A New Containment Method For Rigorous Shadowing

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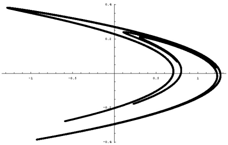
Continuous state discrete time dynamical system

Definitions

- Dynamical system: $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- Initial value: $\mathbf{y}_0 \in \mathbb{R}^n$
- Initial value problem (IVP): estimate the orbit $\mathbf{f}^k(\mathbf{y}_0)$



Aims

- Prove rigorously that the system is chaotic
- We revisit a technique proposed by Stoffer and Palmer in 1999

$\longrightarrow$ Prove that the system contains a full shift on two symbols

Forward and backward error analysis

Pseudo orbits

- δ-pseudo orbit: $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots)$ such that $||\mathbf{y}_{k+1} - \mathbf{f}(\mathbf{y}_k)|| \leq \delta$

Forward error analysis

- Watch the *global error* $||\mathbf{y}_k - \mathbf{f}^k(\mathbf{y}_0)||$
- Chaotic system: $||\mathbf{y}_k - \mathbf{f}^k(\mathbf{y}_0)||$ growth exponentially

$\longrightarrow$ Forward error analysis useless

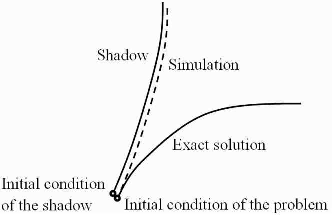
Backward error analysis

- Find a new IVP $(\tilde{\mathbf{f}}, \tilde{\mathbf{y}}_0) \approx (\mathbf{f}, \mathbf{y}_0)$ for which $(\mathbf{y}_k)$ is an accurate approximate solution

Shadowing

Backward error analysis

- Shadowing
  - Keep the dynamical system unchanged
  - Change the initial value



Application

- Shadows are exact trajectories
- Chaotic shadows $\Rightarrow$ chaotic system

Outline

Outline

1. Interval analysis
   - Interval enclosures
   - Existence proof
2. Containment for rigorous shadowing
   - A new containment theorem
   - Parallelepipeds for containment
   - Contracting and expanding directions
3. Applications
   - Plotting strange attractors
   - Embedded shift map
   - Real world applications
4. Conclusion

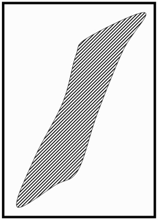
Interval enclosure

From finite to uncountable

- Computations with floating point numbers

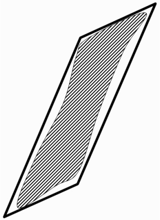
$\longrightarrow$ Rigorous proof of some properties on an uncountable set of reals

- Basic usage
  - Given a set $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n : P(\mathbf{x})\}$
  - Compute an enclosure $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \supseteq \mathbb{X}$ ($\rightarrow 2n$ floating point numbers)



The wrapping effect

- Intervals are too crude to enclose accurately
- Parallelepiped can drastically improve the enclosure



Interval extensions

Definition

- Let $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $[\mathbf{f}] : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$
- $[\mathbf{f}]$ is an interval extension of $\mathbf{f}$ iff $[\mathbf{f}]([\mathbf{x}]) \supseteq \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in [\mathbf{x}]\}$

Computation

- Interval arithmetic: $[\underline{x}, \overline{x}] \circ [\underline{y}, \overline{y}] := \{x \circ y : x \in [\underline{x}, \overline{x}], y \in [\underline{y}, \overline{y}]\}$
- Natural extension: replace real operators by interval operators

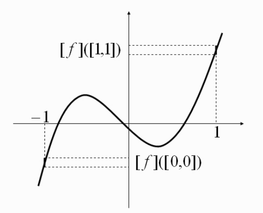
$$[\mathbf{x}]^2 - [\mathbf{x}] + 1 \supseteq \{x^2 - x + 1 : x \in [\mathbf{x}]\}$$

- Mean-value extension: use derivatives to potentially improve the enclosure

Solution of systems of equations

Intermediate value theorem

$$f(-1) \leq 0 \wedge f(1) \geq 0 \implies (\exists x \in [-1, 1])(f(x) = 0)$$



Usage of interval extensions

- $[f]([-1, -1]) \leq 0 \Rightarrow f(-1) \leq 0$
- $[f]([1, 1]) \geq 0 \Rightarrow f(1) \geq 0$

$\rightarrow$ $(\exists x \in [-1, 1])(f(x) = 0)$

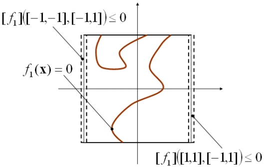
Solution for a system of equations

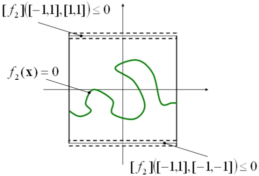
Poincaré-Miranda theorem

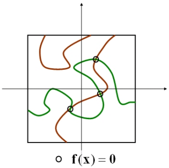
- Poincaré-Miranda theorem ($\approx$ Brouwer fixes point theorem)
- Check signs of function taken on the sides of the boxes

$$\begin{pmatrix} \forall i \in \{1,\dots,n\} \\ \forall \mathbf{x} \in [-1,1]^n \end{pmatrix} \begin{pmatrix} x_i = -1 \Rightarrow f_i(\mathbf{x}) \leq 0 \\ x_i = 1 \Rightarrow f_i(\mathbf{x}) \geq 0 \end{pmatrix}$$

$$\Longrightarrow (\exists \mathbf{x} \in [-1,1]^n)(\mathbf{f}(\mathbf{x}) = \mathbf{0})$$

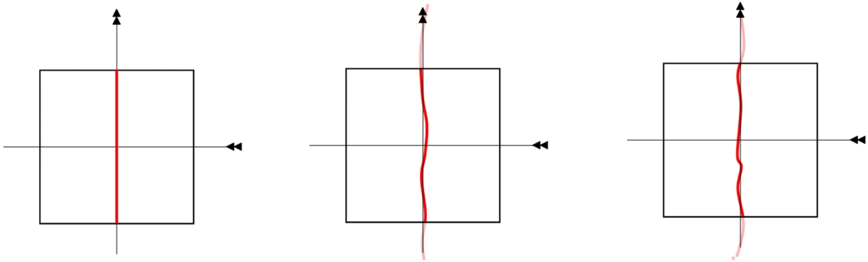






Outline

The idea of inductive containment



Propagation of one dimensional path

- Choose a line in the expanding direction
- Each application of the map stretches the line vertically
- We keep only the part that remains in the next box

A new containment theorem

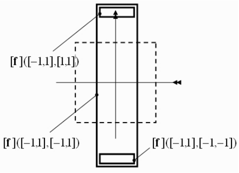
Simplified Inductive Containment Property (ICP)

- Pseudo trajectory around **0**
- Axis aligned stable and unstable directions
- More general situation $\rightarrow$ change of basis

Definition: $(\mathcal{S},\mathcal{U})$-ICP

Let $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and $\mathcal{S}$ and $\mathcal{U}$ be a partition of $\{1, \ldots, n\}$. Then, $\mathbf{f}$ satisfies the $(\mathcal{S},\mathcal{U})$-ICP iff

- $\forall j \in \mathcal{U}$ and for all $\mathbf{x} \in [-1, 1]^n$
  - $x_j = -1 \implies f_j(\mathbf{x}) < -1$
  - $x_j = 1 \implies f_j(\mathbf{x}) > 1.$
- $\forall j \in \mathcal{S}$, $\mathbf{x} \in [-1, 1]^n \implies -1 < f_j(\mathbf{x}) < 1.$



A new containment theorem

Containment Theorem (CT)

Let $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i \in \{1, \dots, m-1\}$, be some continuous maps satisfying the $(\mathcal{S},\mathcal{U})$-ICP
Then there exists an exact orbit $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m)$ such that $\mathbf{x}_i \in [-1, 1]^n$

proof

- $\mathbf{x}_1 = \mathbf{f}_0(\mathbf{x}_0) \wedge \mathbf{x}_2 = \mathbf{f}_1(\mathbf{x}_1) \wedge \cdots \wedge \mathbf{x}_m = \mathbf{f}_{m-1}(\mathbf{x}_{m-1})$

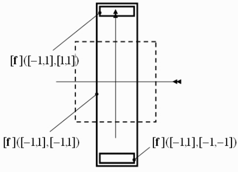
$\rightarrow$ $(m-1)n$ equations, mn unknowns

- Fix $x_{0,k}$ for $k \in \mathcal{U}$ and $x_{n,k}$ for $k \in \mathcal{S}$ (*E.g. in dimension* 2 *where we consider a line in the expanding direction*)
- Use ICP to check the hypothesis of Poincaré-Miranda theorem inside $[-1, 1]^n$

Example

System

- $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0.5x_1 + 0.1\cos(10(x_1^2 + x_2^2)) \\ 2x_2 + 0.1\sin(10(x_1^2 + x_2^2)) \end{pmatrix}$



Interval evaluations

- Interval evaluations:

$$\begin{aligned} [\mathbf{f}]([-1,1],[-1,1]) &= ([-0.6,0.6],[-2.1,2.1]) \\ [\mathbf{f}]([-1,1],[-1,-1]) &= ([-0.6,0.6],[-2.1,-1.9]) \\ [\mathbf{f}]([-1,1],[1,1]) &= ([-0.6,0.6],[1.9,2.1]) \end{aligned}$$

- Containment Theorem: $\forall m \in \mathbb{N}$, there exists an orbit $(\mathbf{x}_i)_{0\le i\le m}$ with $\mathbf{x}_i \in [-1,1]^2$

General pseudo trajectory

- Pseudo orbit: $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$
- Stable and unstable directions: $A_0, A_1, \ldots, A_m$ ($A_{:j}$ is a stable direction if $j \in \mathcal{S}$; $A_{:j}$ is unstable if $k \in \mathcal{U}$)

$$\begin{tikzcd} \mathbf{x} \arrow[r, "\mathbf{f}_k"] & \mathbf{y} \\ \mathbf{u} \arrow[u, "A_k \cdot \mathbf{u} + \mathbf{x}_k"] \arrow[r, "\tilde{\mathbf{f}}_k"'] & \mathbf{v} \arrow[u, "A_{k+1} \cdot \mathbf{v} + \mathbf{x}_{k+1}"'] \end{tikzcd}$$

$$\tilde{\mathbf{f}}_k = A_{k+1}^{-1} \cdot \Big(\mathbf{f}\big(A_k \cdot \mathbf{u} + \mathbf{x}_k\big) - \mathbf{x}_{k+1}\Big)$$

- $\tilde{\mathbf{f}}_k$ can satisfy the ICP

Computation of expanding and contracting directions

Goal: Find A_{k+1} such that

- A_{k+1} close to $J_k \cdot A_k$
- $||J_k \cdot (A_k)_{:j}|| < ||(A_k)_{:j}||$ for $j \in \mathcal{S}$
- $||J_k \cdot (A_k)_{:j}|| > ||(A_k)_{:j}||$ for $j \in \mathcal{U}$
- A_k have good condition numbers

Problems

- We can't just compute $A_{k+1} = J_k \cdot A_k$ because all directions collapse on the most contracting

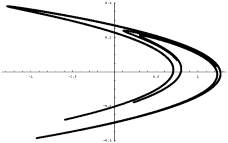
$\longrightarrow$ Compute A_{k+1} using an Gram-Schmidt orthogonalization on $J_k \cdot A_k$

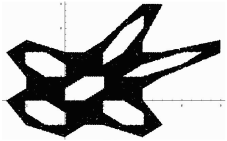
Outline

Plotting strange attractors

Problem

- We think we know Henon's attractor $\rightarrow$ self contradiction!
- The plot uses double precision and several thousands of steps
- Strange attractor $\Rightarrow$ chaos $\Rightarrow$ exponential divergence $\Rightarrow$ plot completely false!





Solution: long term shadowing

- Henon and Gingerbreadman pseudo-orbits $\rightarrow$ shadowed for millions of steps
- The plotted attractor is accurate for a close initial condition $\Rightarrow$ good representation of the attractor

δ-pseudo periodic orbit

Infinite length shadow

- δ-pseudo periodic orbit: $||\mathbf{x}_0 - \mathbf{f}(\mathbf{x}_m)|| \leq \delta$

$\longrightarrow$ Infinite length δ-pseudo orbit:

$$(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_0, \mathbf{x}_1, \ldots)$$

- Containment rigorously proved for $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_0)$ also valid for the infinite length pseudo orbit
- Note that the shadow may not be periodic!

Branching δ-pseudo periodic orbits

Following Stoffer and Palmer 1999

- Two δ-pseudo periodic orbit: $||\mathbf{x}_0 - \mathbf{f}(\mathbf{x}_m)|| \leq \delta$ and $||\mathbf{y}_0 - \mathbf{f}(\mathbf{y}_m)|| \leq \delta$ such that $\mathbf{x}_0 \approx \mathbf{y}_0$
- Containment rigorously proved for $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_0)$ and $(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$

$\longrightarrow$ also valid for all infinite length pseudo orbits

$$(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{x}_0, \mathbf{x}_1, \ldots)$$

- Any sequence of symbols $XXXYXYYYXY \cdots$ gives rise to a specific exact orbit
- Applying $\mathbf{f}^m$ is a shift on these symbols
- In particular: P_n (number of periodic orbits of length n) grows exponentially with n

Real world applications

ODE

- Validation of galaxy simulation (Wayne Hayes) **initial condition chosen randomly**
- Non rigorous Fixed motion shadowing
  - Give some confidence in simulation
  - Explain why simulations look so close to cosmos observations

PDE

- Validation of fluid mixing: Monte-Carlo simulations show mixing quality **initial condition chosen randomly**
- Shadowing $\rightarrow$ validation of simulations
  - ODE: $\partial_t \mathbf{u}(t, \mathbf{x}) = \mathbf{f}(\mathbf{u})$
  - PDE: **f** defined as the solution of a PDE

Conclusion

New containment method

- More simple framework (new proof of the containment theorem using Poincaré-Miranda theorem), more simple algorithm
- Very efficient

Forthcoming work

- Attack more dynamical systems, including continuous time

Discussion

- Possible application in automatic?