Examen Inertiel, 2A Rob.

Luc Jaulin, ENSTA-Bretagne

Monday 7 December 2020. Appareils électroniques interdits.

Polycopié et notes manuscrites autorisés

Durée: 1h30.

Exercise 1. We have a body rotating with a constant rotation vector. Its rotation matrix $\mathbf{R}(t)$ is $\mathbf{R}_0$ at time $t = 0$ and $\mathbf{R}_2$ at time $t = 2$.

1) Give an expression of its rotation vector with respect to $\mathbf{R}_0$ and $\mathbf{R}_2$.

2) Give an expression of $\mathbf{R}(t)$.

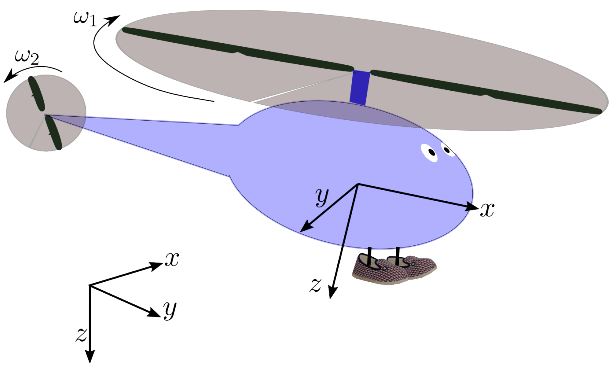
Exercise 2. Consider an axis-aligned parallelepiped with length a, b, c and mass m.

Recall that the inertia matrix of a solid body occupying the volume V is

$$\mathbf{I} = \int_V \rho(x,y,z) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dx\, dy\, dz$$

where $\rho(x, y, z)$, the density is assumed to be constant. Draw, the parallelepiped and compute the inertia matrix $\mathbf{I}$ with respect to a, b, c, ρ.

Exercise 3 (Helicopter). We consider the robot described by the figure below. Propose a state space model for this vehicle.



Correction, Examen Inertiel, 2A Rob
Monday 7 December 2020

Exercise 1. We have a body rotating with a constant rotation vector. Its rotation matrix $\mathbf{R}(t)$ is $\mathbf{R}_0$ at time $t = 0$ and $\mathbf{R}_2$ at time $t = 2$.

1) Give an expression of its rotation vector with respect to $\mathbf{R}_0$ and $\mathbf{R}_2$

2) Give an expression of $\mathbf{R}(t)$.

Solution.
1) The rotation vector is

$$\boldsymbol{\omega} = \frac{1}{2}\wedge^{-1}\log(\mathbf{R}_2\mathbf{R}_0^{\mathrm{T}})$$

2) We have

$$\mathbf{R}(t) = e^{t\boldsymbol{\omega}\wedge}\cdot\mathbf{R}_0$$

Let is check this result for $t = 2$:

$$\mathbf{R}(2) = e^{2\boldsymbol{\omega}\wedge}\cdot\mathbf{R}_0 = e^{2\frac{1}{2}(\wedge^{-1}\log(\mathbf{R}_2\mathbf{R}_0^{\mathrm{T}}))\wedge}\cdot\mathbf{R}_0 = e^{\log(\mathbf{R}_2\mathbf{R}_0^{\mathrm{T}})}\cdot\mathbf{R}_0 = \mathbf{R}_2\mathbf{R}_0^{\mathrm{T}}\mathbf{R}_0 = \mathbf{R}_2$$

Exercise 2. Recall that the inertia matrix of a solid body occupying the volume V is

$$\mathbf{I} = \int_V \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dx\,dy\,dz$$

where $\rho(x,y,z)$, the density is assumed to be constant. Draw, the parallelepiped and compute the inertia matrix $\mathbf{I}$ with respect to a, b, c, ρ.

Solution. The first entry of the matrix is

$$\begin{array}{lll} I_{11} & = & \int_{[-\frac{a}{2},\frac{a}{2}]\times[-\frac{b}{2},\frac{b}{2}]\times[-\frac{c}{2},\frac{c}{2}]}\rho\left(y^2+z^2\right)\,dx\,dy\,dz \\ & = & \rho a\int_{[-\frac{b}{2},\frac{b}{2}]\times[-\frac{c}{2},\frac{c}{2}]}\left(y^2+z^2\right)\,dy\,dz \\ & = & \rho a\int_{[-\frac{b}{2},\frac{b}{2}]\times[-\frac{c}{2},\frac{c}{2}]}y^2\,dy\,dz + \rho a\int_{[-\frac{b}{2},\frac{b}{2}]\times[-\frac{c}{2},\frac{c}{2}]}z^2\,dy\,dz \\ & = & \rho ac\int_{[-\frac{b}{2},\frac{b}{2}]}y^2\,dy + \rho ab\int_{[-\frac{c}{2},\frac{c}{2}]}z^2\,dz \\ & = & \rho ac\left[\frac{1}{3}y^3\right]_{-\frac{b}{2}}^{\frac{b}{2}} + \rho ab\left[\frac{1}{3}z^3\right]_{-\frac{c}{2}}^{\frac{c}{2}} \\ & = & \rho ac\frac{1}{3}\frac{b^3}{4} + \rho ab\frac{1}{3}\frac{c^3}{4} \\ & = & \frac{m}{12}\left(b^2+c^2\right) \end{array}$$

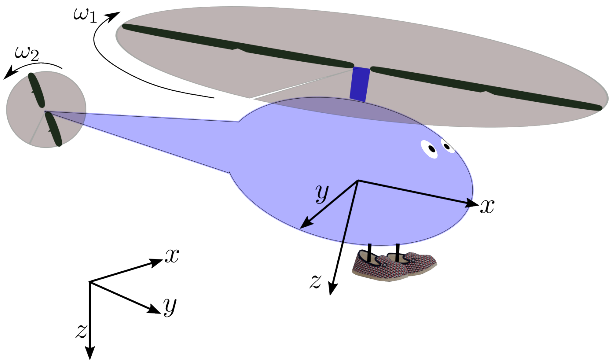
For the second entry, we get

$$\begin{array}{lll} I_{12} & = & -\int_{[-\frac{a}{2},\frac{a}{2}]\times[-\frac{b}{2},\frac{b}{2}]\times[-\frac{c}{2},\frac{c}{2}]}\rho xy\,dx\,dy\,dz \\ & = & -\rho c\int_{[-\frac{a}{2},\frac{a}{2}]}\underbrace{\left(\int_{[-\frac{b}{2},\frac{b}{2}]}y\,dy\right)}_{=0}x\,dx \\ & = & 0 \end{array}$$

We deduce all other components by permutation. We get

$$\mathbf{I} = \frac{m}{12}\cdot\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Exercise 3 (Helicopter). We consider the robot described by the figure below. Propose a state space model for this vehicle.



Solution.

An helicopter has a main rotor on top and a small rotor on the back. In order to control the helicopter, one can tilt the angle of each blade on both rotors. The main rotor is used to move the helicopter up and down, and to make the helicopter tilt forward, backward, left, or right. By tilting a blade to increase the blade's angle of attack, the pilot can increase the force of lift that is pushing up on that blade.

We can take the state space model given by

$$\begin{cases} \dot{\mathbf{p}} = \mathbf{R} \cdot \mathbf{v}_r & (i) \\ \dot{\mathbf{R}} = \mathbf{R} \cdot (\boldsymbol{\omega}_r \wedge) & (ii) \\ \dot{\mathbf{v}}_r = \mathbf{R}^{\mathrm{T}} \cdot \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} + \frac{1}{m} \cdot \mathbf{f}_r - \boldsymbol{\omega}_r \wedge \mathbf{v}_r & (iii) \\ \dot{\boldsymbol{\omega}}_r = \mathbf{I}^{-1} \cdot (\boldsymbol{\tau}_r - \boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r)) & (iv) \end{cases}$$

where

$$\begin{pmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} \beta_1 \omega_1^2 & 0 & 0 & 0 \\ 0 & \beta_2 \omega_1^2 & 0 & 0 \\ 0 & 0 & \beta_3 \omega_1^2 & 0 \\ -\delta_1 \omega_1^2 & 0 & 0 & -\beta_4 \ell \omega_2^2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$\mathbf{f}_r = \begin{pmatrix} 0 \\ 0 \\ -\tau_0 \end{pmatrix} \text{ and } \boldsymbol{\tau}_r = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}.$$

In this model, $\mathbf{R}$ corresponds to the orientation, $\mathbf{p}$, the position, $\mathbf{v}_r$ the speed vector expressed in the robot frame, $\boldsymbol{\omega}_r$ the rotation vector also expressed in the robot frame, τ_0 is the total thrust generated by the main rotor, τ_1 is the roll torque, τ_2 is the pitch torque, τ_3 is head torque generated by the back rotor. We may take for instance: $\omega_1 = \omega_2 = 100$, $\beta_1 = 0.02$, $\beta_2 = \beta_3 = \frac{\beta_1}{10}$, $\beta_4 = 0.002$, $\delta_1 = \frac{\beta_1}{5}$ all expressed in the international unit system.