The Lorenz attractor exists

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The Lorenz equations

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$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= \varrho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2,\end{aligned}$$

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$$C^{\pm} = (\pm\sqrt{\beta(\varrho - 1)}, \pm\sqrt{\beta(\varrho - 1)}, \varrho - 1).$$

Stability: The origin is a saddle point with eigenvalues

$$0 < -\lambda_3 < \lambda_1 < -\lambda_2.$$

The two symmetric fixed points $C^{\pm}$ are unstable spirals.

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Constant divergence:

$$\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = -(\sigma + \beta + 1).$$

The volume of a solid at time t can be expressed as

$$V(t) = V(0)e^{-(\sigma+\beta+1)t} \approx V(0)e^{-13.7t},$$

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Maximal invariant set:

$$\mathcal{A} = \bigcap_{t \geq 0} \varphi(\mathcal{U}, t).$$

$\mathcal{A}$ must have zero volume, and $W^u(0) \subseteq \mathcal{A}$.

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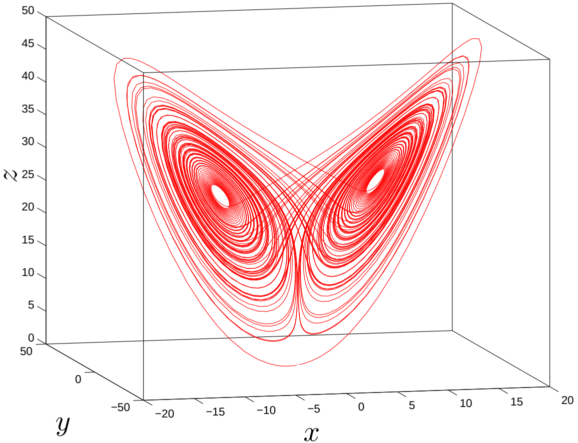
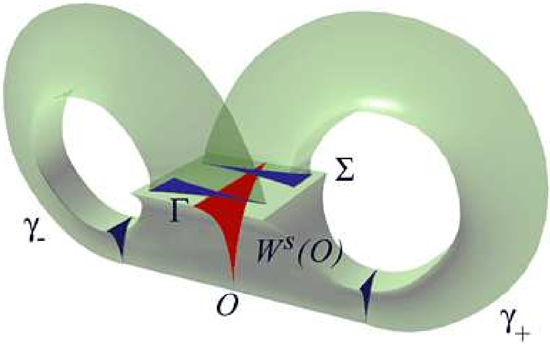


The geometric model:

Introduced by Guckenheimer and Williams (1979)



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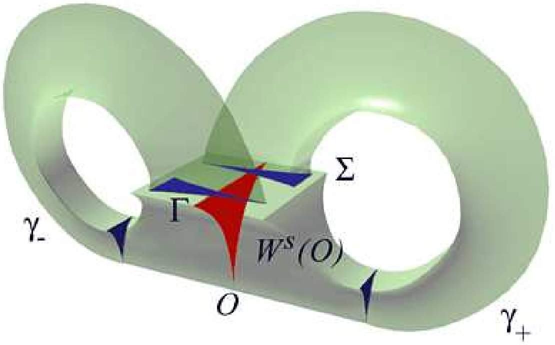

Return map: $R\colon \Sigma \setminus \Gamma \to \Sigma$.

The return plane Σ is foliated by stable leaves.

Projecting along these leaves gives a 1-d function:

$$f\colon [-1,1] \to [-1,1]$$

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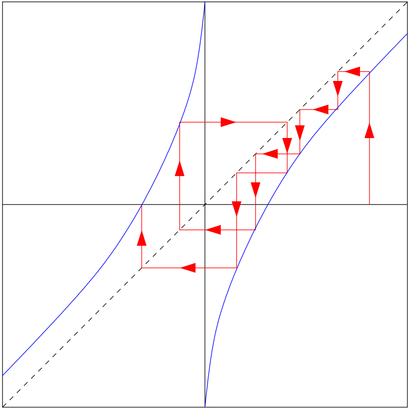

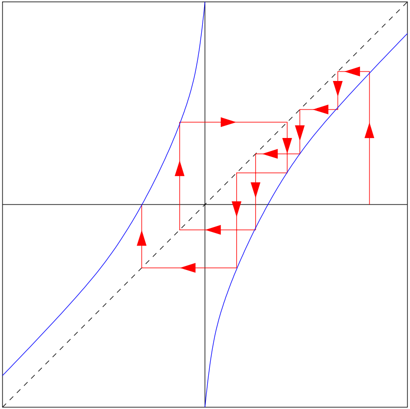

Properties: The function $f : [-1, 1] \to [-1, 1]$ satisfies:

[1] $f(-x) = -f(x)$;

[2] $\lim_{x\to 0} f'(x) = +\infty$;

[3] $f''(x) < 0$ on $(0, 1]$;

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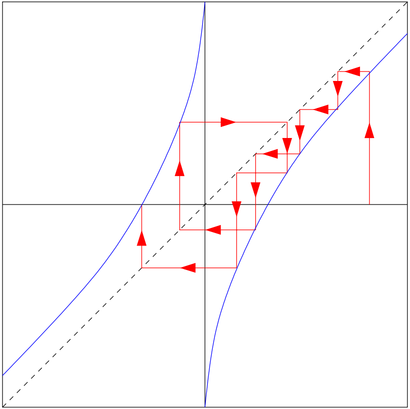

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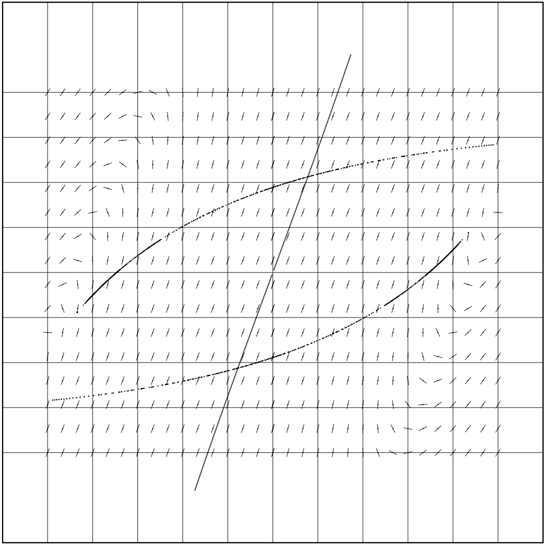
Theorem: [1] - [4] $\Rightarrow$ f is topologically transitive on $[-1, 1]$.



The real attractor seen from above Σ.

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More history:

1989 C. Robinson; M. Rychlik

Constructed *explicit* families of ODEs with geometric Lorenz attractors.

[*] Extra terms of degree 3 were needed,

[*] Arbitrarily small unfoldings,

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1995 K. Mischaikow & M. Mrozek

Computer-aided proof $\Rightarrow$ horseshoe.

[*] Non-classical parameter values,
[*] Objects have measure zero,
[*] Objects are not attracting.

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Open conditions - Perfect for interval methods!

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Theorem: For the classical parameter values, the Lorenz equations support a robust strange attractor $\mathcal{A}$ – the Lorenz attractor!

By robust, we mean that a strange attractor exists in an open neighbourhood of the classical parameter values.

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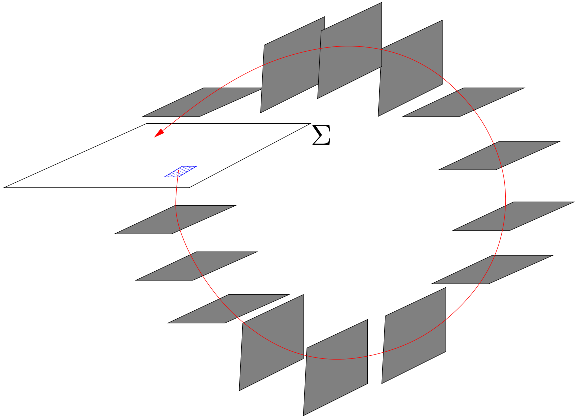
- The linearizing process is very sensitive to changes in parameters.
- Don't linearize, but make the flow closer to linear (normal form).

The flowing process

Let $N = \cup_{i=1}^{k} N_i$, and flow each initial rectangle N_i between several codimension-1 surfaces.



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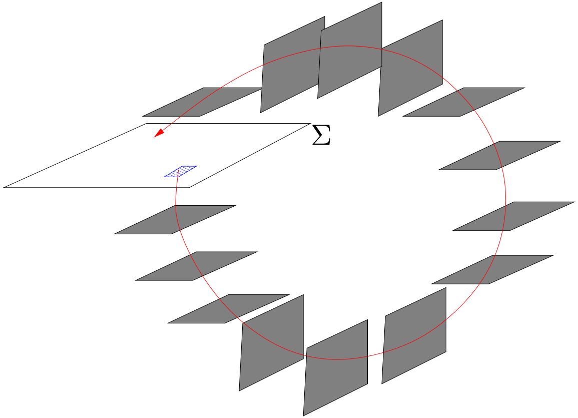
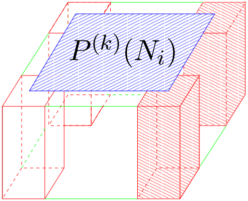


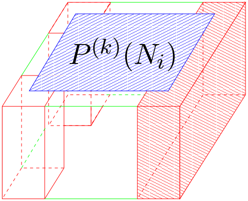
The return of N_i is given by composing several distance-d maps:

$$R(N_i) \subset \Pi^{(k(i))} \circ \cdots \circ \Pi^{(0)}(N_i).$$

The flowing process...

Use the fact that $\Pi^{(k)}$ – the "distance-d map" – often is monotone. This allows us to shrink the flow regions.





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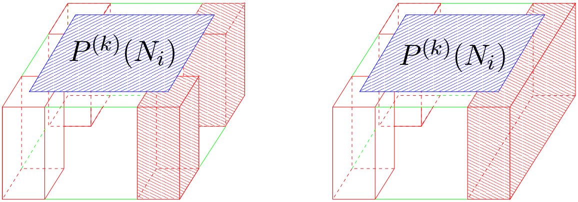
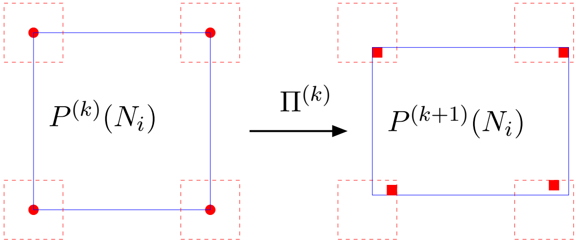


Flowing one step (seen from above):



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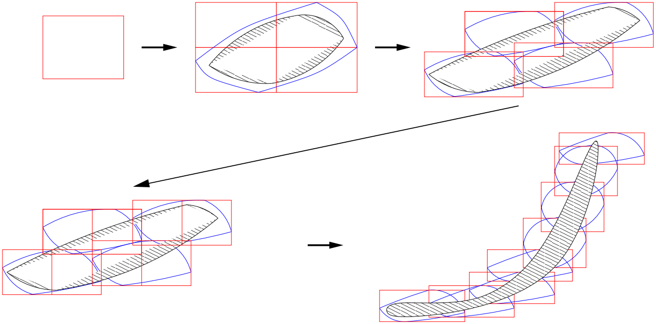
The partitioning process

Idea: Dynamically split large images into smaller rectangles, and flow them separately.



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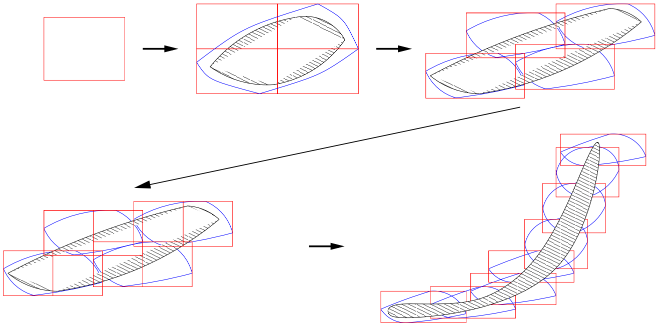

After k steps the image of $N_i \subset \Sigma$ is enclosed by the union of many smaller rectangles:

$$P^{(k)}(N_i) \subseteq \bigcup_{j=1}^{n(i,k)} Q_{i,j}^{(k)}.$$

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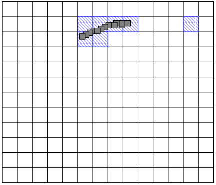
Finding the invariant set

At the return to Σ we have information of the type

$$R(N_i) \subseteq \bigcup_{j=1}^{n(i)} Q_{i,j} \subseteq \bigcup_{j=1}^{m(i)} N_j.$$



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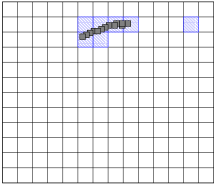

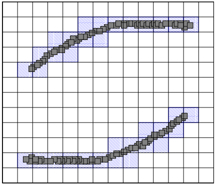

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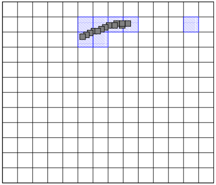

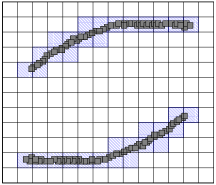

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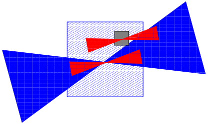
Verify the cone condition:

$Q_{i,j} \cap N_k \neq \emptyset \Rightarrow \mathfrak{C}(Q_{i,j}) \subset \mathfrak{C}(N_k).$



Local theory and normal forms

Notation:

$$x = (x_1, x_2, x_3), \quad x^n = x_1^{n_1} x_2^{n_2} x_3^{n_3}.$$

$$|x| = \max\{|x_i| \colon i = 1, 2, 3\}, \qquad \|f\|_r = \max\{|f(x)| \colon |x| \leq r\}.$$

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Flatness of order p:

$$x^n \in \mathcal{O}^p(x_1) \cap \mathcal{O}^p(x_2, x_3)$$

if $n \in \mathbb{U}_p \stackrel{\text{def}}{=} \{n \in \mathbb{N}^3 \colon n_1 \ge p$ and $n_2 + n_3 \ge p\}$.

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Change of variables:

$$\underbrace{\dot{x} = Ax + F(x)}_{\text{original Lorenz}} \stackrel{x=y+\phi(y)}{\longrightarrow} \underbrace{\dot{y} = Ay + G(y)}_{\text{normal form}}$$

where $G(y) \in \mathcal{O}^{10}(y_1) \cap \mathcal{O}^{10}(y_2, y_3)$. G is *almost* linear.

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Local theory and normal forms...

We find $\phi(y) = \sum a_n y^n$ by a simple power series substitution:

$$L_A \phi(y) = \{F(y + \phi(y))\}_{\mathbb{V}_{10}},$$

where $\mathbb{V}_{10} = \mathbb{N}^3 \setminus \mathbb{U}_{10}$, and

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Existence of a formal ϕ:

Lemma: Let $n \in \mathbb{V}_{10}$. Then, for $|n| \in [2, 57]$, we have $|n\lambda - \lambda_i| \geq 0.0112$. For $|n| \geq 58$, we have $|n\lambda - \lambda_i| \geq \frac{8}{3}|n|$. The proof requires the computation of the 19.386 first divisors (using interval arithmetic).

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$$L_A\phi(y) = \{F(y + \phi(y))\}_{\mathbb{V}_{10}},$$

where $\mathbb{V}_{10} = \mathbb{N}^3 \setminus \mathbb{U}_{10}$, and

$$L_{A,i}(a_{i,n}y^n) = \overbrace{(n\lambda - \lambda_i)}^{\text{divisor}} a_{i,n}y^n.$$

Can we formally solve for the coefficients?

Existence of a formal ϕ:

Lemma: Let $n \in \mathbb{V}_{10}$. Then, for $|n| \in [2, 57]$, we have $|n\lambda - \lambda_i| \geq 0.0112$. For $|n| \geq 58$, we have $|n\lambda - \lambda_i| \geq \frac{8}{3}|n|$. The proof requires the computation of the 19.386 first divisors (using interval arithmetic).

OK, what about convergence of ϕ?

Convergence of ϕ:

Majorants: Find a $\hat{F}\colon \mathbb{R} \to \mathbb{R}$ such that $|F_i(r,r,r)| \le \hat{F}(r)$, and let

$$\Omega(k) = \min_{|n|=k} \min_i \{|n\lambda - \lambda_i| \colon n \in \mathbb{V}_{10}\}.$$

Then ϕ converges whenever $\Psi(r) = \sum c_k r^k$ does, where

$$c_k = \frac{1}{\Omega(k)} \big[\hat{F}(r + \sum_{j=2}^{k-1} c_j r^j)\big]_k.$$

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Proposition: The change of variables satisfies

$$\|\phi\|_r \le \frac{r^2}{2} \qquad r \le 1,$$

and the normal form satisfies

$$\|G\|_r \le 7 \cdot 10^{-9} \frac{r^{20}}{1-3r} \qquad r < \frac{1}{3}.$$

For the proof we need the 186.576 first coefficients of ϕ.

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- I am very grateful to Jacob Palis and Lennart Carleson for suggesting this problem to me.

References

J. Guckenheimer and R. F. Williams, *Structural Stability of Lorenz Attractors*, Publ. Math. IHES **50**, 307–320, 1979.

S. P. Hastings and W. C. Troy, *A Shooting Approach to the Lorenz Equations*, Bull. Amer. Math. Soc. **27:2**, 298–303, 1992.

E. N. Lorenz, *Deterministic Non-periodic Flow*, J. Atmos. Sci. **20**, 130–141, 1963.

K. Mischaikow and M. Mrozek, *Chaos in the Lorenz equations: a computer-assisted proof*, Bull. Amer. Math. Soc. **32:1**, 66–72, 1995.

R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20:2**, 7–15, 1998.

W. Tucker, *The Lorenz attractor exists*, C. R. Acad. Sci. Paris, t. 328, Série I, 1197–1202, 1999.

W. Tucker, *A Rigorous ODE Solver and Smale's 14th Problem*, Found. Comput. Math. **2:1**, 53–117, 2002.