Interval methods: a tutorial

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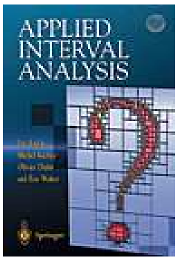


1 Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}]\,,\, f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= [1,8], \\ [-1,3]\cdot[2,5] &= [-5,15], \\ \operatorname{abs}([-7,1]) &= [0,7] \end{aligned}$$

The interval extension of

$$\begin{aligned} f(x_1, x_2) &= x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 \\ &\quad + \sin x_1 \cdot \sin x_2 + 2 \end{aligned}$$

is

$$\begin{aligned} [f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ &\quad + \sin [x_1] \cdot \sin [x_2] + 2. \end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0$$

2 Computing with sets

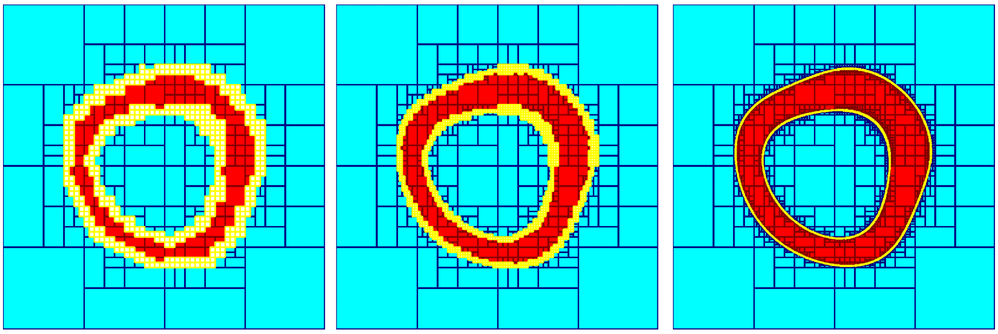
Subsets $\mathbb{X} \subset \mathbb{R}^n$ can be bracketted by subpavings :

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

which can be obtained using interval calculus

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}.$$



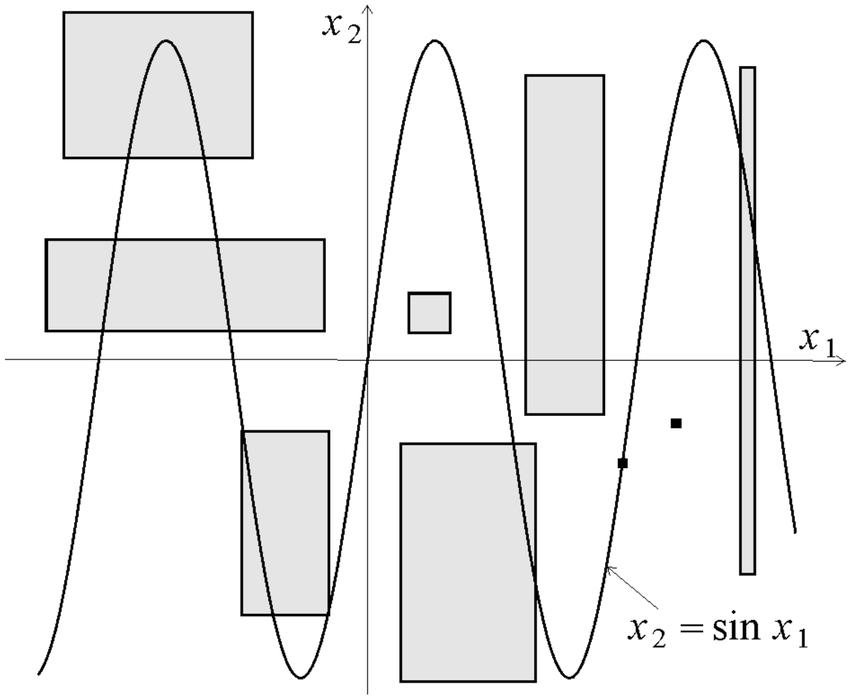
3 Contractors

The operator $\mathcal{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

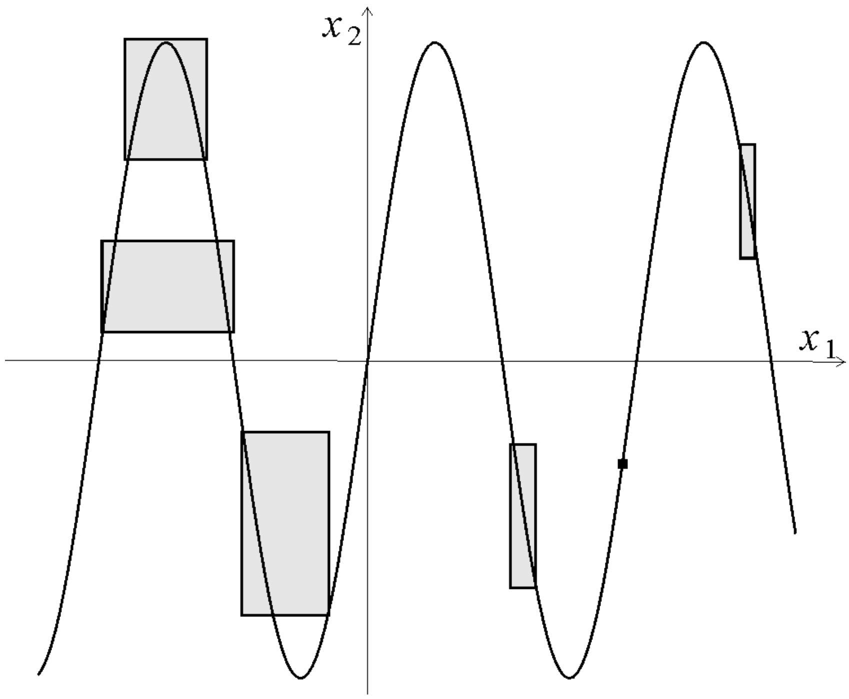
$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

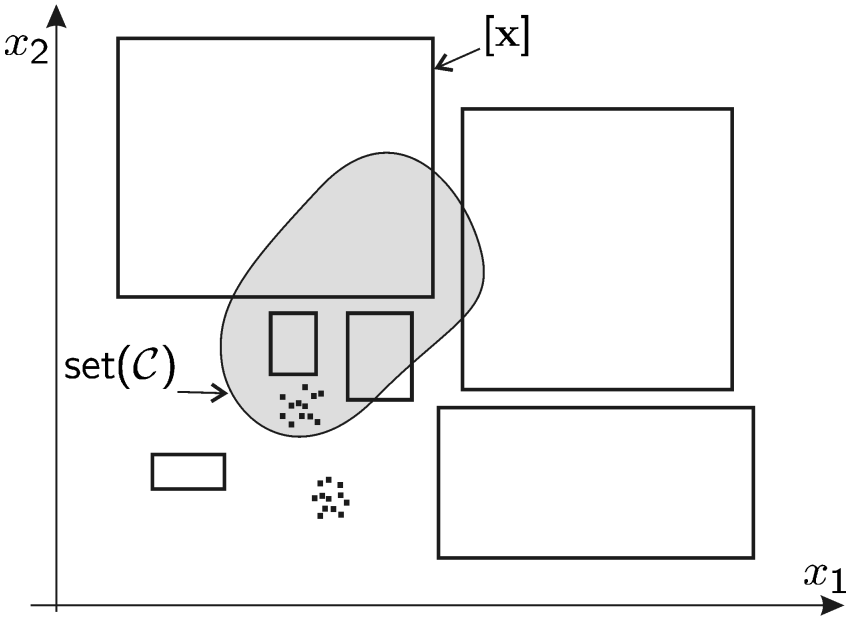
Example. Consider the primitive equation:

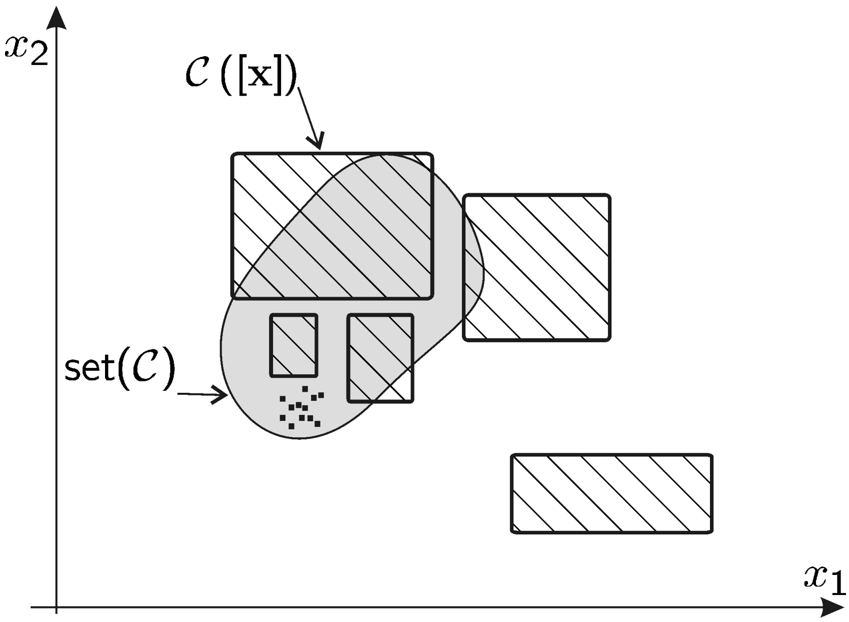
$$x_2 = \sin x_1.$$



x_2
x_1
$x_2 = \sin x_1$


x_2
x_1


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
set$(\mathcal{C})$
x_1

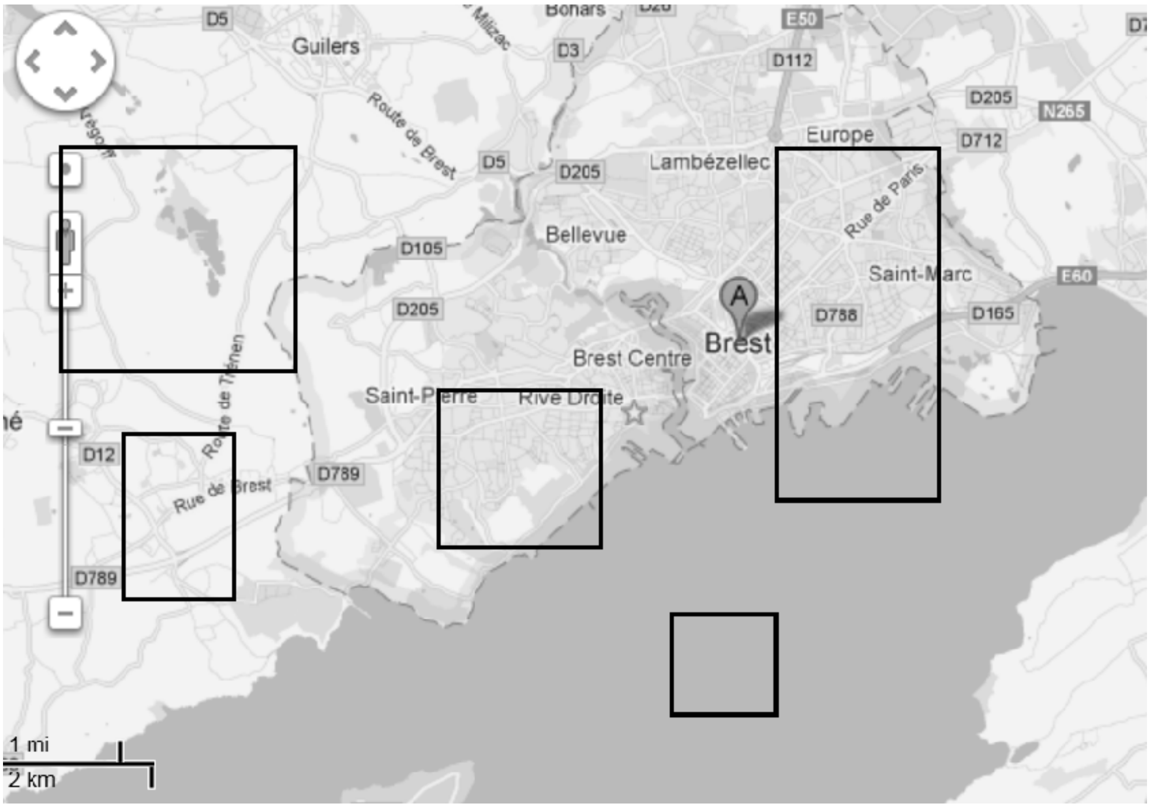
$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *idempotent* if	$\mathcal{C}\left(\mathcal{C}([\mathbf{x}])\right) = \mathcal{C}([\mathbf{x}])$

Contractor algebra

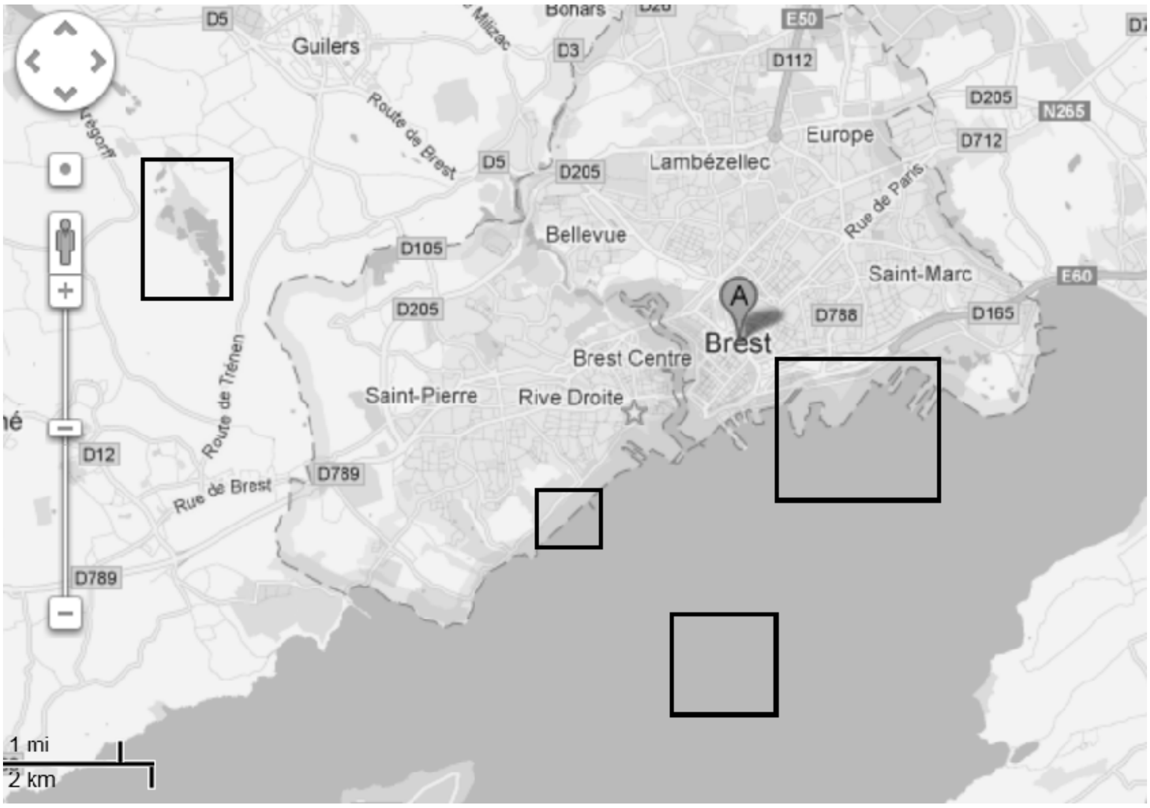
intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Contractor associated with a database

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Bohars
D3
E50
D112
D205
N265
Route de Brest
D5
D205
Lambézellec
Europe
D712
Bellevue
Rue de Paris
D105
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
D12
Rue de Brest
D789
D789
1 mi
2 km



Guilers
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D105
E60
D788
D165
D12
D789
1 mi
2 km

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$\begin{aligned} x_3 = x_1 + x_2 \Rightarrow \quad & x_3 \in \quad [x_3] \cap ([x_1] + [x_2]) \quad && // \text{ forward} \\ x_1 = x_3 - x_2 \Rightarrow \quad & x_1 \in \quad [x_1] \cap ([x_3] - [x_2]) \quad && // \text{ backward} \\ x_2 = x_3 - x_1 \Rightarrow \quad & x_2 \in \quad [x_2] \cap ([x_3] - [x_1]) \quad && // \text{ backward} \end{aligned}$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

4 Solver

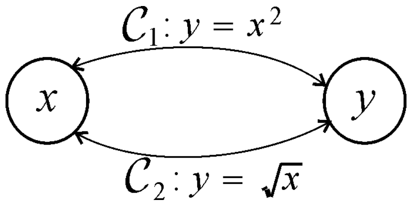
Example. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

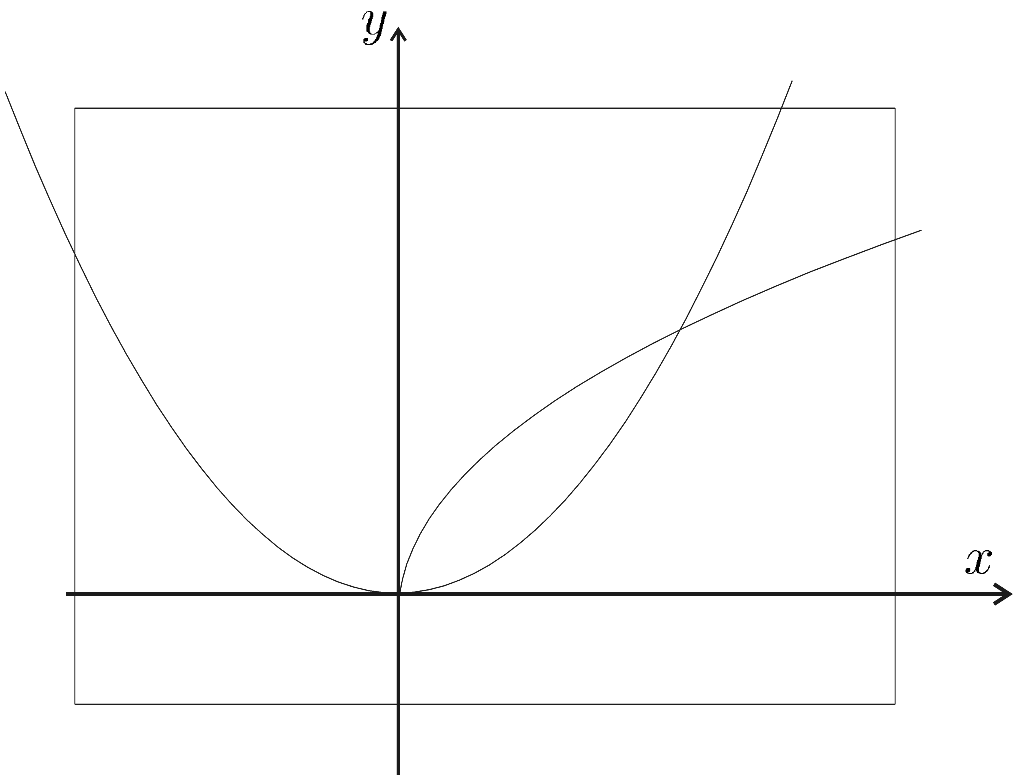
We build two contractors

$$\mathcal{C}_1 : \left\{ \begin{array}{l} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{array} \right. \quad \text{associated with } y = x^2$$

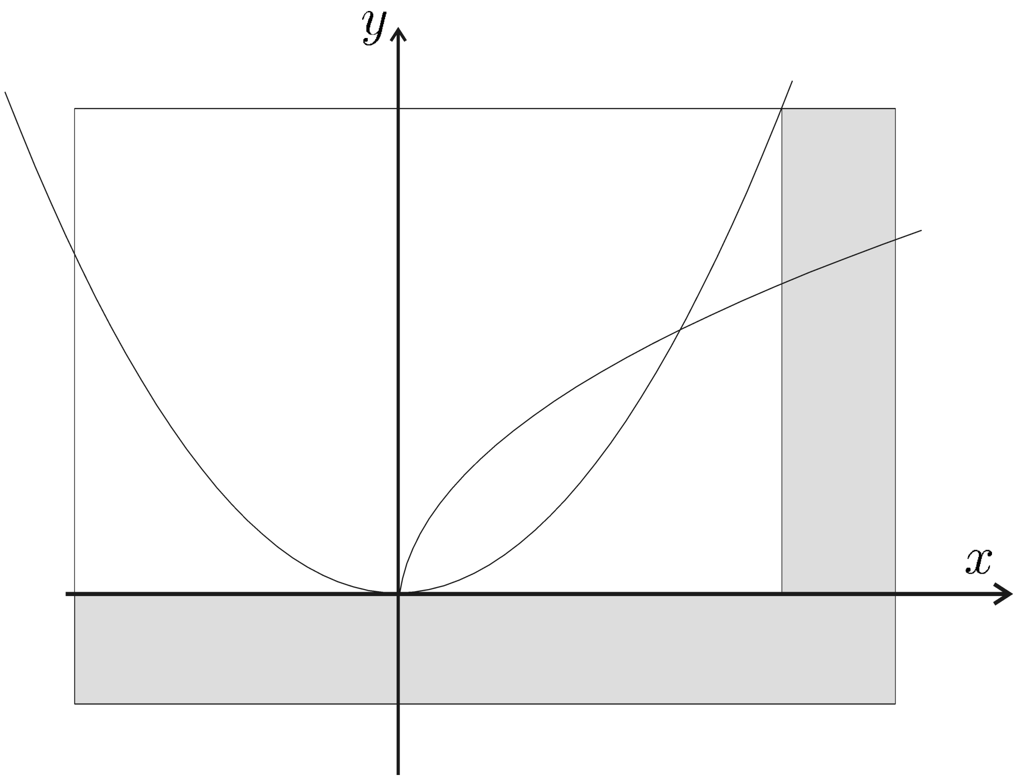
$$\mathcal{C}_2 : \left\{ \begin{array}{l} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{array} \right. \quad \text{associated with } y = \sqrt{x}$$



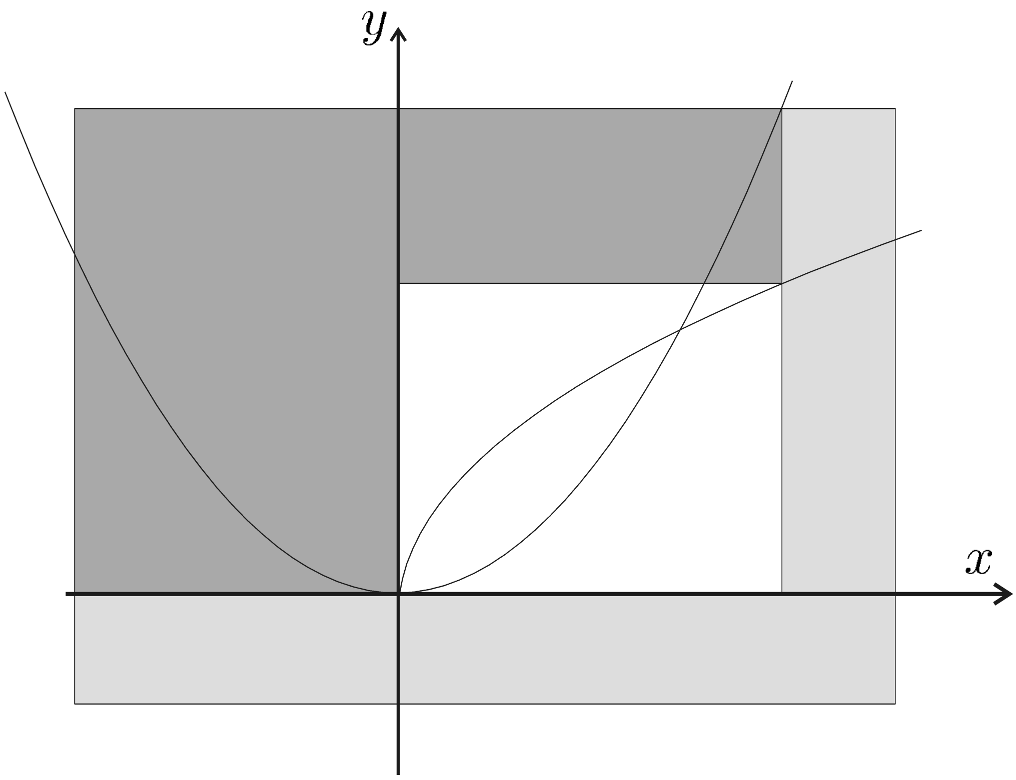
Contractor graph



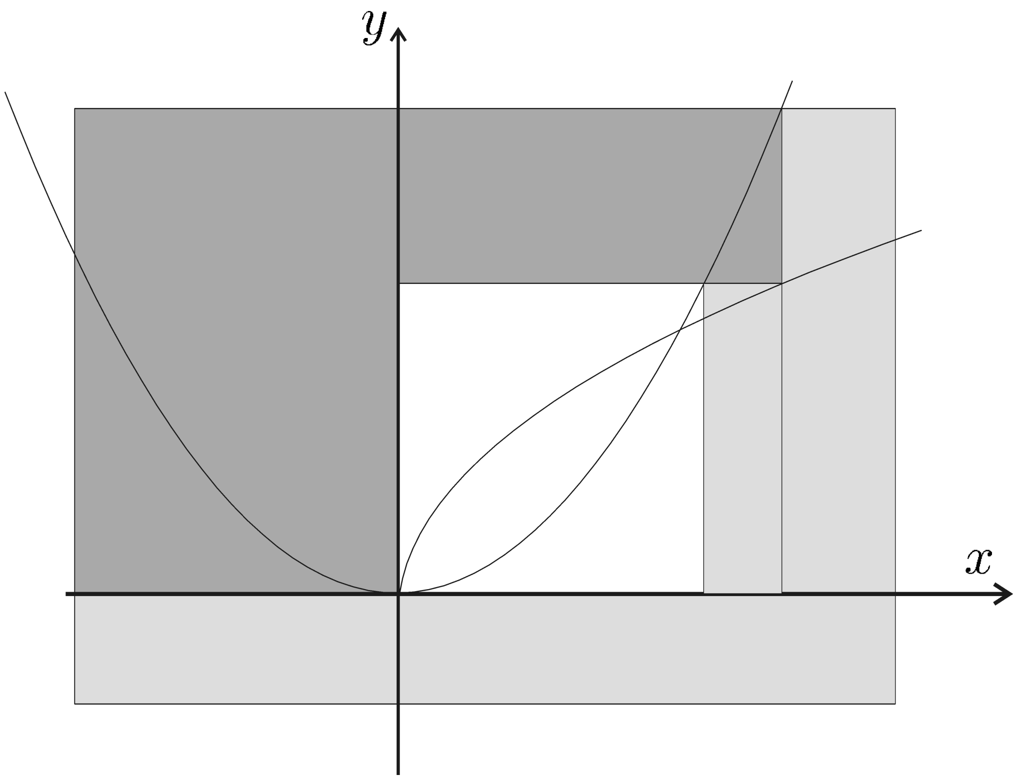
y
x

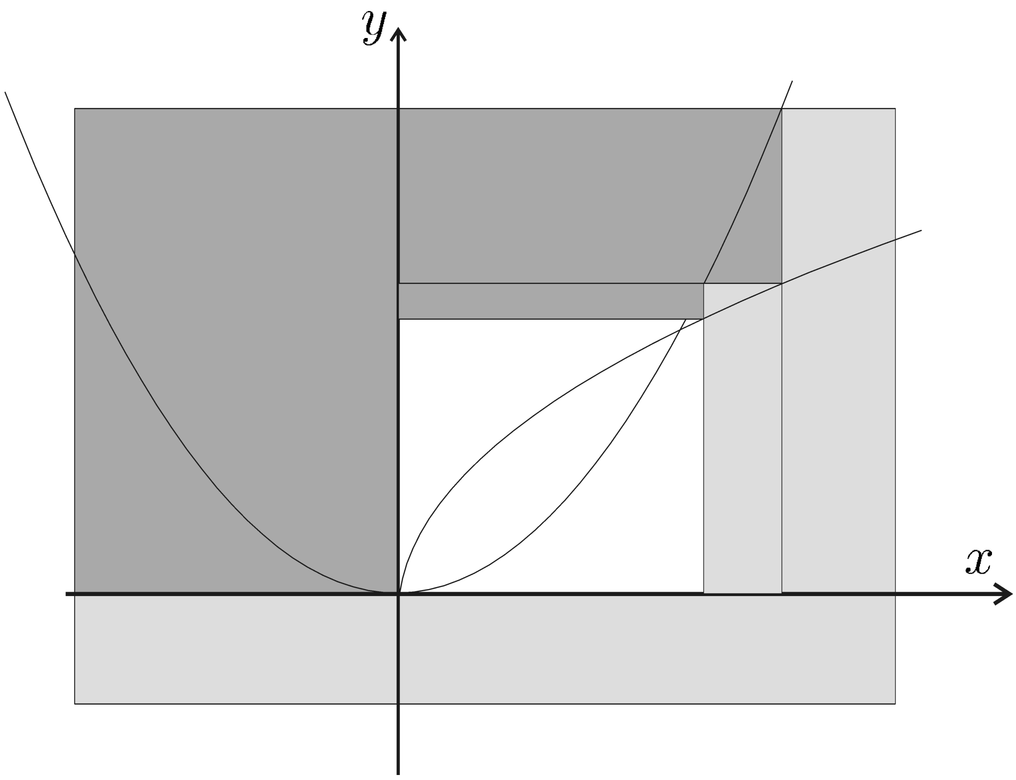


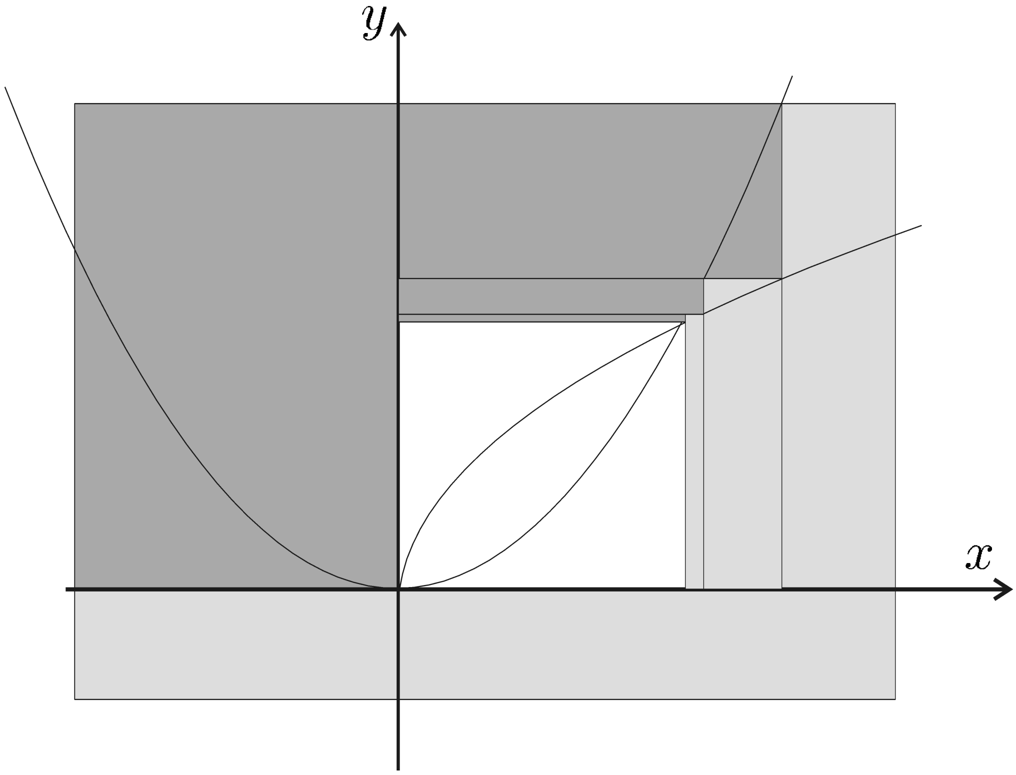
y
x



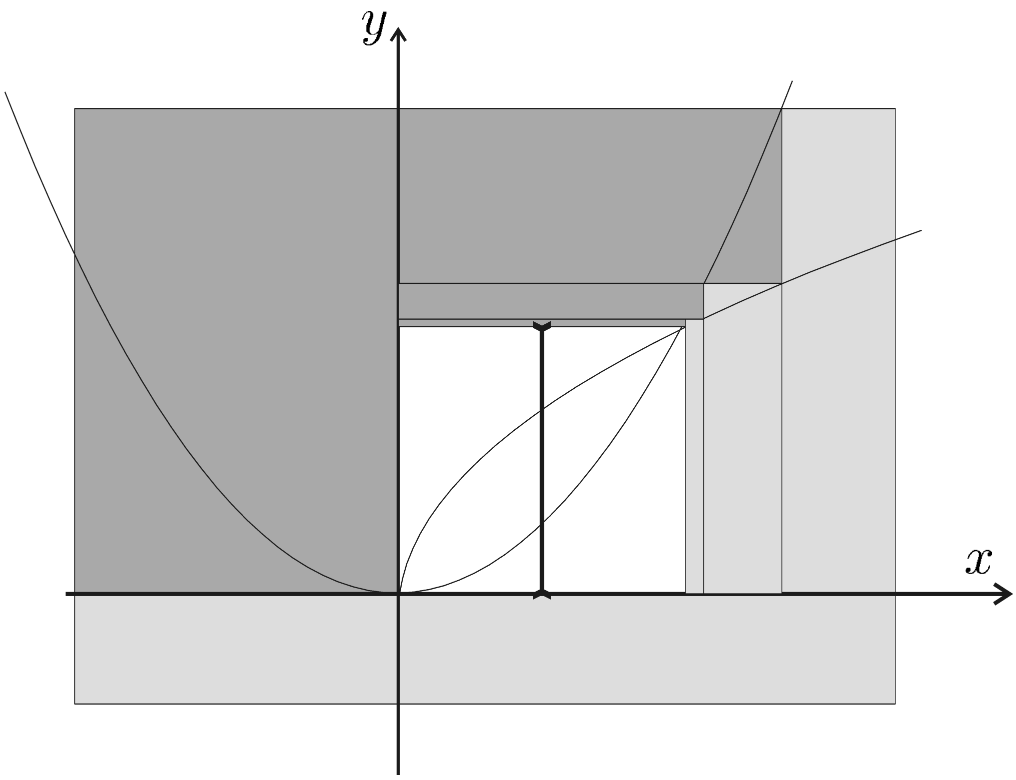
y
x

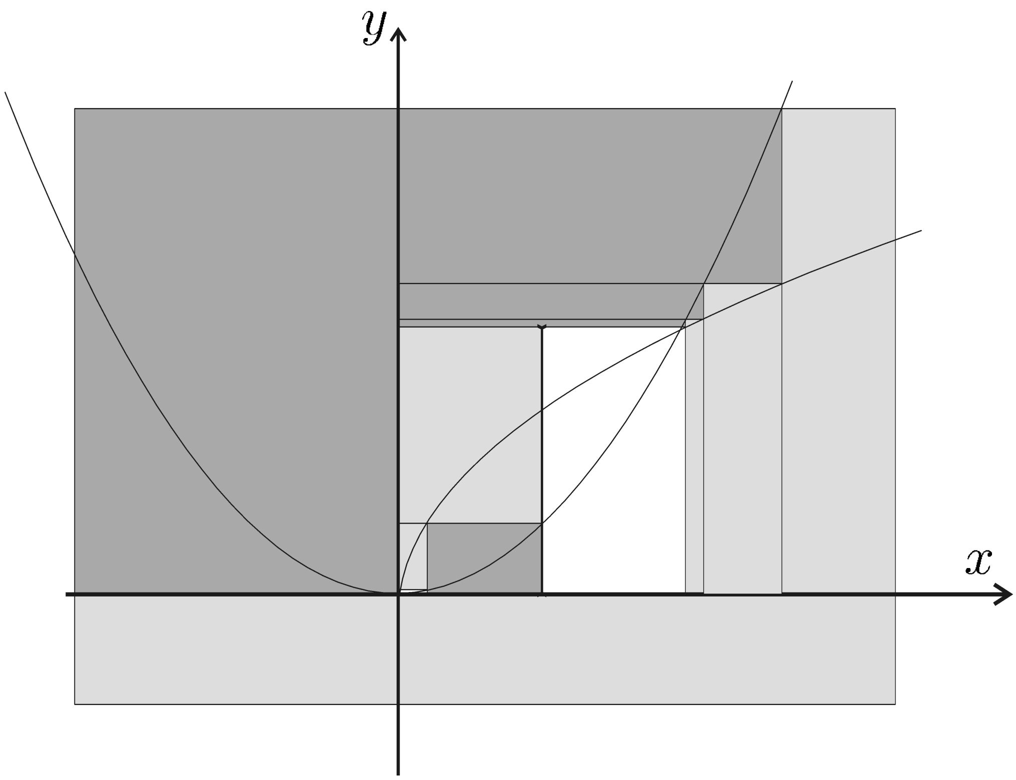

y
x


y
x

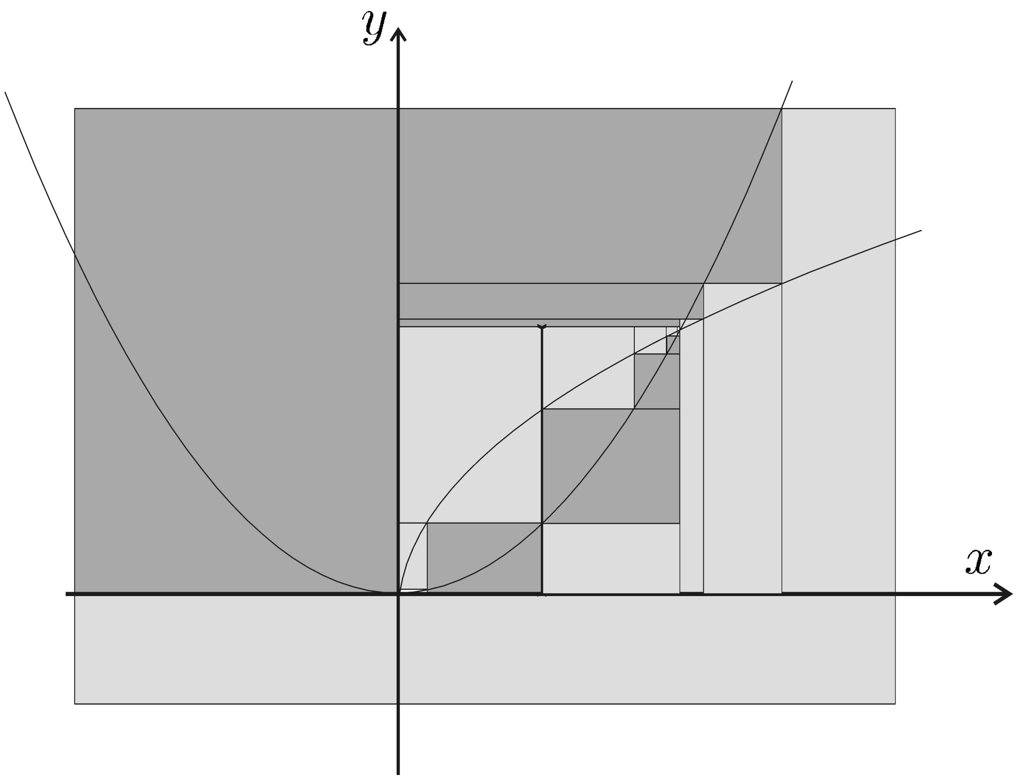


y
x


y
x



y
x



y
x