Fleeting state estimation

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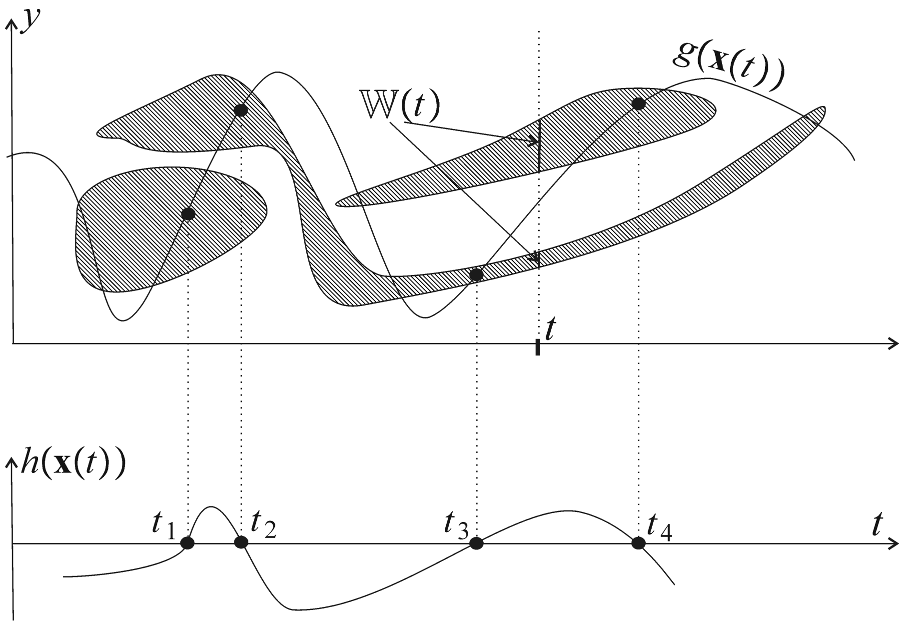
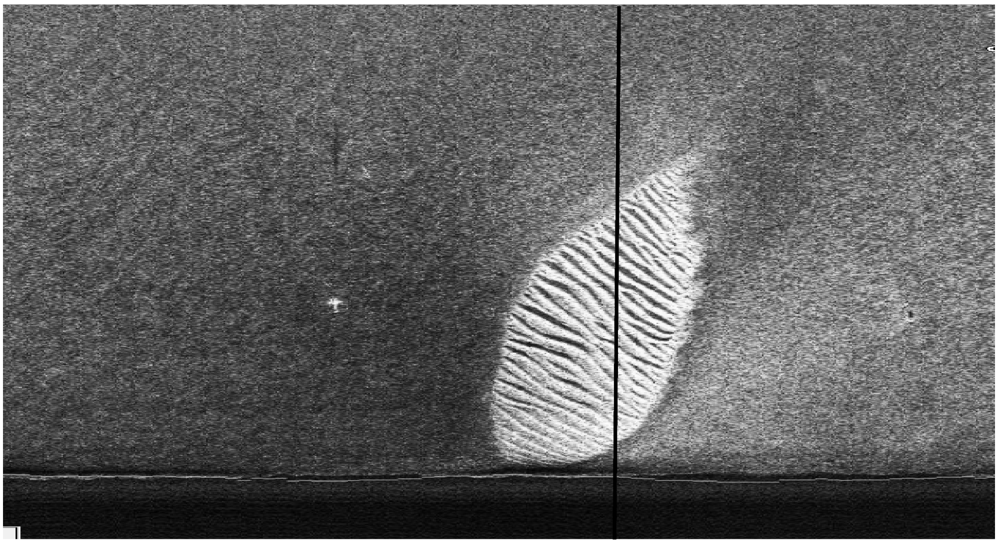
1 Problem

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) + \mathbf{b}(t) & \text{(state equations)} \\ h(\mathbf{x}(t)) = 0 \Rightarrow g(\mathbf{x}(t)) \in \mathbb{W}(t) & \text{(visibility conditions)} \end{cases}$$

The function $\mathbb{W}(t) \subset \mathbb{R}$ is the *waterfall.*

A *fleeting data* is a pair $(t, g(\mathbf{x}(t)))$ with $h(\mathbf{x}(t)) = 0$.





Part of the waterfall collected by the portside lateral sonar of the Redermor, GESMA

2 Tubes

Our state estimation problem is a CSP

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- Variables are the trajectories

$$\mathbf{x}\left(t\right),\dot{\mathbf{x}}\left(t\right),\mathbf{b}(t).$$

- Domains are interval trajectories ot *tubes*

$$\left[\mathbf{x}\right]\left(t\right),\left[\dot{\mathbf{x}}\right]\left(t\right),\left[\mathbf{b}\right]\left(t\right).$$

- Constraints

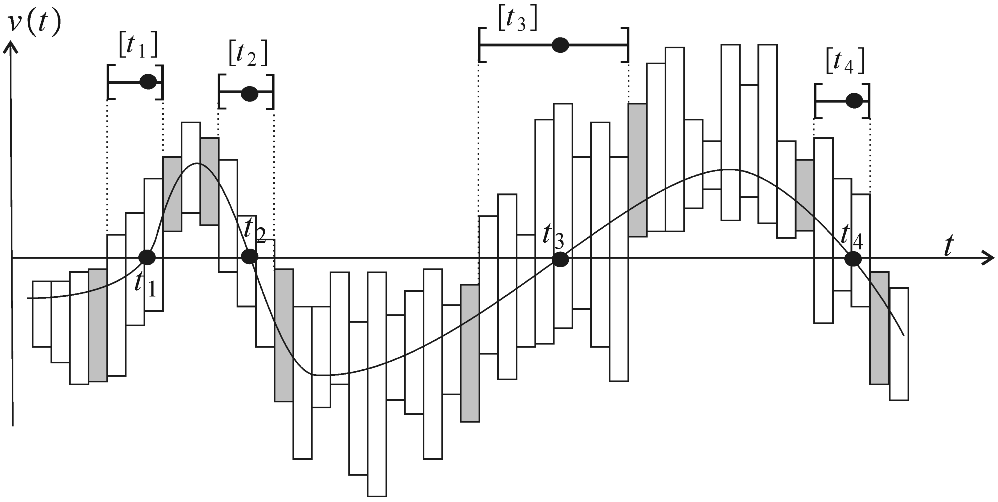
$$\left\{\begin{array}{l}\dot{\mathbf{x}}\left(t\right)=\mathbf{f}\left(\mathbf{x}\left(t\right),t\right)+\mathbf{b}(t)\\ h\left(\mathbf{x}\left(t\right)\right)=0\Rightarrow g\left(\mathbf{x}\left(t\right)\right)\in\mathbb{W}\left(t\right).\end{array}\right.$$

Decomposition

$$\left\{\begin{array}{l} \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),t)+\mathbf{b}(t) \\ h(\mathbf{x}(t))=0 \\ \qquad\quad \Rightarrow g(\mathbf{x}(t))\in\mathbb{W}(t) \end{array}\right. \Rightarrow \left\{\begin{array}{l} \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),t)+\mathbf{b}(t) \\ v(t)=h(\mathbf{x}(t)) \\ \dot{y}(t)=\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}(t)).\dot{\mathbf{x}}(t) \\ y(t)=g(\mathbf{x}(t)) \\ v(t)=0\Rightarrow y(t)\in\mathbb{W}(t) \end{array}\right.$$

The set of functions from $\mathbb{R} \to \mathbb{R}^n$ is a lattice. Interval methods can thus be used.

A *machine tube* $[\mathbf{x}](t)$, with a sampling time $\delta > 0$, is a box-valued function which is constant inside intervals $[k\delta, k\delta + \delta]$, $k \in \mathbb{N}$.



A tube encloses an uncertain trajectory

Define the *index correspondence function* κ as follows

$$\kappa\left([t_a, t_b]\right) = \left\{k \in \mathbb{N}, \exists t \in [t_a, t_b], t \in [k\delta, k\delta + \delta]\right\}.$$

Integral.

$$\int_{t_0}^{t} [\mathbf{x}](\tau)\, d\tau = \sum_{k \in \kappa([t_0,t])} \delta * [\mathbf{x}](k),$$

We have

$$\mathbf{x}(t) \in [\mathbf{x}](t) \Rightarrow \int_{t_0}^{t} \mathbf{x}(\tau)\, d\tau \in \int_{t_0}^{t} [\mathbf{x}](\tau)\, d\tau$$

and that the quantity $\int_{t_0}^{t} [\mathbf{x}](\tau)\, d\tau$ is a tube.

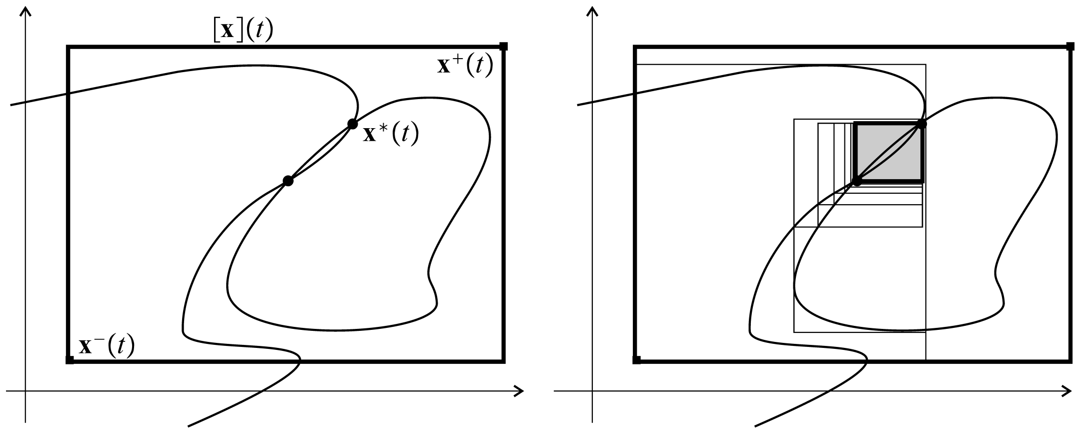
The derivative of a tube cannot be defined. Now, since

$$\begin{array}{rcl}\dot{\mathbf{x}}(t) & := & \mathbf{f}\left(\mathbf{x}\left(t\right),t\right)+\mathbf{b}(t)\\ \dot{y}(t) & := & \frac{\partial g}{\partial \mathbf{x}}\left(\mathbf{x}\left(t\right)\right).\dot{\mathbf{x}}(t)\\ \dot{v}(t) & := & \frac{\partial h}{\partial \mathbf{x}}\left(\mathbf{x}\left(t\right)\right).\dot{\mathbf{x}}(t).\end{array}$$

tubes enclosing the $\dot{\mathbf{x}}\left(t\right)$, $\dot{y}\left(t\right)$ and $\dot{v}\left(t\right)$ are

$$\begin{array}{rcl}[\dot{\mathbf{x}}](t) & := & [\mathbf{f}]\left([\mathbf{x}]\left(t\right),t\right)+[\mathbf{b}]\left(t\right)\\ [\dot{y}]\left(t\right) & := & \left[\frac{\partial g}{\partial \mathbf{x}}\right]\left([\mathbf{x}]\left(t\right)\right).[\dot{\mathbf{x}}](t)\\ [\dot{v}]\left(t\right) & := & \left[\frac{\partial h}{\partial \mathbf{x}}\right]\left([\mathbf{x}]\left(t\right)\right).[\dot{\mathbf{x}}](t).\end{array}$$

Propagation.



Left: a tube $[\mathbf{x}]\,(t) = \left[\mathbf{x}^-(t), \mathbf{x}^+(t)\right] \ni \mathbf{x}^*(t)$.
Right: propagation.

3 Visibility contractor

Contract tubes $[v](t), [y](t)$ with respect to

$$v(t) = 0 \Rightarrow y(t) \in \mathbb{W}(t).$$

3.1 Contraction of $[y](t)$

$$v(t) = 0 \Rightarrow y(t) \in \mathbb{W}(t) \, .$$

Theorem 1. If $0 \in v([t])$ then for all t,

$$y(t) \in \bigcup_{\tau \in [t]} \left(\left(\mathbb{W}(\tau) \cap [y](\tau) \right) + \int_{\tau}^{t} [\dot{y}](\alpha) .d\alpha \right).$$

Proof. If $0 \in v([t])$, then $\exists \tau \in [t], v(\tau) = 0$. Now,

$$y(\tau) \in \mathbb{W}(\tau) \cap [y](\tau)$$

Since $y(t) = y(\tau) + \int_{\tau}^{t} \dot{y}(\alpha) .d\alpha$, we get

$$y(t) \in \left(\mathbb{W}(\tau) \cap [y](\tau) \right) + \int_{\tau}^{t} [\dot{y}](\alpha) .d\alpha. \blacksquare$$

Corollary. If $0 \in v([t])$ then for all t,

$$\begin{aligned} y(t) &\in \bigcup_{\tau \in [t]} \left(\left(\mathbb{W}(\tau) \cap [y](\tau) \right) + [y_0](t) - [y_0](\tau) \right). \\ [y_0] &= \int_0^t [\dot{y}](\alpha) \, . d\alpha. \end{aligned}$$

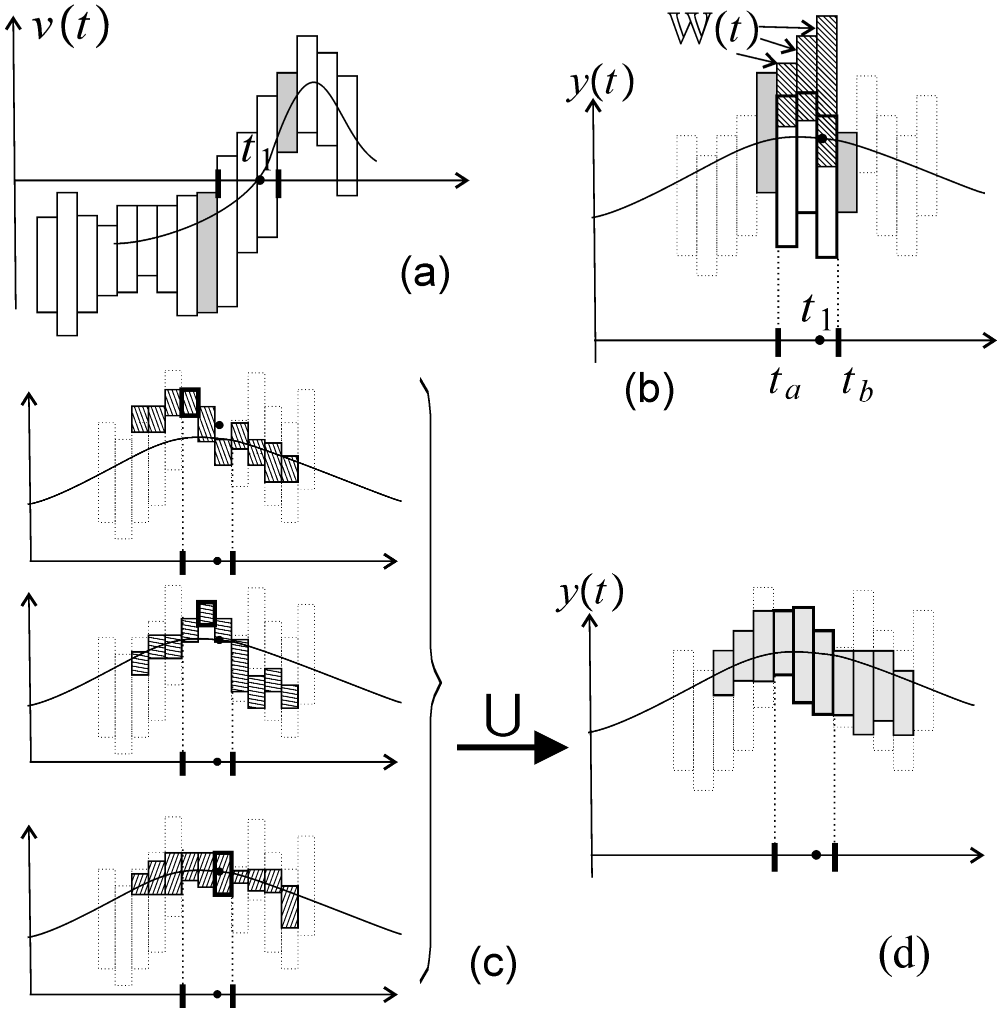
Proof. y_0 is the primitive of $\dot{y}$ such that $y_0(0) = 0$.■

Contraction of $[y](t)$.

- Compute the tube $[y_0] = \int_{\tau}^{t} [\dot{y}](\alpha) \, .d\alpha$.

- Perform the contraction

$$\begin{aligned} [y](t) &= [y](t) \cap \\ & \bigcup_{\tau \in [t]} \left\{ (\mathbb{W}(\tau) \cap [y](\tau)) + [y_0](t) - [y_0](\tau) \right\} \end{aligned}$$



Contraction of $[y]$

3.2 Contraction of $[v](t)$

$$v(t) = 0 \Rightarrow y(t) \in \mathbb{W}(t)\,.$$

Theorem 2. We have

$$\forall t \in [t], [y](t) \cap \mathbb{W}(t) = \emptyset \Rightarrow 0 \notin v([t]).$$

Proof. If $0 \in v([t])$, then $\exists \tau \in [t], v(\tau) = 0$ and thus $y(\tau) \in \mathbb{W}(\tau)$. Now $y(\tau) \in [y](\tau)$ which is in contradiction with the assumption.■

Contraction of $[v]\,(t)$:

- Find a long interval $[t]$ of $\mathbb{R}$ such that $\forall t \in [t], [y]\,(t) \cap \mathbb{W}\,(t) = \emptyset$.

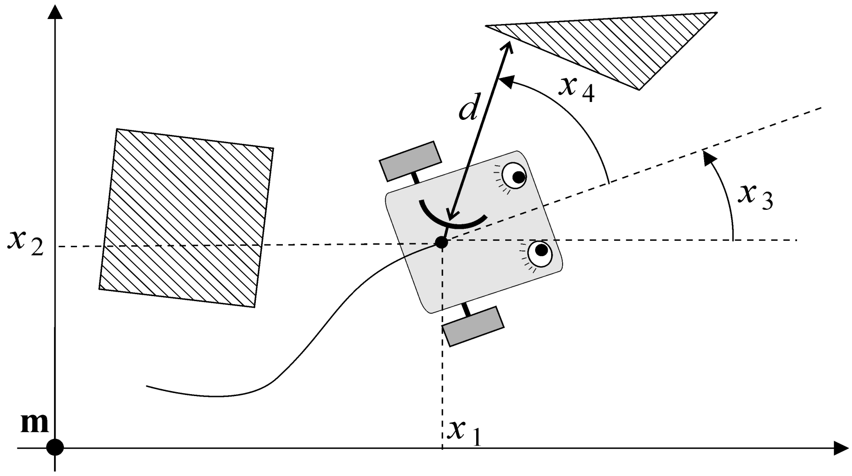
- Find, if any, $t_1 \in [t]$ such that $[v]\,(t_1) \neq 0$

$$\begin{array}{lll} \text{if } [v]\,(t_1) > 0 & \text{, then} & [v]\,([t]) := [v]\,(t) \cap [0, \infty] \\ \text{if } [v]\,(t_1) < 0 & \text{, then} & [v]\,([t]) := [v]\,([t]) \cap [-\infty, 0]. \end{array}$$

4 Test-case

Consider a robot moving on a plane and equipped with a directive laser rotating telemeter

$$\begin{cases} \dot{x}_1 & = & \cos x_3 \quad + b_1 \\ \dot{x}_2 & = & \sin x_3 \quad + b_2 \\ \dot{x}_3 & = & u + b_3 \\ \dot{x}_4 & = & 2 + b_4. \end{cases}$$



The robot with a rotating telemeter.
The location of $\mathbf{m}$ is known.

- A mark $\mathbf{m}$ is located at coordinates $(0, 0)$.

- The distance to the mark is measured with an accuracy of ± 0.01.

- The scope of the telemeter corresponds to the interval $[s] = [s^-, s^+] = [1, 10]$.

We have

$$\begin{cases} h(\mathbf{x}) &= x_1 \sin(x_3 + x_4) - x_2 \cos(x_3 + x_4) & \text{(visibility)} \\ g(\mathbf{x}) &= -x_1 \cos(x_3 + x_4) - x_2 \sin(x_3 + x_4) & \text{(distance)} \end{cases}$$

Note that both $h(\mathbf{x})$ and $g(\mathbf{x})$ are differentiable.

We have

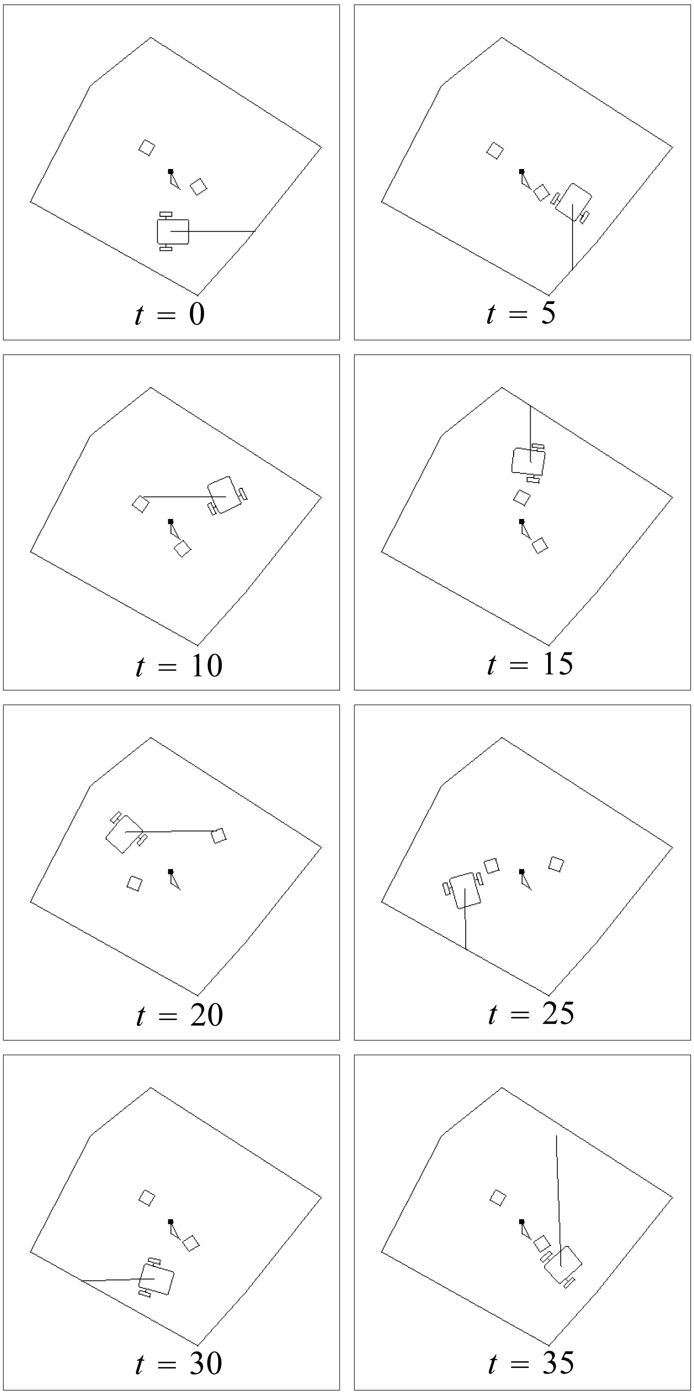
$$h(\mathbf{x}) = 0 \text{ and } g(\mathbf{x}) \in [s] \cap [-\infty, d] \Rightarrow d = g(\mathbf{x}).$$

Since

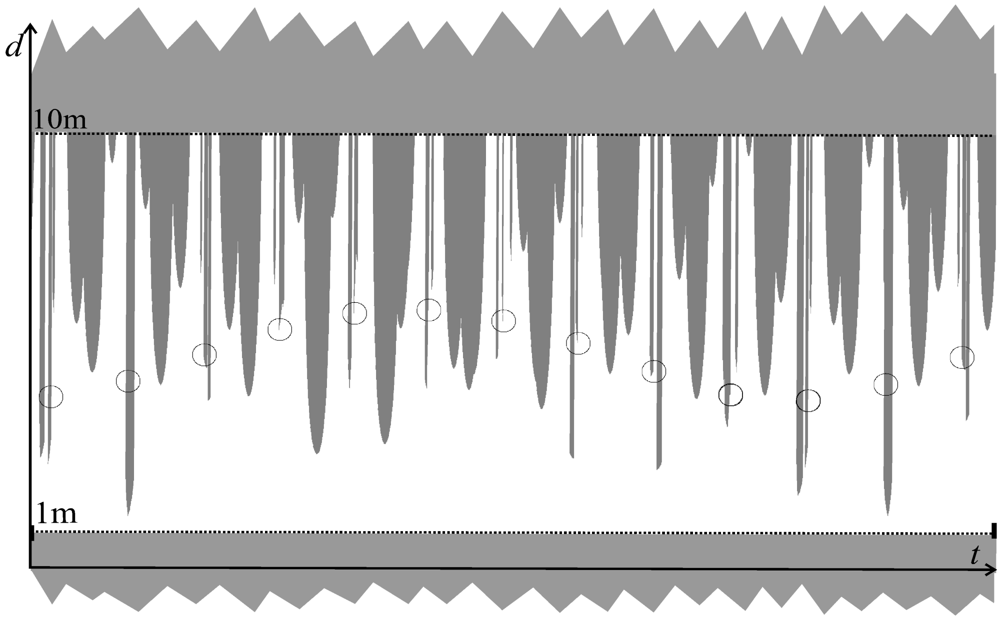
$$(A \wedge B \Rightarrow C) \Leftrightarrow (A \Rightarrow \neg B \vee C),$$

we get

$$\begin{aligned} h(\mathbf{x}) = 0 &\Rightarrow g(\mathbf{x}) \notin ([s] \cap [-\infty, d]) \text{ or } d = g(\mathbf{x}). \\ &\Leftrightarrow g(\mathbf{x}(t)) \in [-\infty, s^-] \cup [s^+, \infty] \cup [d, \infty] \\ &\Rightarrow g(\mathbf{x}(t)) \in \underbrace{[-\infty, s^-] \cup [s^+, \infty] \cup [d^-, \infty]}_{\mathbb{W}(t)} \end{aligned}$$

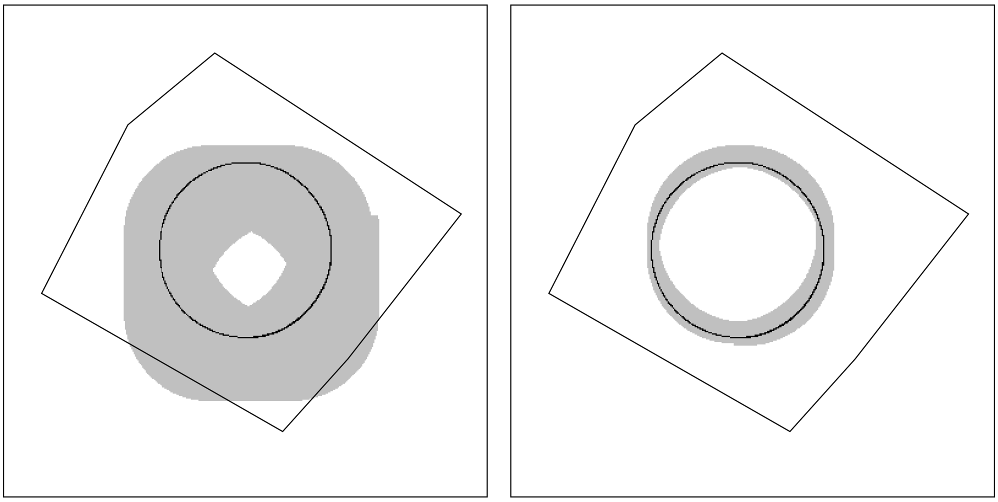


Robot for different t during the mission



The robot only knows that the fleeting points $(t, g(\mathbf{x}(t)))$
lie inside $\mathbb{W}(t)$

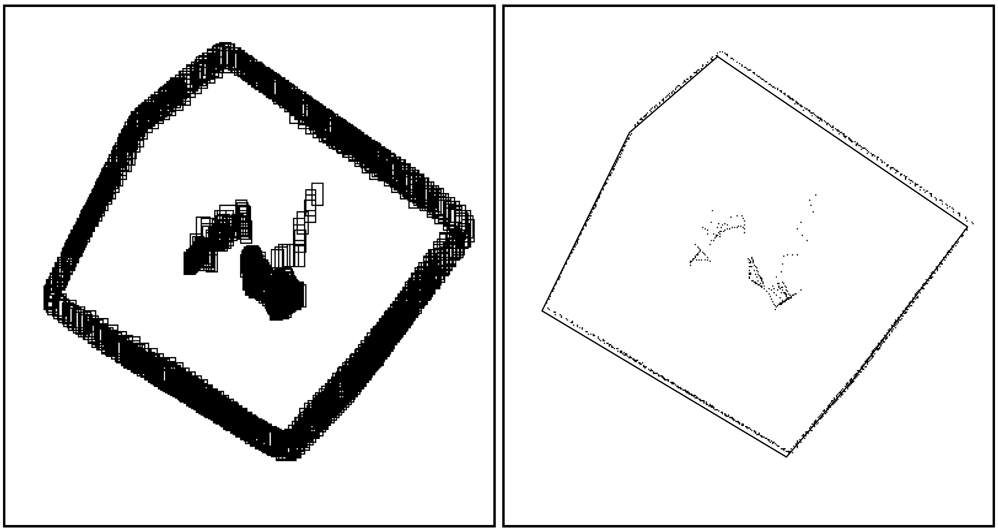
Envelope of the trajectory.



(left) envelope without using the telemeter;
(right) envelope using the telemeter

Constructing the map.

$$
\begin{array}{rcll}
\mathbb{M} & = & \{ & (z_1, z_2), \exists t \in [0, 40], \exists \mathbf{x} \in [\mathbf{x}](t) \\
& & & z_1 = x_1(t) + d(t).\cos(x_3(t) + x_4(t)), \\
& & & z_2 = x_2(t) + d(t).\sin(x_3(t) + x_4(t)) \quad \}
\end{array}
$$



(left) set of boxes enclosing the map;
(right) approximation of the map