intervals :

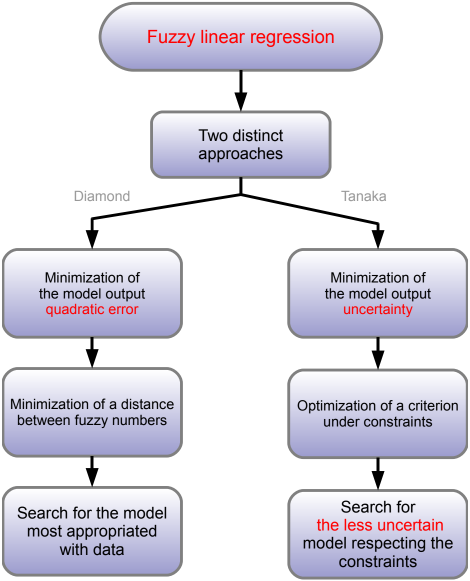
$$A \subseteq B \Leftrightarrow K_A \subseteq K_B \text{ et } S_A \subseteq S_B$$

- in the case of symmetrical triangular fuzzy intervals :

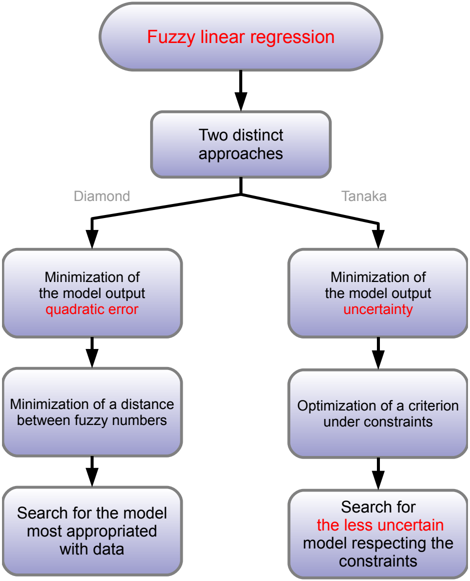
$\Rightarrow$ equality of the Kernel values



Fuzzy linear regression



Fuzzy linear regression

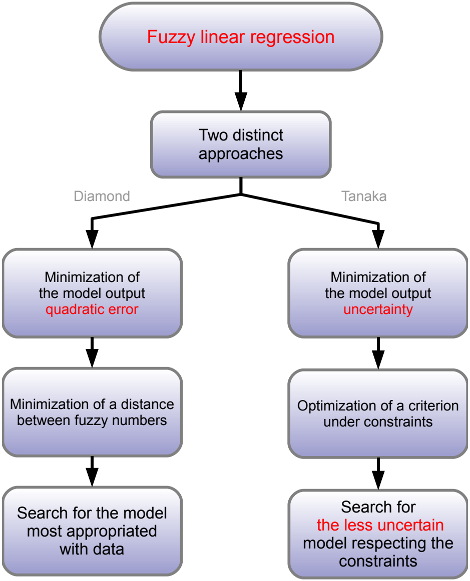


Which constraints ?

Relationship between observed and predicted outputs :

- *necessity* model : predicted outputs included in observed ones
- *conjunction* model : no empty intersection between predicted and observed outputs

Fuzzy linear regression



Which constraints ?

Relationship between observed and predicted outputs :

- *necessity* model : predicted outputs included in observed ones
- *conjunction* model : no empty intersection between predicted and observed outputs
- *possibility* model : observed outputs included in predicted ones

$\Rightarrow$ data total uncertainty is taken in care

Context of the study

- observed outputs : symmetrical triangular fuzzy intervals

  $\Rightarrow$ identified model with triangular parameters
- study leaded for the identification of single input models :

$$\hat{Y}(x) = A_0 \oplus A_1 x$$

with $\oplus$ sum of fuzzy intervals

  $\surd$ *concepts can be extended to the multi inputs case*
- possibility model : observed inputs included in predicted ones

Fuzzy linear regression : possibilistic approach

Basic concepts

- use of α-cuts

  $\Rightarrow$ conventionnal intervals are handled
- minimization of a linear criterion under constraints :

  $\Rightarrow$ criterion : a representation of the model uncertainty

  $\Rightarrow$ constraints : inclusion of conventional intervals at the level α

$\hookrightarrow$ What are the limits of this approach ?

Fuzzy linear regression : possibilistic approach

The use of α-cuts

- inclusion constraints defined for this level α :

$$[Y_j]_\alpha \subseteq [\hat{Y}_j]_\alpha$$

$$\Leftrightarrow \begin{cases} K_{\hat{Y}_j} + (1-\alpha)R_{\hat{Y}_j} \geq K_{Y_j} + (1-\alpha)R_{Y_j} \\ K_{\hat{Y}_j} - (1-\alpha)R_{\hat{Y}_j} \leq K_{Y_j} - (1-\alpha)R_{Y_j} \end{cases}$$

- after the identification : parameters considered as valid $\forall\alpha \in [0,1]$
- BUT : triangular identified model, *equality of the Kernels necessary* for the total inclusion
- Total inclusion $\forall\alpha \in [0,1]$ not guaranteed !

Fuzzy linear regression : possibilistic approach

Conventionnal criteria

- Minimization of the model uncertainty

  $\Rightarrow$ How can it be quantified ?

$\hookrightarrow$ Most used criterion (Tanaka) : sum of the predicted intervals radius

$$Somme = M \cdot R_{A_0} + R_{A_1} \cdot \sum_{j=1}^{M} | x_j |$$

- BUT : minimization at the observed points

$\Rightarrow$ strongly dependant on the learning points : weak robustness

Model representativity

Conventionnal linear model : study of the output variation on the domain

- kernel : $M(\hat{Y}(x)) = K_{\hat{Y}(x)} = K_{A_0} + K_{A_1} \cdot x$

$\Rightarrow$ variation according to the sign of K_{A_1} $\rightarrow$ **any variation**

- radius : $R(\hat{Y}(x)) = R_{\hat{Y}(x)} = R_{A_0} + R_{A_1} \cdot |x|$

$\Rightarrow$ Radius always positive !

$\Rightarrow$ **Variation of the output radius limited by the input sign**

Propositions

Points to improve in the previous method : summary

- Is it possible to consider a model respecting the inclusion $\forall\alpha \in [0, 1]$?
- Is it possible to identify such a model with a more robust criterion ?
- Is it possible to impove the representativity of a fuzzy linear model ?

Propositions

Solution to the inclusion problem

- identification of a trapezoidal fuzzy model

$\longrightarrow$ inclusion constraints at two levels α : $\alpha = 0$ and $\alpha = 1$

- $\alpha = 1 : K_{Y_j} \in [K^-_{\hat{Y}_j}, K^+_{\hat{Y}_j}]$
- $\alpha = 0 : \left[K_{Y_j} - R_{Y_j}, K_{Y_j} + R_{Y_j}\right] \subseteq [S^-_{\hat{Y}_j}, S^+_{\hat{Y}_j}]$

$\longrightarrow$ linear membership function

$\Rightarrow$ inclusion guaranteed $\forall \alpha \in [0, 1]$

Remark : model output for a data j :

$$\hat{Y}_j = ([K^-_{\hat{Y}_j}, K^+_{\hat{Y}_j}], [S^-_{\hat{Y}_j}, S^+_{\hat{Y}_j}])$$

Propositions

A more robust criterion

- Objective : make the criterion independant of data

$\longrightarrow$ definition of a total uncertainty on the domain D

$\longrightarrow$ vertical dimension (trapezes) taken into account

$\Longrightarrow$ Minimization of Volume delimited by the model on D

$$Volume = R(K_{A_0}) + R(S_{A_0}) + (R(K_{A_1}) + R(S_{A_1})) \cdot |M(D)|$$

$\hookrightarrow$ independant of observed inputs x_j

$\hookrightarrow$ still linear criterion

Propositions

A model more representative of the data

- To have any kind of the output radius variation :
  - $\longrightarrow$ the input variable sign must be modified
  - $\longrightarrow$ the linear behavior must be kept

$\Rightarrow$ The origin of the model is set on a bound of $D = [x_{min}, x_{max}]$

$\Rightarrow$ New model defined on the domain D :

$$\hat{Y}(x) = A_0 \oplus A_1(x - shift)$$

Study of the variation of this new model output radius

$$R([S_{\hat{Y}}]) = R([S_{A_0}]) + R([S_{A_1}]) \cdot |x - shift|$$

$\longrightarrow$ $x - shift \geq 0 \ \forall x \in D \Rightarrow$ increasing radius

$\longrightarrow$ $x - shift \leq 0 \ \forall x \in D \Rightarrow$ decreasing radius

output spread variation	$\nearrow$	$\searrow$
Used model	$A_0 \oplus A_1(x - x_{min})$	$A_0 \oplus A_1(x - x_{max})$

Identification method : summary

Main steps

- What is the global tendency of the oberved outputs radius on D ?

$\Rightarrow$ Choice of the appropriated value of shift

- Identification of trapezoidal coefficients :

$\Rightarrow$ Minimization of the criterion **Volume** under the inclusion constraints :

  - at $\alpha = 0$ and $\alpha = 1 \rightarrow$ total inclusion guaranteed

Optimization linear program

$$Min\ R(K_{A_0}) + R(S_{A_0}) + (R(K_{A_1}) + R(S_{A_1})) \cdot |M(D)|$$

$$s.c. \begin{cases} K_{Y_j} \in [K^-_{\hat{Y}_j}, K^+_{\hat{Y}_j}] \\ \left[K_{Y_j} - R_{Y_j}, K_{Y_j} + R_{Y_j}\right] \subseteq [S^-_{\hat{Y}_j}, S^+_{\hat{Y}_j}] \end{cases}$$

Performance indicators

Comparison of two models : it is necessary to use performance indicators

Two approaches :

- indicators about the model uncertainty :
  - **Sum** of the radius
  - **Volume** delimited by the model on D
- indicators about the model fitting with data
  - **Compatibility** between observed and predicted data : inclusion degree
  - **Distance** between observed and predicted data : quadratic error

Extension to piecewise linear models

Global model :

- shifted submodels with trapezoidal coefficients
- each submodel defined on its own domain

⇒ independant submodels for identification and using

Identification method :

- data segmentation :
  - on observed kernels
  - on observed radius

  ⇒ More representative models !
- optimization of the parameters with the criterion **Volume**

⇒ identification of a gobal model with a minimal uncertainty and respecting the total inclusion

Generalization to multi-inputs models

Previous concepts applicable to multi-inputs ($x_i, i = 1, ..., N$)

- Trapezoidal coefficients
- Appropriated shift for each x_i
- Criterion **Volume** can be extended, still linear

$\Rightarrow$ identification of the optimal model for the total ouput uncertainty on the domain

Case of a piecewise model

- Data segmentation : cartesian product
- Identification of submodels

$\Rightarrow$ Independant submodels on each region

Numerical examples : inclusion problem

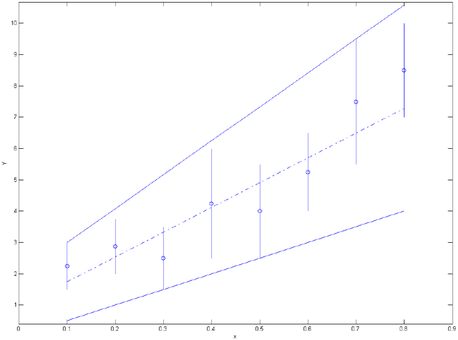
j	x_j	Y_j
1	0.1	(2.25, 0.75)
2	0.2	(2.875, 0.875)
3	0.3	(2.5, 1)
4	0.4	(4.25, 1.75)
5	0.5	(4.0, 1.5)
6	0.6	(5.25, 1.25)
7	0.7	(7.5, 2)
8	0.8	(8.5, 1.5)

- 8 observed data, $D = [0.1, 0.8]$
- crisp inputs
- observed outputs : symmetrical triangular fuzzy numbers
- form of the identified model :

$$\hat{Y}(x) = A_0 \oplus A_1 x$$

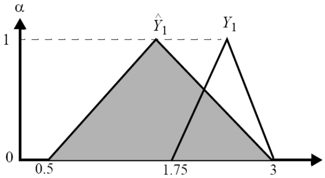
Numerical examples : inclusion problem

Triangular coefficients : identification at $\alpha = 0$



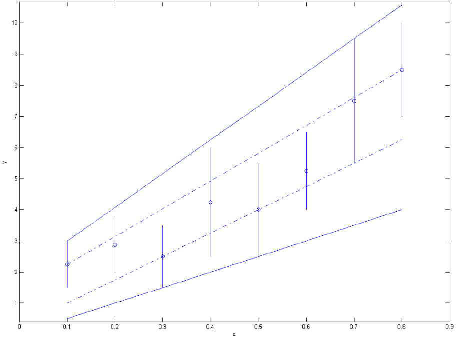
	modèle flou triangulaire
A_0	(0.96, 0.96)
A_1	(7.92, 2.92)
Compatibilité	0.83

study of the inclusion : data $j = 1$



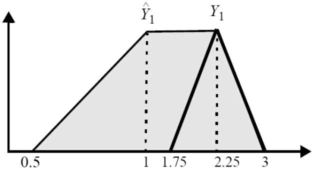
- inclusion is respected at $\alpha = 0$
- inclusion is not guaranteed $\forall \alpha$

Trapezoidal coefficients



	modèle flou trapézoidal
A_0	([0.25 1.36],[0 1.92])
A_1	([7.5 8.93],[5 10.83])
compatibilité	1

study of the inclusion : data $j = 1$



- inclusion is respected $\forall\alpha$!

Numerical examples : criterion robustness

presentation of the data set

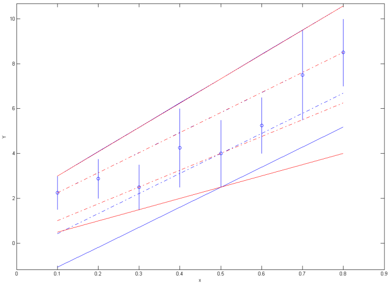
- data $j = 8$ duplicated three times, $D = [0.1, 0.8]$

$\Rightarrow$ study of the influence of data redundancy

- trapezoidal identified model
- 2 minimized criteria : *Sum* and *Volume*

	Somme	*Volume*
A_0	([-0.46 1.36],[-1.96 1.92])	([0.25 1.36],[0 1.92])
A_1	([8.93 8.93],[8.93 10.83])	([7.5 8.93],[5 10.83])
distance	71.05	91.29
somme	37.08	38.42
volume	3.28	3.15

Identified models : *Sum (blue)* and *Volume (red)*



- initial model identified again for *Volume*
- Minimal total uncertainty
- the indicators *distance* and *sum* are not robust !

Numerical examples : output representativity

presentation of the data set

- initial data shifted to have negative inputs

$\Rightarrow$ increasing outputs radius

- trapezoidal identified models, for the minimum *Volume* :
  - conventionnal model : $\hat{Y}(x) = A_0 \oplus A_1 x$
  - shifted model : $\hat{Y}(x) = A_0 \oplus A_1(x - shift)$

j	x_j	Y_j
1	-0.8	(2.25, 0.75)
2	-0.7	(2.875, 0.875)
3	-0.6	(2.5, 1)
4	-0.5	(4.25, 1.75)
5	-0.4	(4.0, 1.5)
6	-0.3	(5.25, 1.25)
7	-0.2	(7.5, 2)
8	-0.1	(8.5, 1.5)

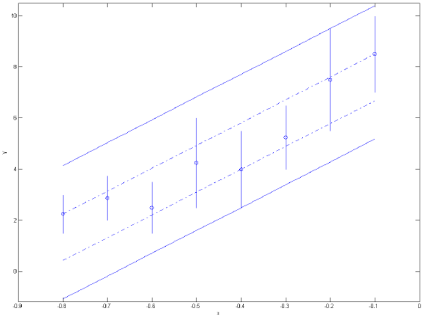
	$\hat{Y}(x) = A_0 \oplus A_1 \cdot x$	$\hat{Y}(x) = A_0 \oplus A_1(x - shift)$
A_0	([7.57 9.39],[0.07 11.29])	([1 2.25],[0.5 3])
A_1	8.93	([7.5 8.93],[5 10.83])
distance	58.83	48.08
somme	28.14	25.17
volume	3.52	3.15

All the indicators are better for the shifted model !

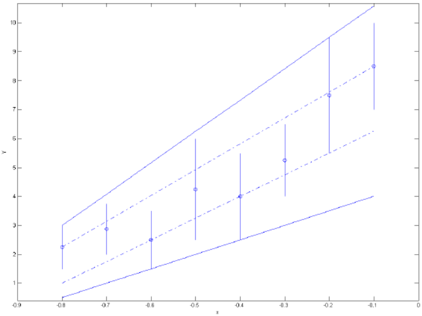
Numerical examples : output representativity

	$\hat{Y}(x) = A_0 \oplus A_1 \cdot x$	$\hat{Y}(x) = A_0 \oplus A_1(x - shift)$
A_0	([7.57 9.39],[0.07 11.29])	([1 2.25],[0.5 3])
A_1	8.93	([7.5 8.93],[5 10.83])
distance	58.83	48.08
somme	28.14	25.17
volume	3.52	3.15

Conventionnal model



Model with shifted input

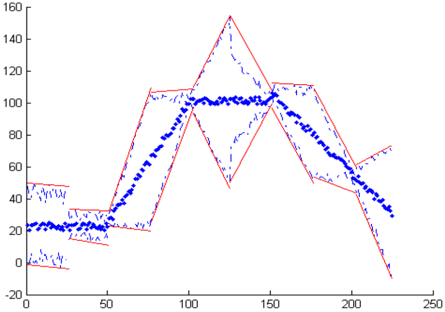


Numerical examples : piecewise model

- Great number of data
- Various tendencies of kernel and radius variations, noise : segmentation

$\Rightarrow$ Identification of shifted trapezoidal multi-inputs submodels

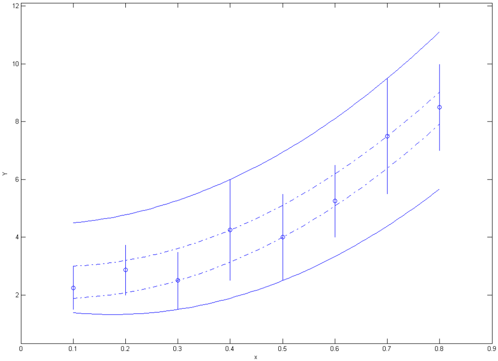
$\Rightarrow$ Identification of the optimal global model for the minimal uncertainty (**Volume**)



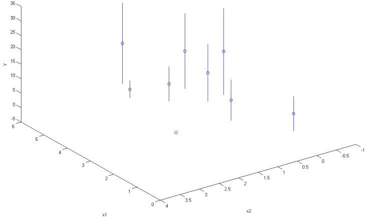
Numerical examples : higher order model

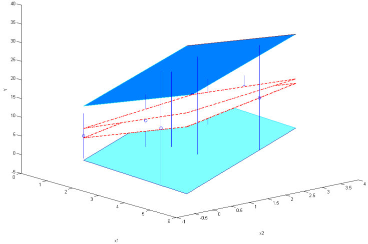
- Initial data set
- 2-order identified model :

$$\hat{Y}(x) = A_0 \oplus A_1 x \oplus A_2 x^2$$



Numerical examples : multi-inputs model





- Two inputs → two appropriated values of shifts :
  - Increasing radius on x_1 and decreasing one on x_2
- Identification of the trapezoidal model for **Volume**

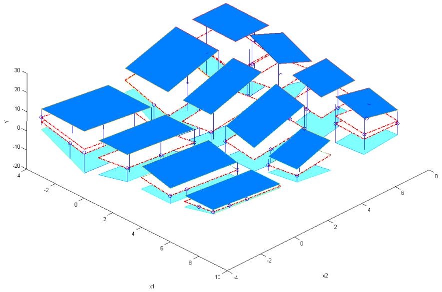
⇒ "Planes" with total inclusion of data

Numerical examples : multi-inputs piecewise model

- Various tendencies of kernel and radius variations : segmentation

⇒ Identification of shifted trapezoidal multi-inputs submodels

⇒ Identification of the optimal global model for the minimal uncertainty (**Volume**)



Conclusion

Propositions

- Identification of trapezoidal models : total inclusion is guaranteed
- Improvement of the model representativity
- Identification for the minimal total uncertainty : increased robustness
- Application of these concepts on piecewise linear regression problems

Conclusion

Future work

- Imprecise observed inputs
  - Uncertainty comes from :
    - fuzzy parameters
    - fuzzy inputs
  - Lack of representativity of conventionnal space (X, Y)

$\Rightarrow$ Representation of the model in the space $(Mid(X), Rad(X), Y)$

$\Rightarrow$ Identification of the model in this space

END

Questions ?...