Combining interval analysis and nonlinear predictive control to compute capture tubes

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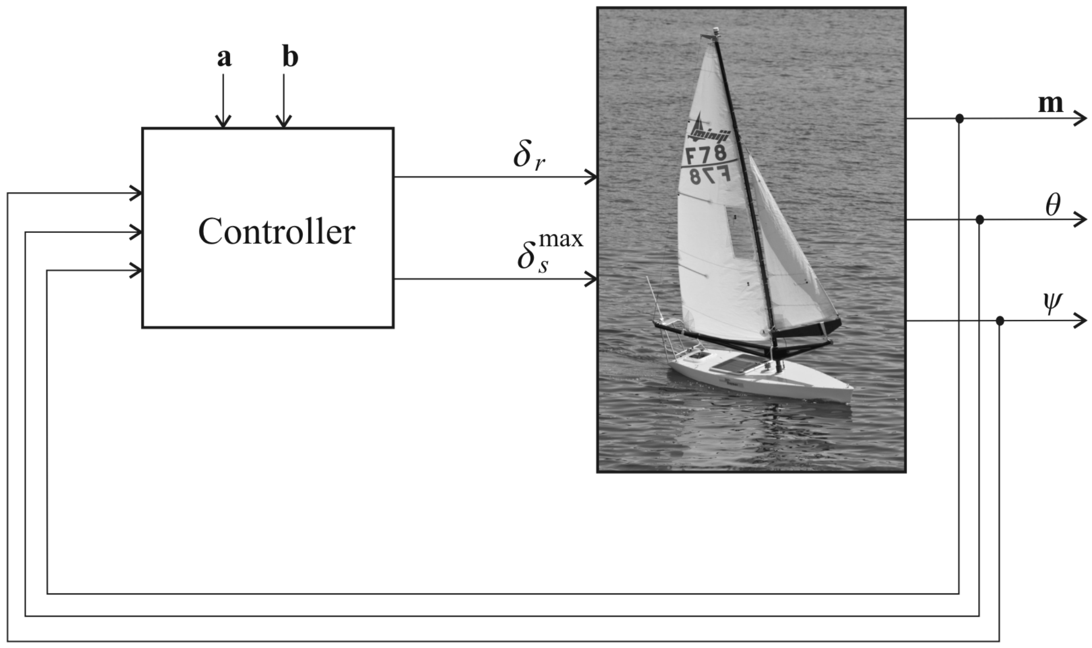
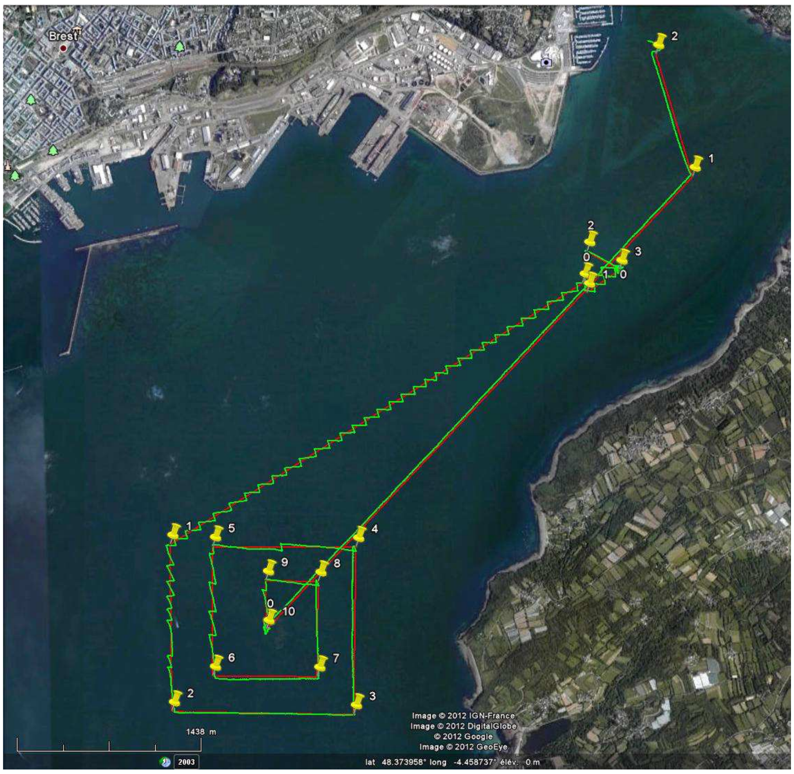
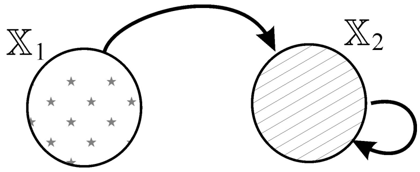
1 V-stability

Vaimos (IFREMER and ENSTA)


a
b
Controller
δ_r
$\delta_s^{\max}$
F78
m
θ
ψ



Brest
1438 m
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
lat 48.373958° long -4.458737° élév 0 m



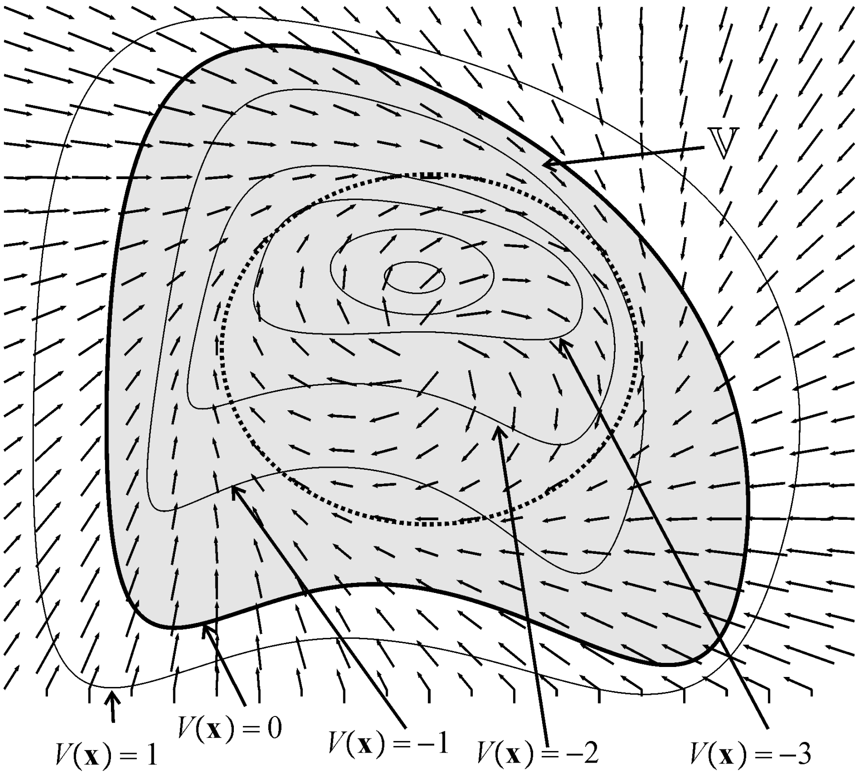
$\mathbb{X}_1$: outside the corridor.

$\mathbb{X}_2$: inside the corridor.

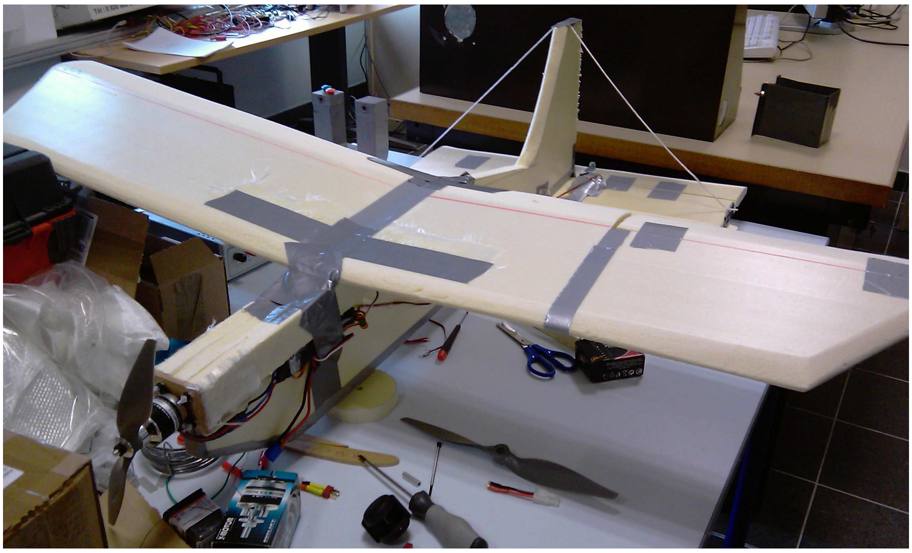
Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right).$$

Checking the V-stability can be done using using interval analysis.



$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$



Système non-holonomes

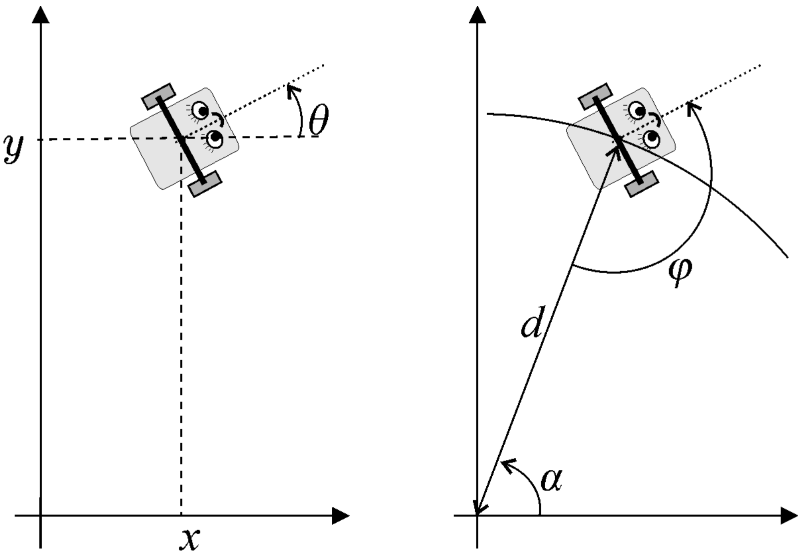
2 Station keeping problem

The problem of *station keeping* for a robot is to stay inside a disk around origin.

Consider a non holonomous robot described by

$$\begin{cases} \dot{x} &= \cos\theta \\ \dot{y} &= \sin\theta \\ \dot{\theta} &= u. \end{cases}$$

Since $\dot{x}^2 + \dot{y}^2 = 1$, this robot cannot stop.



Transformation from Cartesian to polar

The polar form for the state equations is

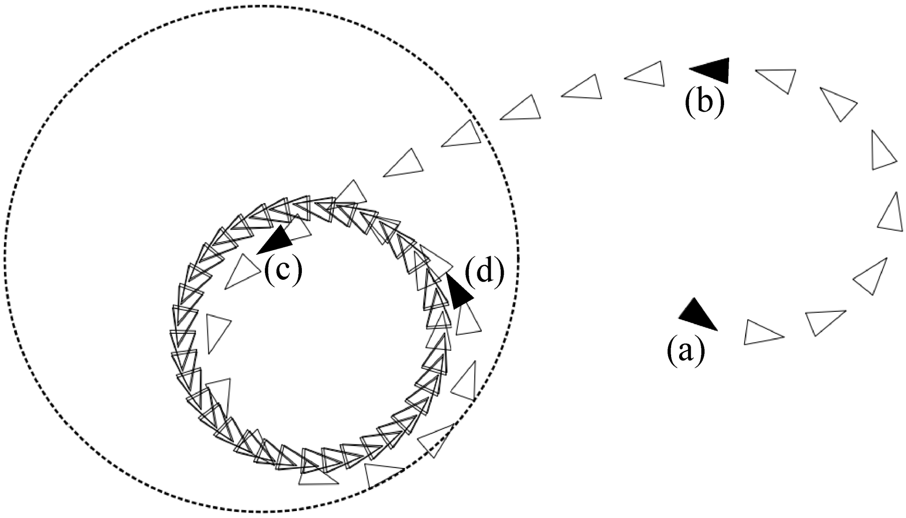
$$\begin{cases} \text{(i)} & \dot{\varphi} &= \frac{\sin\varphi}{d} + u \\ \text{(ii)} & \dot{d} &= -\cos\varphi. \\ \text{(iii)} & \dot{\alpha} &= -\frac{\sin\varphi}{d}. \end{cases}$$

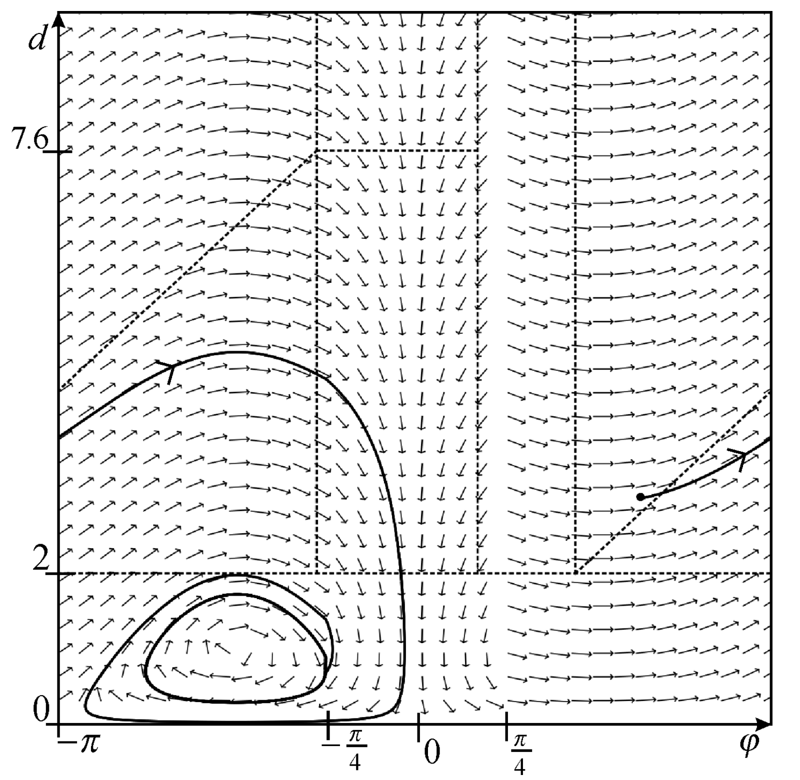
We propose here the following control

$$u = \begin{cases} +1 & \text{if } \cos\varphi \leq \frac{1}{\sqrt{2}} \quad \text{(the robot turns left)} \\ -\sin\varphi & \text{otherwise} \quad \text{(the robot goes toward zero)} \end{cases}$$

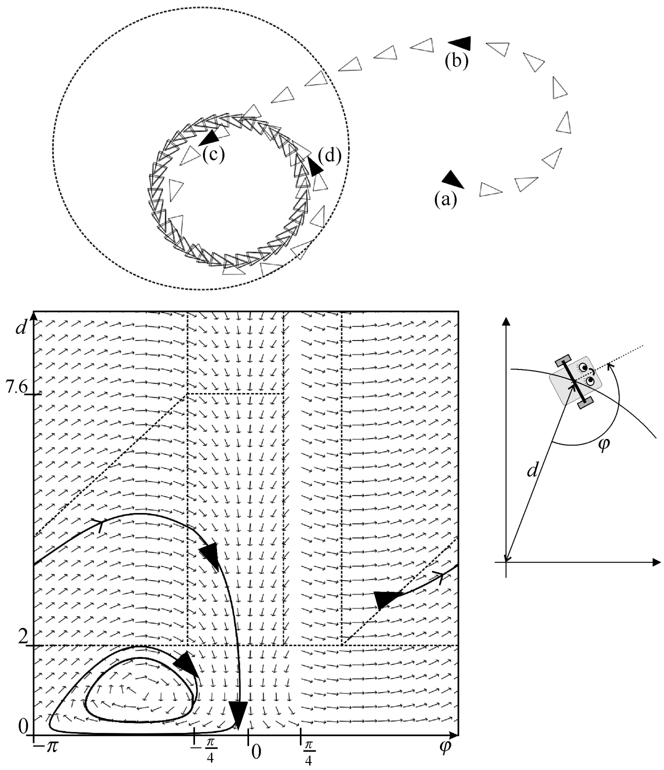
The closed loop state equations are

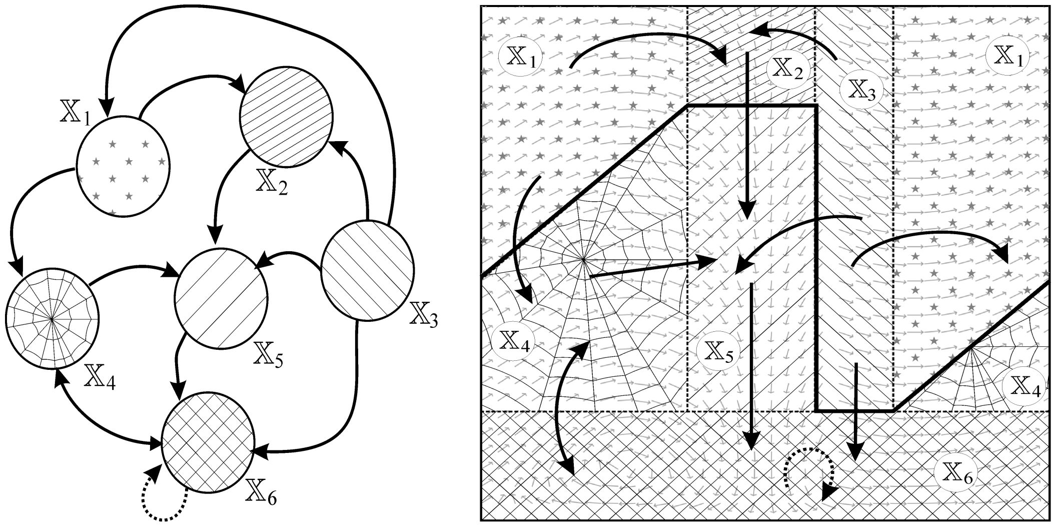
$$\begin{cases} \text{(i)} & \dot{\varphi} & = \begin{cases} \frac{\sin\varphi}{d} + 1 & \text{if } \cos\varphi \leq \frac{1}{\sqrt{2}} \\ \left(\frac{1}{d} - 1\right)\sin\varphi & \text{otherwise} \end{cases} \\ \text{(ii)} & \dot{d} & = -\cos\varphi. \end{cases}$$


(a)
(b)
(c)
(d)



d
7.6
2
0
$-\pi$
$-\frac{\pi}{4}$
0
$\frac{\pi}{4}$
φ


(a)
(b)
(c)
(d)
d
7.6
2
0
−π
−π/4
0
π/4
φ
d
φ


$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_1$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_4$
$\mathbb{X}_6$

3 Capture tubes

Consider the time dependant system

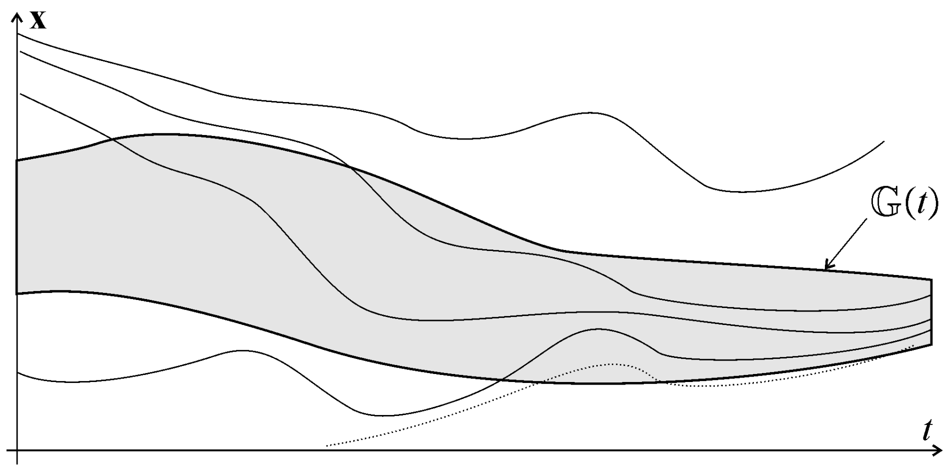
$$\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t\right)$$

and a *tube*

$$\mathbb{G}(t) \subset \mathbb{R}^n, t \in \mathbb{R}.$$

The tube $\mathbb{G}(t)$ is said to be a *capture tube* if

$$\mathbf{x}(t) \in \mathbb{G}(t), \tau > 0 \Rightarrow \mathbf{x}(t+\tau) \in \mathbb{G}(t+\tau).$$



Theorem. Consider the tube

$$\mathbb{G}\left(t\right)=\left\{\mathbf{x},\mathbf{g}\left(\mathbf{x},t\right)\leq\mathbf{0}\right\}$$

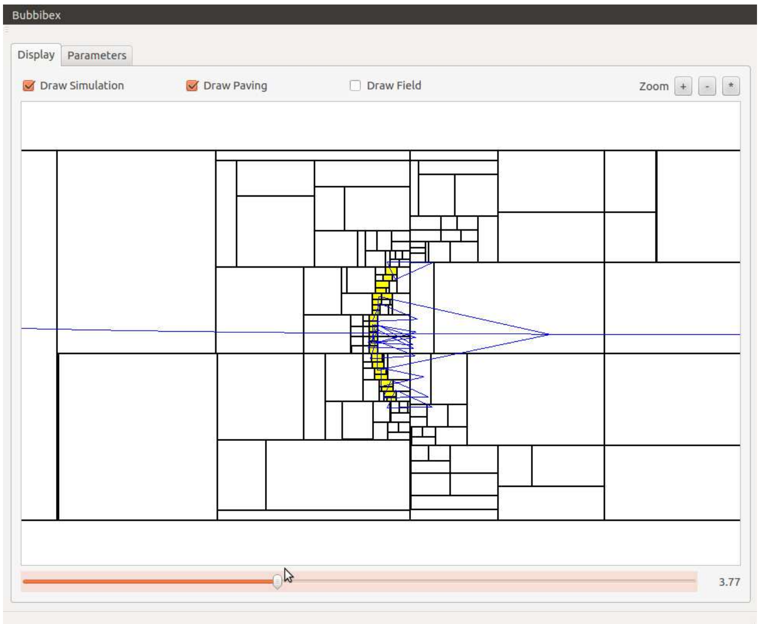
where $\mathbf{g}:\mathbb{R}^{n}\times\mathbb{R}\rightarrow\mathbb{R}^{m}$. If the system

$$\begin{cases}\text{(i)} & \underbrace{\dfrac{\partial g_i}{\partial\mathbf{x}}\left(\mathbf{x},t\right).\mathbf{f}(\mathbf{x},t)+\dfrac{\partial g_i}{\partial t}\left(\mathbf{x},t\right)}_{\dot{g}_i(\mathbf{x},t)}\geq 0\\ \text{(ii)} & g_i\left(\mathbf{x},t\right)=0\\ \text{(iii)} & \mathbf{g}\left(\mathbf{x},t\right)\leq 0\end{cases}$$

is inconsistent for all $(\mathbf{x},t,i)$, then $\mathbb{G}\left(t\right)$ is a capture tube for $\mathcal{S}:\dot{\mathbf{x}}=\mathbf{f}\left(\mathbf{x},t\right)$

A software Bubbibex has been built by students from ENSTA Bretagne.

Bubbibex uses interval analysis to prove the inconsistency.



4 Test-case

Robot

$$\begin{cases} \dot{x} &=& u_1 \\ \dot{y} &=& u_2 \\ \dot{\theta} &=& -\theta. \end{cases}$$

Target $(x_d, y_d) = (t, 0)$. We choose the control

$$u_1 = -x + t,\ u_2 = -y.$$

Remark. We have

$$\dot{x} = -x + t$$

i.e.

$$x(t) = e^{-t}(x_0 + 1) + t - 1$$

The error on x is

$$e_x(t) = e^{-t}(x_0 + 1) - 1$$

The closed loop system satisfies

$$\begin{cases} \dot{x} & = & -x + t \\ \dot{y} & = & -y \\ \dot{\theta} & = & -\theta. \end{cases}$$

Target tube. The tube we want is

$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\},$$

with

$$\begin{cases} g_1(\mathbf{x}, t) &= (x_1 - t)^2 + x_2^2 - r^2 \\ g_2(\mathbf{x}, t) &= (\cos x_3 - 1)^2 + \sin^2 x_3 - 0.2. \end{cases}$$

For $r = 4$, Bubbibex proves that $\mathbb{G}(t)$ is a capture tube.

For $r < 1$, some trajectories leave $\mathbb{G}(t)$ forever.

5 Lattice and capture tubes

Consider $\mathcal{S}:\dot{\mathbf{x}}=\mathbf{f}\left(\mathbf{x},t\right)$.

If $\mathbb{T}$ is the set of tubes and $\mathbb{T}_c$ is the set of all capture tubes of $\mathcal{S}$ then $(\mathbb{T}_c,\subset)$ is a sublattice of $(\mathbb{T},\subset)$.

We have indeed

$$\left\{\begin{array}{lcl}\mathbb{G}_1(t) & \in & \mathbb{T}_c \\ \mathbb{G}_2(t) & \in & \mathbb{T}_c\end{array}\right. \Rightarrow \left\{\begin{array}{lcl}\mathbb{G}_1(t)\cap\mathbb{G}_2(t) & \in & \mathbb{T}_c \\ \mathbb{G}_1(t)\cup\mathbb{G}_2(t) & \in & \mathbb{T}_c\end{array}\right.$$

Remark. If $\mathbb{G}(t) \in \mathbb{T}$, define

$$\operatorname{capt}\left(\mathbb{G}(t)\right) = \bigcap \left\{ \overline{\mathbb{G}}(t) \in \mathbb{T}_c \mid \mathbb{G}(t) \subset \overline{\mathbb{G}}(t) \right\}.$$

This set corresponds to the smallest capture tube which encloses $\mathbb{G}(t)$.

6 Computing capture tubes

Problem. Given $\mathbb{G}(t) \in \mathbb{T}$, compute an interval $\left[\mathbb{G}^{-}(t), \mathbb{G}^{+}(t)\right] \in \mathbb{IT}$ such that

$$\mathrm{capt}\left(\mathbb{G}(t)\right) \in \left[\mathbb{G}^{-}(t), \mathbb{G}^{+}(t)\right].$$

Proposition 1. We have

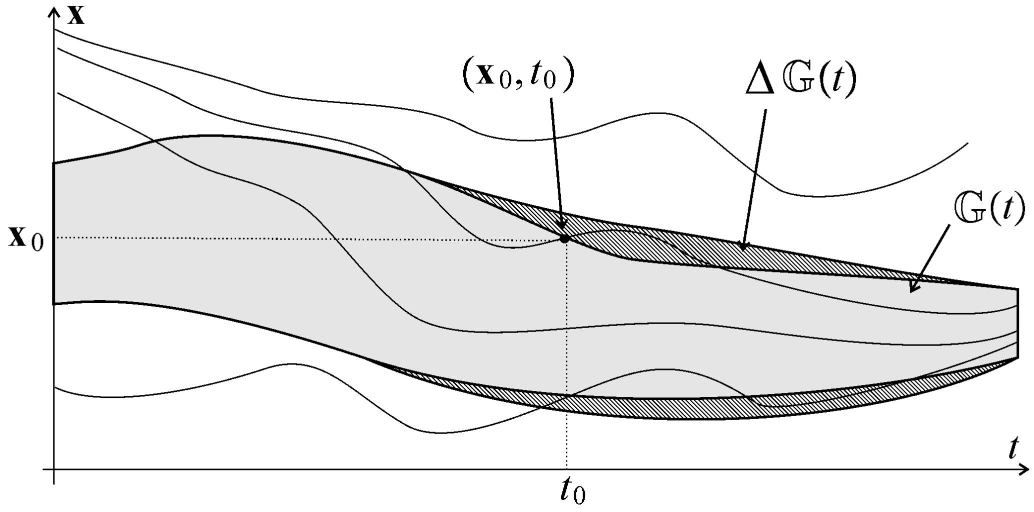
$$\mathsf{capt}\left(\mathbb{G}(t)\right) = \; \{(\mathbf{x},t) \;\mid\; \exists\,(\mathbf{x}_0,t_0)\,,\; \mathbf{x}_0 \in \mathbb{G}(t_0) \\ t \geq t_0,\; \mathbf{x} = \boldsymbol{\phi}_{t-t_0}(\mathbf{x}_0,t_0) \;\}.$$

Proposition 2. We have

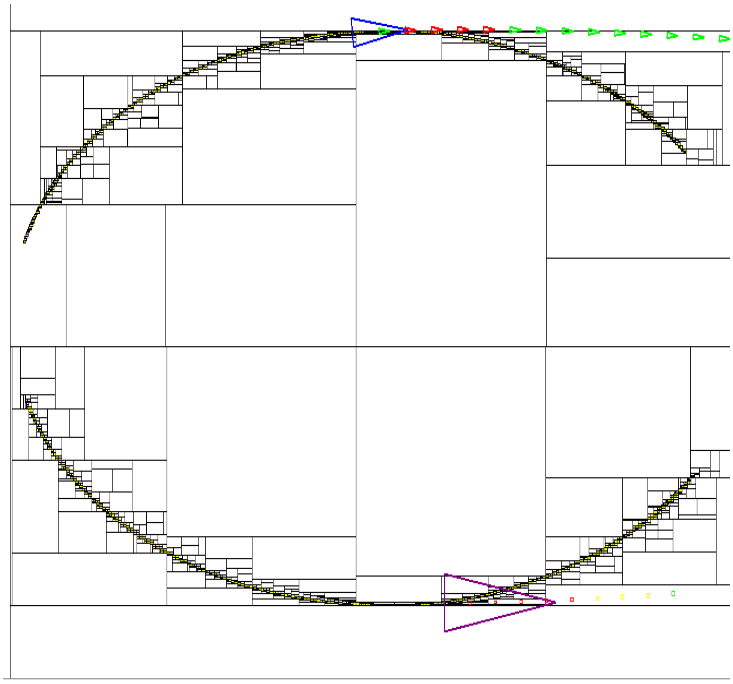
$$\text{capt}\,(\mathbb{G}(t)) = \mathbb{G}(t) \cup \mathbf{\Delta}\mathbb{G}(t),$$

with

$$\begin{aligned} \mathbf{\Delta}\mathbb{G}(t) = \ \{(\mathbf{x}, t) \quad | \quad & \exists\,(\mathbf{x}_0, t_0)\,,\ \mathbf{x}_0 \in \partial\mathbb{G}(t_0), \\ & t \geq t_0,\ \mathbf{x} = \boldsymbol{\phi}_{t-t_0}\,(\mathbf{x}_0, t_0) \\ & \boldsymbol{\phi}_{]t_0,t]}\,(\mathbf{x}_0, t_0) \notin \mathbb{G}(t) \qquad\qquad \} \end{aligned}$$



$\mathbf{x}$
$(\mathbf{x}_0, t_0)$
$\Delta\,\mathbb{G}(t)$
$\mathbb{G}(t)$
$\mathbf{x}_0$
t
t_0



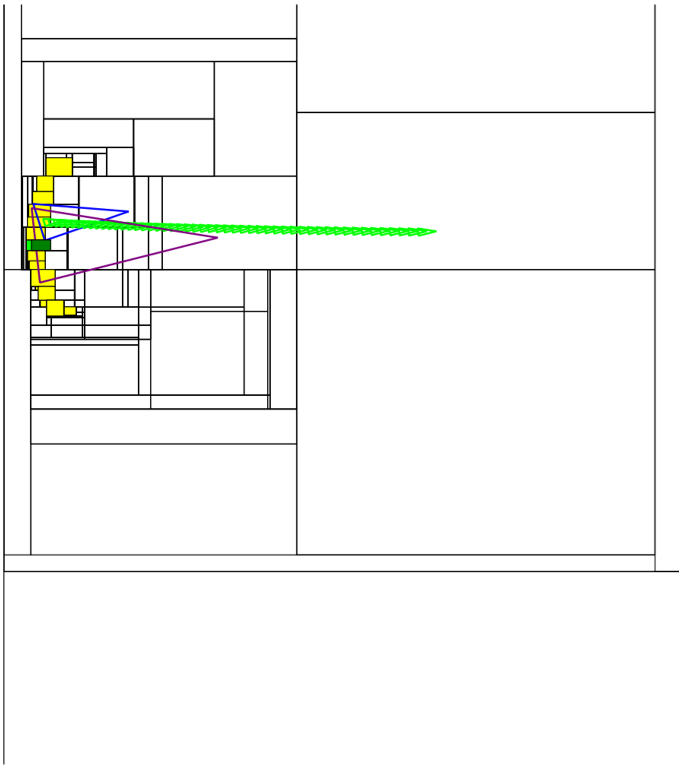
7 Test case

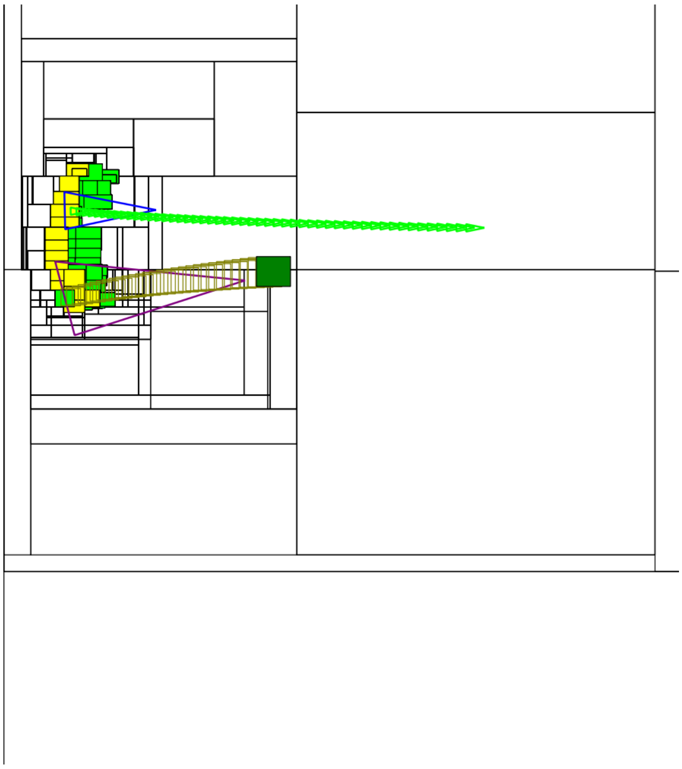
$$\mathcal{S} : \begin{cases} \dot{x} &=& -x + t \\ \dot{y} &=& -y \\ \dot{\theta} &=& -\theta. \end{cases}$$

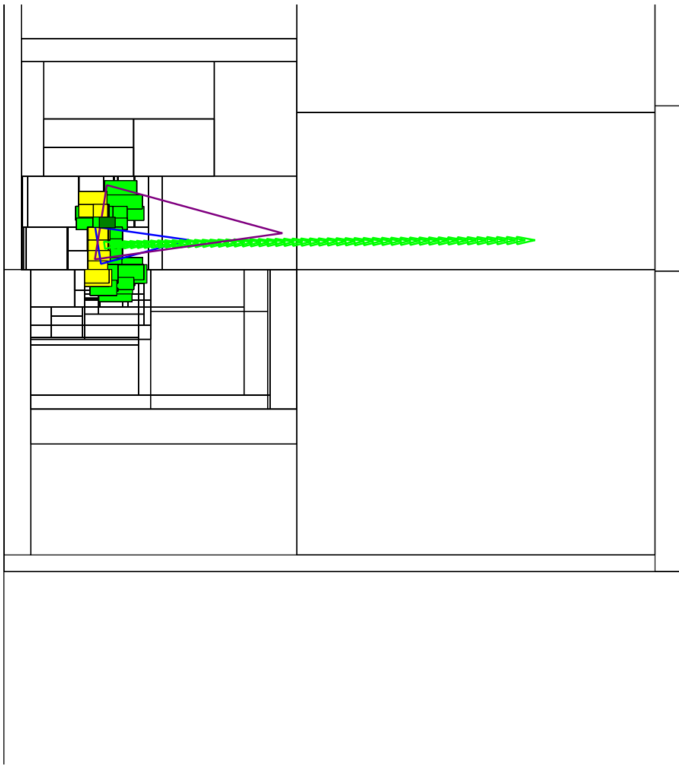
$$\mathbb{G}(t) : \begin{cases} g_1(\mathbf{x}, t) &=& (x_1 - t)^2 + x_2^2 - r(t) \\ g_2(\mathbf{x}, t) &=& (\cos x_3 - 1)^2 + \sin^2 x_3 - 0.2. \end{cases}$$

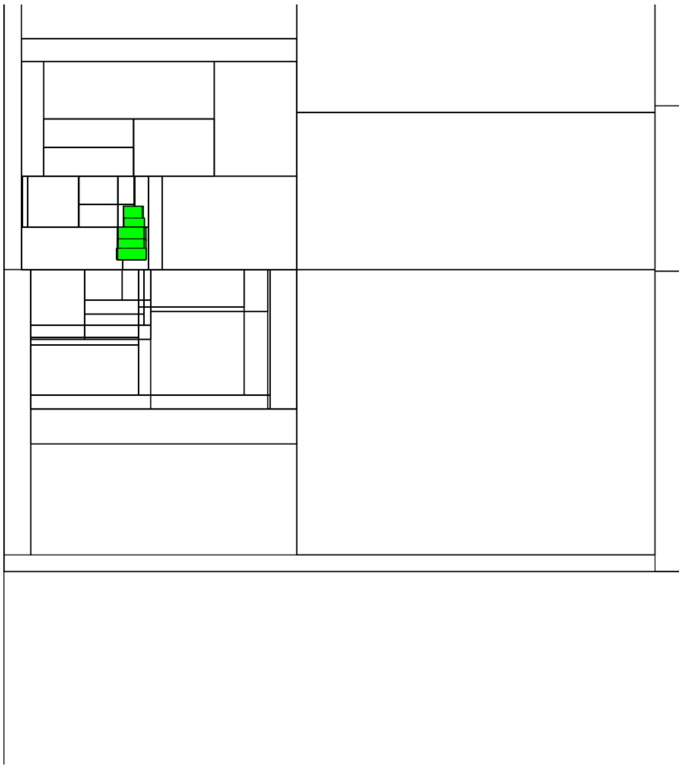
$$r(t) = 0.2 \cdot (t + 1)^2 .$$

Some trajectories leave and come back to $\mathbb{G}(t)$.









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