Universität Rostock

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Chair of Mechatronics

Experimental Validation of Interval Sliding Mode Observers for Nonlinear Systems with Bounded Measurement and Parameter Uncertainty

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Contents

- Motivation
- Interval Sliding Mode Observer (ISMO)
- Lyapunov functions
- Optimal input design
- Experimental setup
- Results
- Conclusions and outlook on further work

1 Motivation

2 ISMO

3 Lyapunov Functions

4 Optimal Input Design

5 Experiment

6 Results

7 Conclusions

Motivation

- Characterization of nonlinear dynamic systems
- Common situation: non-measurable states and unknown or uncertain parameters

Uncertainty

- Lack of knowledge about system parameters
- Inaccurate measurements
- Manufacturing tolerances

- Simultaneous state estimation and parameter identification necessary
- State-of-the-art sliding mode techniques have to satisfy restrictive matching conditions

Interval Sliding Mode Observer

- Intervals defining tolerance bounds for parameters and measured data $\rightarrow$ advantage: reduction of chattering
- Suitable candidates for Lyapunov functions
  - Guarantee for asymptotic stability
  - Used for calculation of switching amplitude
- Adaptation of switching amplitude of the observer's variable structure part $\rightarrow$ reduction of amplification of measurement noise
- Simultaneous state estimation and parameter identification
- Implementation using C++ S-functions in MATLAB with software library C-XSC
- Optimal input design for improved parameter estimation[1]

[1] Senkel, Luise; Rauh, Andreas; Aschemann, Harald: *Optimal Input Design for Online State and Parameter Estimation using Interval Sliding Mode Observers*, 52nd IEEE Conference on Decision and Control CDC 2013, Firenze, Italy, 2013. Under review.

1. Motivation
2. ISMO
3. Lyapunov Functions
4. Optimal Input Design
5. Experiment
6. Results
7. Conclusions

Classical Sliding Mode Observer (1)

Subdivision into two parts: continuous and variable structure

Continuous structure (model of the dynamic system)

- Assume a dynamic system $\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$
- Representation by set of state equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{S} \cdot \mathbf{w}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t)$$

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$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t)$$

- $\mathbf{x}(t)$ – state vector (contains uncertain but bounded parameters)
- $\mathbf{A}$, $\mathbf{B}$ – constant system and input matrices
- $\mathbf{u}(t)$ – vector-valued control signal
- $\mathbf{y}(t)$ – (linear) system output with constant matrix $\mathbf{C}$

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$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t)$$

- $\mathbf{S} \in \mathbb{R}^{n \times q}$ – influence of a-priori unknown terms on system dynamics, condition $\|\mathbf{w}\left(\mathbf{x}, \mathbf{u}\right)\| \leq \overline{\mathbf{w}}$ (fixed upper bound for the vector norm)
- $\mathbf{S} \cdot \mathbf{w}\left(\mathbf{x}(t), \mathbf{u}(\mathbf{t})\right)$ contains all nonlinearities

Classical Sliding Mode Observer (2)

Variable structure observer representation (used for estimation)

$$\dot{\hat{\mathbf{x}}}(t) = \underbrace{\hat{\mathbf{A}} \cdot \hat{\mathbf{x}}(t) + \hat{\mathbf{B}} \cdot \mathbf{u}(t)}_{=\hat{\mathbf{f}}(\hat{\mathbf{x}}(t),\mathbf{u}(t))} + h_s \cdot \mathbf{S} \cdot \tilde{\mathbf{e}} + \mathbf{H}_p \cdot \Big(\mathbf{y}_m(t) - \hat{\mathbf{C}} \cdot \hat{\mathbf{x}}(t) \Big)$$

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- $\hat{\mathbf{x}}$, $\hat{\mathbf{A}} = \mathbf{A}(\hat{\mathbf{x}})$, $\hat{\mathbf{B}} = \mathbf{B}(\hat{\mathbf{x}})$ and $\hat{\mathbf{C}} = \mathbf{C}(\hat{\mathbf{x}})$ − corresponding state vector and matrices of observer parallel model
- h_s − scaling factor → guarantees asymptotic stability in spite of nonlinearities and uncertainties
- $\mathbf{H}_p$ − observer gain matrix
  - ▷ stabilizing error dynamics of the linear part
  - ▷ usually determined by pole assignment
- $\mathbf{y}_m(t)$ − measured system outputs

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- Error vector $\tilde{\mathbf{e}} = \frac{\mathbf{S}^T \mathbf{P}(\mathbf{x}-\hat{\mathbf{x}})}{\|\mathbf{S}^T \mathbf{P}(\mathbf{x}-\hat{\mathbf{x}})\|}$ accounts for deviations between true and estimated system states
- Matrix $\mathbf{P}$ results from solving the Lyapunov equation $\mathbf{A}_O \cdot \mathbf{P} + \mathbf{P} \cdot \mathbf{A}_O^T + \mathbf{Q} = \mathbf{0}$ with $\mathbf{A}_O = \hat{\mathbf{A}} - \mathbf{H}_p \cdot \hat{\mathbf{C}}$
- Requirements for applicability of observer:
  - Pair $\left(\hat{\mathbf{A}}, \hat{\mathbf{C}}\right)$ is observable
  - Unknown and nonlinear terms included in $\mathbf{S} \cdot \mathbf{w}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ are bounded

Classical Sliding Mode Observer (3)

Adaptation of observer differential equation

- If change of sign in $\mathbf{C}(\mathbf{x}-\hat{\mathbf{x}})$: term $\mathbf{w}$ is reproduced approximately by the variable structure part of the observer
- (Matching) condition: $\mathbf{S}\cdot\mathbf{w}\approx h_s\cdot\mathbf{S}\cdot\tilde{\mathbf{e}}\approx\tilde{\mathbf{S}}\cdot h_s\cdot\mathrm{sign}\left(\mathbf{y}_m-\hat{\mathbf{C}}\hat{\mathbf{x}}\right)$, identical structure of $\mathbf{S}$ and $\tilde{\mathbf{S}}$
- Stabilization of the error dynamics in spite of nonlinearities

$$\dot{\hat{\mathbf{x}}}=\underbrace{\hat{\mathbf{A}}\cdot\hat{\mathbf{x}}+\hat{\mathbf{B}}\cdot\mathbf{u}}_{=\hat{\mathbf{f}}(\hat{\mathbf{x}},\mathbf{u})}+\tilde{\mathbf{S}}\;h_s\,\mathrm{sign}\left(\mathbf{y}_m-\hat{\mathbf{C}}\hat{\mathbf{x}}\right)+\mathbf{H}_p\left(\mathbf{y}_m-\hat{\mathbf{C}}\hat{\mathbf{x}}\right)$$

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Observer parallel model, locally valid and linear

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Switching term

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Observer parallel model, locally valid and linear

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Observer gain matrix for linear part of the system

Interval Sliding Mode Observer (1)

- Goal: extension of classical observer
  $\Rightarrow$ simultaneous estimation of system states and identification of constant parameters

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- New observer structure to handle
  - Uncertainty (caused by a lack of knowledge about specific parameters)
  - Inaccuracies (due to inevitable design and manufacturing tolerances)
  - Unavoidable external disturbances

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  - Unavoidable external disturbances
- Interval variables for
  - Uncertain parameters
  - Disturbances
  - Measurement, estimation and control errors

$\rightarrow$ Range description in which true values are located

Interval Sliding Mode Observer (2)

Classical Sliding Mode Observer

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{B}} \cdot \mathbf{u} + \tilde{\mathbf{S}}\, h_s \operatorname{sign}\left(\mathbf{y} - \hat{\mathbf{C}}\hat{\mathbf{x}}\right) + \mathbf{H}_p \left(\mathbf{y}_m - \hat{\mathbf{C}}\hat{\mathbf{x}}\right)$$

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Interval Sliding Mode Observer: description by the set of ODEs

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\left(\hat{\mathbf{x}}(t)\right) \cdot \hat{\mathbf{x}}(t) + \mathbf{B}\left(\hat{\mathbf{x}}(t)\right) \cdot \mathbf{u}(t) + \mathbf{H}_p \cdot \mathbf{e}_m(t) + \mathbf{P}^{+} \cdot \mathbf{C}\left(\hat{\mathbf{x}}(t)\right) \cdot \mathbf{H}_s \cdot \mathrm{sign}\left(\mathbf{e}_m(t)\right)$$

Interval Sliding Mode Observer (2)

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$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{B}} \cdot \mathbf{u} + \tilde{\mathbf{S}}\, h_s \operatorname{sign}\left(\mathbf{y} - \hat{\mathbf{C}}\hat{\mathbf{x}}\right) + \mathbf{H}_p \left(\mathbf{y}_m - \hat{\mathbf{C}}\hat{\mathbf{x}}\right)$$

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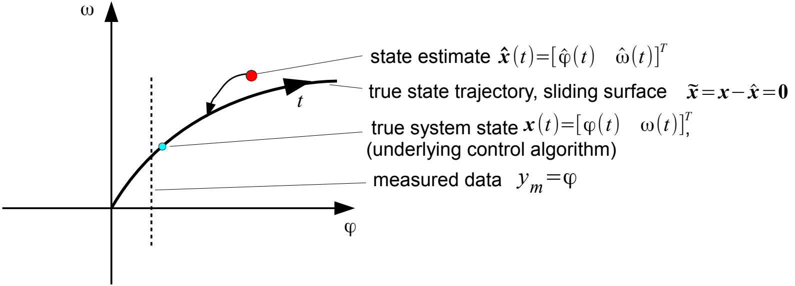
- Matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are now no longer assumed to be constant
- Difference between measured and estimated output $\mathbf{e}_m(t) = \left(\mathbf{y}_m(t) - \hat{\mathbf{C}}\left(\hat{\mathbf{x}}(t)\right) \cdot \hat{\mathbf{x}}(t)\right)$
- Switching amplitude $\mathbf{H}_s = \operatorname{diag}\left(\mathbf{h}_s\right)$
- More than just one switching amplitude (depends on number of outputs)

Interval Sliding Mode Observer (3)

- Example: system with two states; angle φ and angular velocity $\omega = \frac{d\ \varphi}{d\ t}$
- Goal: location of estimated states near sliding surface $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}} = \mathbf{0}$



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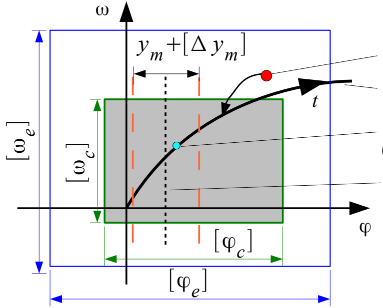

state estimate $\hat{\boldsymbol{x}}(t)=[\hat{\varphi}(t) \quad \hat{\omega}(t)]^T$

true state trajectory, sliding surface $\tilde{\boldsymbol{x}}=\boldsymbol{x}-\hat{\boldsymbol{x}}=\mathbf{0}$

true system state $\boldsymbol{x}(t)=[\varphi(t) \quad \omega(t)]^T$, located in $[\varphi_c],[\omega_c]$ (underlying control algorithm)

measured data $y_m=\varphi$

control error intervals $[\varphi_c]=\varphi_d+[\Delta\varphi_c]$ $[\omega_c]=\omega_d+[\Delta\omega_c]$

estimation error intervals $[\varphi_e]=\hat{\varphi}+[\Delta\varphi_e]$ $[\omega_e]=\hat{\omega}+[\Delta\omega_e]$
-stabilize error dynamics
-worst-case bounds despite bounded measurement noise

bounded measurement noise $[\Delta y_m]$

tolerances $[\Delta\varphi_e],[\Delta\varphi_c],[\Delta\omega_e],[\Delta\omega_c]$

interval variables $v\in[\underline{v};\overline{v}]=[\inf(v);\sup(v)]$

interval vectors $\boldsymbol{v}\in[\underline{\boldsymbol{v}};\overline{\boldsymbol{v}}]$ $(\underline{v}\leq\overline{v})$

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Lyapunov Functions: Stability Proof (1)

- Calculation of observer gain $\mathbf{H}_p$ for quasi-linear system part as a constant matrix for a fixed operating point
- On this basis: construction of a suitable Lyapunov function
- Goal: ensure stability by online computation of the switching amplitude $\mathbf{h}_s$

Lyapunov function

$$V(t) = \frac{1}{2}\mathbf{e}(t)^T\mathbf{P}\mathbf{e}(t) > 0 \;\; \text{with} \;\; \mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t), \;\; \mathbf{P} = \mathbf{P}^T$$

Time derivative of the Lyapunov function

$$\dot{V}(t) = \mathbf{e}(t)^T\,\mathbf{P}\,\dot{\mathbf{e}}(t) = (\mathbf{x}(t) - \hat{\mathbf{x}}(t))^T\mathbf{P}(\dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(\mathbf{t}))$$

Lyapunov Functions: Stability Proof (2)

Time derivative of the Lyapunov function

$$\dot{V}(t) = \mathbf{e}(t)^T \, \mathbf{P} \, \dot{\mathbf{e}}(t) = (\mathbf{x}(t) - \hat{\mathbf{x}}(t))^T \mathbf{P} (\dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(\mathbf{t}))$$

- Stability proof is successful if $\dot{V}(t) < 0$ holds
- Evaluation of $\dot{\mathbf{x}}(t)$ and $\dot{\hat{\mathbf{x}}}(t)$ in real-time for all possible parameters and states
- Intervals for parameters, control, estimation and measurement errors included

Lyapunov Functions: Switching Amplitude (1)

Reformulation of the time derivative of the Lyapunov function

$$\dot{V} = \mathbf{e}^T\mathbf{P}\cdot\left(\mathbf{f}-\hat{\mathbf{f}}-\mathbf{H}_p\mathbf{e}_m-\mathbf{P}^+\mathbf{C}^T\mathbf{H}_s\cdot\mathrm{sign}\left(\mathbf{e}_m\right)\right)$$
$$= \underbrace{\mathbf{e}^T\mathbf{P}\cdot\left(\mathbf{f}-\hat{\mathbf{f}}-\mathbf{H}_p\mathbf{e}_m\right)}_{\dot{V}_a\in\left[\dot{V}_a\right]} + \mathbf{h}_s^T\cdot\underbrace{\left(-\mathbf{CPP}^+\mathbf{C}^T\mathbf{C}\cdot\mathrm{diag}\{\mathbf{e}\}\cdot\mathrm{sign}\left(\mathbf{e}\right)\right)}_{\dot{\mathbf{V}}_b=-|\mathbf{e}_m(t)|\in-|[\mathbf{e}_m(t)]|}$$

- Matrix $\mathbf{P}$ results from solving the Lyapunov equation $\mathbf{A}_O\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{A}_O^T+\mathbf{Q}=\mathbf{0}$ with $\mathbf{A}_O=\hat{\mathbf{A}}-\mathbf{H}_p\cdot\hat{\mathbf{C}}$
- Worst-case bounds for the error vector $\mathbf{e}$ correspond to $[\mathbf{e}]=[\mathbf{x}_c]-[\mathbf{x}_e]$
- $\dot{\mathbf{V}}_b=-\left|\mathbf{e}_m(t)\right|$ holds with $\mathbf{e}_m=\mathbf{C}\cdot\mathbf{e}$, if $\mathbf{C}$ describes the direct measurement of state variables

Lyapunov Functions: Switching Amplitude (2)

Calculation of the switching amplitude

$\dot{V} = \dot{V}_a + \mathbf{h}_s^T \cdot \dot{\mathbf{V}}_b = \dot{V}_a - \mathbf{h}_s^T \cdot |[\mathbf{e}_m]| < 0$

$$\mathbf{h}_s \begin{cases} = \mathbf{0} \ , & \text{if } 0 \in |[\mathbf{e}_m]|^T \, |[\mathbf{e}_m]| \\ \geq \sup \left(|[\mathbf{e}_m]|^+ \cdot \left[\dot{V}_a \right] \right) \ , & \text{else} \end{cases}$$

Two cases because of denominator of interval pseudo inverse $|[\mathbf{e}_m]|^+ = \left(|[\mathbf{e}_m]|^T \, |[\mathbf{e}_m]| \right)^{-1} |[\mathbf{e}_m]|^T$ with the interval $[\mathbf{e}_m] = \mathbf{e}_m(t) + [\Delta \mathbf{y}_m]$ and $\mathbf{e}_m(t) = \mathbf{y}_m(t) - \hat{\mathbf{y}}_m(t)$

$\mathbf{h}_s = \mathbf{0}$, if

- $\mathbf{0} \in [\tilde{\mathbf{x}}] = [\mathbf{x}] - [\hat{\mathbf{x}}]$ or
- $0 \in |[\mathbf{e}_m]|^T \, |[\mathbf{e}_m]|$

$\Rightarrow$ Corresponds to deactivation of the variable structure part

$\Rightarrow$ Continuous part $\dot{V}_a(t)$ has to stabilize the system

Lyapunov Functions: Switching Amplitude (3)

- Online adaptation of the switching amplitude in each discretization step t_k
- Iterative adjustment of $\mathbf{h}_s$ as long as $\sup\left(\left[\dot{V}(t)\right]\right) > 0$
- Guaranteed stability proof with minimum noise amplification
- Avoids instabilities that might be caused by using a finite discretization period
- Euler discretization of $\dot{V}(t)$ and observer ODEs
- Reason: less time consuming in case of nonlinear high-dimensional processes than online evaluation of $\dot{V}_a(t)$ in which $\mathbf{f}$ and $\hat{\mathbf{f}}$ have to be calculated separately

Lyapunov Functions: Switching Amplitude (4)

If $0 \notin |[\mathbf{e}_m]|^T \, |[\mathbf{e}_m]|$: Overapproximation of $\dot{V}$ by Euler discretization

$$\dot{V}(t_{k+1}) \in \left[\dot{V}(t_{k+1})\right] = [\mathbf{e}(t_{k+1})]^T \, \mathbf{P} \, [\dot{\mathbf{e}}(t_{k+1})]$$

$[\mathbf{e}(t_k)] = [\mathbf{x}_c] - [\mathbf{x}_e]$ with $[\mathbf{x}_c] = [[\varphi_c] \quad [\omega_c]]^T$ and $[\mathbf{x}_e] = [[\varphi_e] \quad [\omega_e]]^T$

$$[\mathbf{e}(t_{k+1})] = [\mathbf{x}(t_{k+1})] - [\hat{\mathbf{x}}(t_{k+1})] + [\mathbf{x}_e]$$

$$[\dot{\mathbf{e}}(t_{k+1})] = \frac{[\mathbf{e}(t_{k+1})] - [\mathbf{e}(t_k)]}{T}$$

$$\mathbf{x}(t_{k+1}) \in \mathbf{x}(t_k) + T \cdot [\dot{\mathbf{x}}(t_k)]$$

$$\hat{\mathbf{x}}(t_{k+1}) \in \hat{\mathbf{x}}(t_k) + T \cdot \left[\dot{\hat{\mathbf{x}}}(t_k)\right]$$

- discretization errors are assumed to be small enough $\Rightarrow$ higher order terms for calculation of $\mathbf{x}(t_{k+1})$ and $\hat{\mathbf{x}}(t_{k+1})$ omitted
- sampling time: $T = 1\text{ms}$

Lyapunov Functions: Switching Amplitude (5)

$$\mathbf{h}_s \begin{cases} = \mathbf{0} \ , & \text{if } 0 \in |[\mathbf{e}_m]|^T \, |[\mathbf{e}_m]| \, , \ \sup \left[\dot{V}(t_{k+1})\right] < 0 \\ = \text{adaptive scheme}^2 \ , & \text{if } 0 \in |[\mathbf{e}_m]|^T \, |[\mathbf{e}_m]| \, , \ \sup \left[\dot{V}(t_{k+1})\right] > 0 \\ \geq \sup \left(|[\mathbf{e}_m]|^+ \cdot \left[\dot{V}(t_{k+1})\right] \right) \ , & \text{else} \end{cases}$$

[2]Heuristic for calculation of switching amplitude in such a way that $\mathbf{h}_s$ is adapted as long as $\sup\left(\left[\dot{V}(t_{k+1})\right]\right) > 0 \rightarrow \mathbf{h}_s$ as small as possible

[2]Senkel, Luise; Rauh, Andreas; Aschemann, Harald: *Interval-Based Sliding Mode Observer Design for Nonlinear Systems with Bounded Measurement and Parameter Uncertainty*, IEEE Intl. Conference on Methods and Models in Automation and Robotics MMAR 2013, Miedzyzdroje, Poland, 2013. Accepted.

Lyapunov Functions: Extensions

Extension 1: Guarantee minimum convergence rate for measured quantities

$$\dot{V}(t) < -\mathbf{e}_m(t)^T \cdot \mathbf{Q} \cdot \mathbf{e}_m(t) < 0,\ \mathbf{Q} > 0$$

$$\mathbf{h}_s \begin{cases} = \mathbf{0}\ , & \text{if } 0 \in |[\mathbf{e}_m]|^T\, |[\mathbf{e}_m]| \\ \geq \sup\left(|[\mathbf{e}_m]|^+ \cdot \left(\left[\dot{V}_a\right] + |[\mathbf{e}_m]|^T\, \mathbf{Q}\, |[\mathbf{e}_m]| \right)\right), & \text{else} \end{cases}$$

Lyapunov Functions: Extensions

Extension 1: Guarantee minimum convergence rate for measured quantities

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Extension 2: Guarantee minimum convergence rate for vector of estimated variables

$$\dot{V}(t) < -\mathbf{e}(t)^T \cdot \mathbf{Q} \cdot \mathbf{e}(t) < 0,\ \mathbf{Q} > 0$$

$$\mathbf{h}_s \begin{cases} = \mathbf{0}\ , & \text{if } 0 \in |[\mathbf{e}_m]|^T\, |[\mathbf{e}_m]| \\ \geq \sup\left(|[\mathbf{e}_m]|^+ \cdot \left(\left[\dot{V}_a\right] + |[\mathbf{e}]|^T\, \mathbf{Q}\, |[\mathbf{e}]| \right) \right), & \text{else} \end{cases}$$

Lyapunov Functions: Extensions

Extension 2: Guarantee minimum convergence rate for vector of estimated variables

$\dot{V}(t) < -\mathbf{e}(t)^T \cdot \mathbf{Q} \cdot \mathbf{e}(t) < 0$, $\mathbf{Q} > 0$

$$\mathbf{h}_s \begin{cases} = \mathbf{0} \;, & \text{if } 0 \in \left|[\mathbf{e}_m]\right|^T \left|[\mathbf{e}_m]\right| \\ \geq \sup\left(\left|[\mathbf{e}_m]\right|^+ \cdot \left(\left[\dot{V}_a\right] + \left|[\mathbf{e}]\right|^T \mathbf{Q} \left|[\mathbf{e}]\right|\right)\right), & \text{else} \end{cases}$$

Extension 3: Linear weighting of the estimation errors

$\dot{V}(t) < -\mathbf{q}^T \cdot |\mathbf{e}_m| < 0$, component-wise strictly positive vector $\mathbf{q}$

$$\mathbf{h}_s \begin{cases} = \mathbf{0} \;, & \text{if } 0 \in \left|[\mathbf{e}_m]\right|^T \left|[\mathbf{e}_m]\right| \\ \geq \sup\left(\left|[\mathbf{e}_m]\right|^+ \cdot \left[\dot{V}_a\right]\right) + \mathbf{q}^T \;, & \text{else} \end{cases}$$

1 Motivation

2 ISMO

3 Lyapunov Functions

4 Optimal Input Design

5 Experiment

6 Results

7 Conclusions

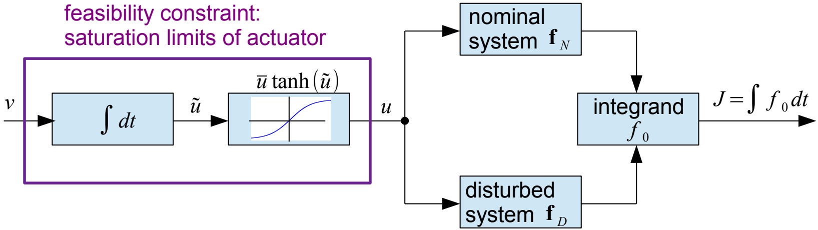
Optimal Input Design for Trajectory Planning (1)

- Goal: Improve observability of the system by a suitable excitation of the system dynamics
- Reason: Some system parameters are slowly varying (e.g. friction coefficient)
- States (angle, angular velocity etc.) vary faster than parameters
- Use of Pontryagin's Maximum Principle[3] to find optimal inputs which maximize the deviation between nominal and disturbed system outputs

[3] Senkel, Luise; Rauh, Andreas; Aschemann, Harald: *Optimal Input Design for Online State and Parameter Estimation using Interval Sliding Mode Observers*, 52nd IEEE Conference on Decision and Control CDC 2013, Firenze, Italy, 2013. Under review.

Optimal Input Design for Trajectory Planning (2)

Pontryagin's Maximum Principle



v leads to parameterization of driving cycle in the experiment

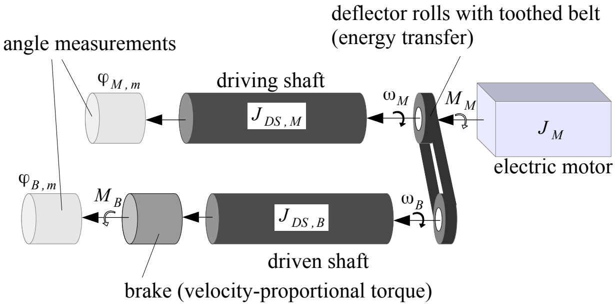
$\tilde{u}$ smooth virtual input

$u = \ddot{\varphi}_d$ actual bounded (optimal) input

Goal: Minimize J by maximization of the deviation between $\boldsymbol{x}_N$ and $\boldsymbol{x}_D$ in f_0

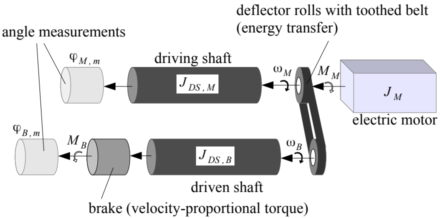
1. Motivation
2. ISMO
3. Lyapunov Functions
4. Optimal Input Design
5. Experiment
6. Results
7. Conclusions

Experimental Setup: Test Rig (1)



- Motor torque M_M, braking torque M_B
- Angular velocity of the motor ω_M
- Measured angles $\varphi_{M,m}$ as well as $\varphi_{B,m}$
- J_{rot} contains all mass moments of inertia $J_{DS,M}$, $J_{DS,B}$, J_M with respect to the driving shaft
- Braking represents a disturbance, that is identified by the observer

Experimental Setup: Test Rig (2)



System model

- ODE $J_{rot} \cdot \dot{\omega}_M = M_M - M_B$
- Motor torque (underlying control for the angle φ_M)
  $M_M = K_2 \cdot (\dot{\varphi}_{M,d} - \dot{\varphi}_M) + K_1 \cdot (\varphi_{M,d} - \varphi_M)$, desired angle $\varphi_{M,d}$, controller gains K_1 and K_2 (chosen by pole placement)
- Braking torque $M_B = k_{D_2} \cdot \omega_B$
- Transmission ratio $k = \frac{\omega_M}{\omega_B}$

Experimental Setup: Test Rig (3)

System Model (φ_M angle of rotation of the motor shaft)

$$\mathbf{f}_N = \begin{bmatrix} \dot{x}_{N1} \\ \dot{x}_{N2} \end{bmatrix} = \begin{bmatrix} \dot{\varphi}_M \\ \dot{\omega}_M \end{bmatrix} = \begin{bmatrix} \omega_M \\ \alpha \cdot \omega_M + \beta \cdot M_M \end{bmatrix}$$

Task for Interval Sliding Mode Observer

- Estimate states φ_M and ω_M
- Identify parameters $\alpha = -\frac{k_D}{J_{rot}}$ and $\beta = \frac{1}{J_{rot}}$ with $k_D = k_{D_1} + \frac{k_{D_2}}{k}$
- Unknown parameters: velocity-proportional friction k_{D_1} and mass moment of inertia J_{rot}
- Braking resistance k_{D_2} (defined by pure feedforward control)
- Software implementation: Interface between MATLAB SIMULINK and C-XSC with *Labview NI Simulation Interface Toolkit*

Experimental Setup: Test Rig (4)

System model (φ_M angle of rotation of the motor shaft)

$$\mathbf{f}_N = \begin{bmatrix} \dot{x}_{N1} \\ \dot{x}_{N2} \end{bmatrix} = \begin{bmatrix} \dot{\varphi}_M \\ \dot{\omega}_M \end{bmatrix} = \begin{bmatrix} \omega_M \\ \alpha \cdot \omega_M + \beta \cdot M_M \end{bmatrix}$$

Assumptions

- Static friction is assumed to be negligibly small
- Implementation using a cascaded observer[4]: 2 subsystems
  - First subsystem estimates φ_M and its derivatives $\rightarrow$ serves as virtual generator of measurements for second subsystem
  - Second subsystem determines the parameters α and β

[4] Senkel, Luise; Rauh, Andreas; Aschemann, Harald: *Interval-Based Sliding Mode Observer Design for Nonlinear Systems with Bounded Measurement and Parameter Uncertainty*, IEEE Intl. Conference on Methods and Models in Automation and Robotics MMAR 2013, Miedzyzdroje, Poland, 2013. Accepted.

1 Motivation

2 ISMO

3 Lyapunov Functions

4 Optimal Input Design

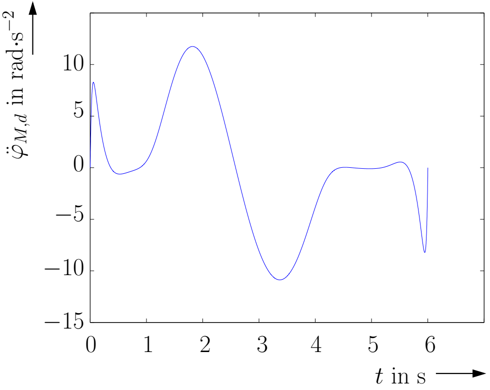
5 Experiment

6 Results

7 Conclusions

Results: Parameter Identification - Optimal Input Trajectory

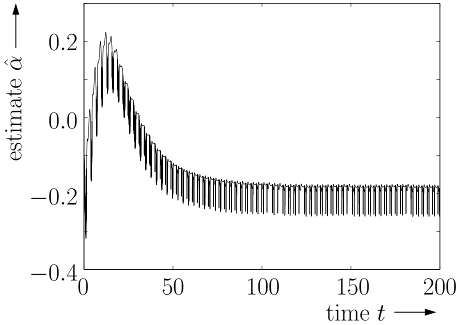
Optimal input trajectory for desired angular acceleration $\ddot{\varphi}_{M,d}$



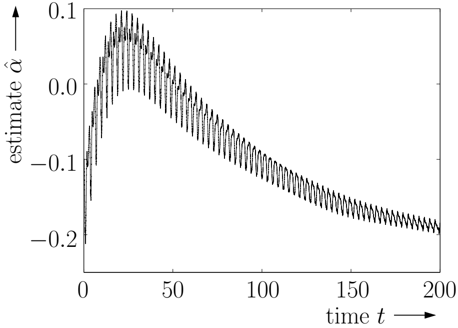
smooth trajectory, no steps, saturation limits

Results: Parameter Identification - Simulation

nominal parameters: $\alpha = -0.2$ and $\beta = 1$



(a) Estimate $\hat{\alpha}$ with ISMO.

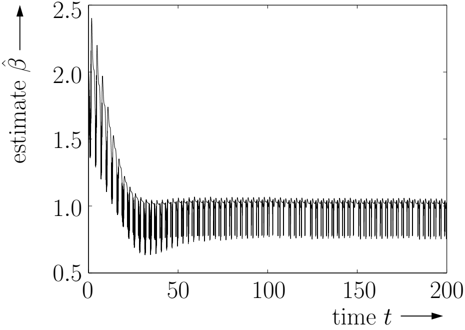


(b) Estimate $\hat{\alpha}$ with Classical SMO.

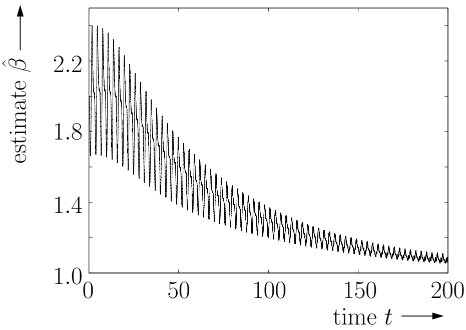
$\rightarrow$ shorter transient phases with ISMO than with classical sliding mode observer

Results: Parameter Identification - Simulation

nominal parameters: $\alpha = -0.2$ and $\beta = 1$



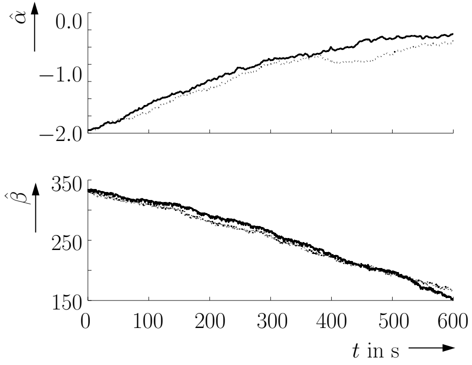
(c) Estimate $\hat{\beta}$ with ISMO.



(d) Estimate $\hat{\beta}$ with Classical SMO.

$\rightarrow$ shorter transient phases with ISMO than with classical sliding mode observer

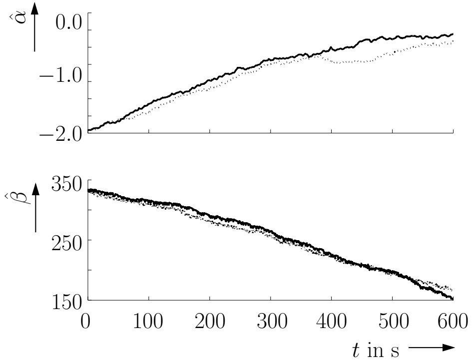
Results: Parameter Identification - Experiment



- Drive cycle length: $t_f = 6$s
- 100 repetitions
- 2 experiments

Nominal parameters (identified by open-loop control, step response analysis): $\alpha = -1.3667$ and $\beta = 166.6667$

Results: Parameter Identification - Experiment



- Drive cycle length: $t_f = 6$s
- 100 repetitions
- 2 experiments

ISMO detects deviations from nominal parameters → possible reasons:

- Phases with sliding friction play major role
- Necessity for a refined control strategy of the test rig
- Thermal dependency of braking resistance k_{D_2}?
- Delayed responding behavior of brake?

1 Motivation

2 ISMO

3 Lyapunov Functions

4 Optimal Input Design

5 Experiment

6 Results

7 Conclusions

Conclusions and Outlook

Conclusion

- Interval sliding mode observer, validated in simulation and experiment
- Identify unknown system parameters, estimate state variables

Outlook on further work

- Third parameter: static friction
- Implementation of extensions for Lyapunov functions
- Closed control loop for reliable compensation of disturbances (e.g. static and sliding friction)
- Combination with linear matrix inequalities (LMIs) for quasi-linear part of the observer
- Experimental validation of interval sliding mode observer for other real-time applications

Thank you for your attention!

$$\dot{V} = \mathbf{e}^T\mathbf{P}\dot{\mathbf{e}} = \mathbf{e}^T\mathbf{P}\cdot\left(\mathbf{f} - \hat{\mathbf{f}} - \mathbf{H}_p\mathbf{e}_m - \mathbf{P}^+\mathbf{C}^T\mathbf{H}_s\cdot\mathrm{sign}\left(\mathbf{e}_m\right)\right)$$

$$\dot{V} = \mathbf{e}^T\mathbf{P}\cdot\left(\mathbf{f} - \hat{\mathbf{f}} - \mathbf{H}_p\mathbf{e}_m\right) - \mathbf{e}^T\mathbf{P}\cdot\left(\mathbf{P}^+\mathbf{C}^T\mathbf{H}_s\cdot\mathrm{sign}\left(\mathbf{e}_m\right)\right)$$

with $\mathbf{P}\mathbf{P}^+ = \mathbf{I}$ and

$$\mathbf{e}^T\cdot\mathbf{C}^T\cdot\mathbf{H}_s\cdot\mathrm{sign}\left(\mathbf{e}_m\right)$$

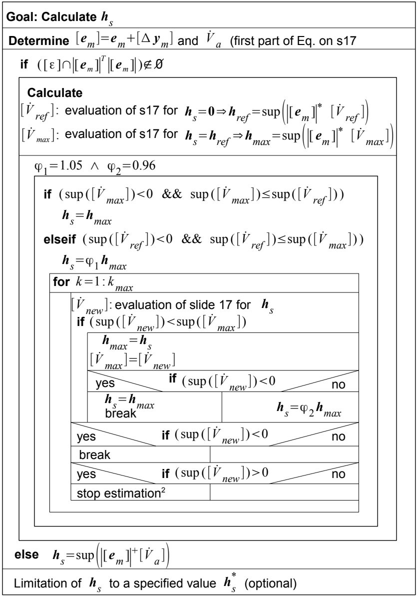
$$= \mathbf{e}^T\cdot\mathbf{C}^T\cdot\begin{bmatrix} h_{s,1}\cdot\mathrm{sign}\left(e_{m,1}\right) & 0 & \cdots & 0 \\ 0 & h_{s,2}\cdot\mathrm{sign}\left(e_{m,2}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & h_{s,n}\cdot\mathrm{sign}\left(e_{m,n}\right) \end{bmatrix}$$

$$= \mathbf{h}_s^T\cdot\mathbf{C}\cdot\mathbf{C}^T\cdot\mathbf{C}\cdot\mathrm{diag}\left(\mathbf{e}\right)\cdot\mathrm{sign}\left(\mathbf{e}\right)$$

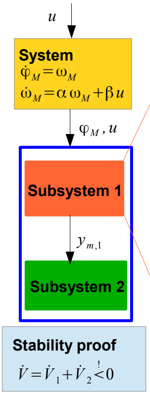
follows to

$$\dot{V} = \underbrace{\mathbf{e}^T\mathbf{P}\cdot\left(\mathbf{f} - \hat{\mathbf{f}} - \mathbf{H}_p\mathbf{e}_m\right)}_{\dot{V}_a\in\left[\dot{V}_a\right]} + \mathbf{h}_s^T\cdot\underbrace{\left(-\mathbf{C}\mathbf{P}\mathbf{P}^+\mathbf{C}^T\mathbf{C}\cdot\mathrm{diag}\{\mathbf{e}\}\cdot\mathrm{sign}\left(\mathbf{e}\right)\right)}_{\dot{\mathbf{V}}_b=-|\mathbf{e}_m(t)|\in-|[\mathbf{e}_m(t)]|}$$

Structure diagram of the guaranteed stabilizing parameterization of the variable-structure observer with a generalization according to $\|[\mathbf{e}_m]\|^* := \left(\left[\delta; \sup\left(\left|[\mathbf{e}_m]\right|^T \left|[\mathbf{e}_m]\right|\right)\right]\right)^{-1} \cdot \left|[\mathbf{e}_m]\right|^T$ with $[\epsilon] = [-\epsilon; \epsilon]$, $\epsilon > 0$ and $\delta > 0$



Structure of the Cascaded Observer



Model

$$\boldsymbol{f}_1 = \begin{bmatrix} \omega_M + [\Delta x_{c,\omega_M,1}] \\ \alpha(\omega_M + [\Delta x_{c,\omega_M,1}]) + \beta u \\ \alpha^2(\omega_M + [\Delta x_{c,\omega_M,1}]) + \alpha\beta u + \beta\dot{u} \\ \alpha^3(\omega_M + [\Delta x_{c,\omega_M,1}]) + \alpha^2\beta u + \alpha\beta\dot{u} + \beta\ddot{u} \\ \alpha^4(\omega_M + [\Delta x_{c,\omega_M,1}]) + \alpha^3\beta u + \alpha^2\beta\dot{u} + \alpha\beta\ddot{u} + \beta\ddot{u} \end{bmatrix}$$

$$y_{m,1} = \omega_M = e_{m,1}$$

Original model, 3 derivatives, model error

Observer parallel model

$$\hat{\boldsymbol{f}}_1 = \begin{bmatrix} \dot{\hat{\omega}}_M + [\Delta x_{e,\dot{\omega}_M,1}] \\ \ddot{\hat{\omega}}_M + [\Delta x_{e,\ddot{\omega}_M,1}] \\ \dddot{\hat{\omega}}_M + [\Delta x_{e,\ddot{\omega}_M,1}] \\ \hat{z}_M + [\Delta x_{e,z,1}] \\ 0 + [\Delta x_{e,\dot{z},1}] \end{bmatrix}$$

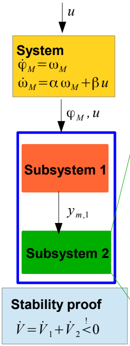
Integrator chain

Interval Sliding-Mode-Observer

$$\dot{\hat{\boldsymbol{x}}}_1 = \boldsymbol{A}_1 \hat{\boldsymbol{x}}_1 + \boldsymbol{b}_1 u + \boldsymbol{h}_{p,1} \boldsymbol{e}_{m,1} + \boldsymbol{P}_1^{+} \boldsymbol{C}_1^{T} \left(h_{s,1} .* \operatorname{sign}(e_{m,1}) \right)$$

Estimation of states: angle, angular velocity, angular acceleration,
third derivative of angular, model error of subsystem 1

Structure of the Cascaded Observer



Model

$$f_2=\begin{bmatrix}\alpha(\omega_M+[\Delta x_{c,\omega_M,2}])+\beta u+(z+[\Delta x_{c,z,2}])\\ \alpha^2(\omega_M+[\Delta x_{c,\omega_M,2}])+\alpha\beta u+\beta\dot{u}\\ 0+[\Delta x_{c,\alpha,2}]\\ 0+[\Delta x_{c,\beta,2}]\\ 0+[\Delta x_{c,z,2}]\end{bmatrix}$$

2.nd equation of original model with model error,
Derivative, integrator disturbance model for α, β, z

$$y_{m,2}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}\omega_M\\ \dot{\omega}_M\\ \alpha\\ \beta\\ z\end{bmatrix}$$

Observer parallel model

$$\hat{f}_2=\begin{bmatrix}[\alpha_I](\hat{\omega}_M+[\Delta x_{e,\omega_M,2}])+[\beta_I]u+(\hat{z}+[\Delta x_{e,z,2}])\\ [\alpha_I]^2(\hat{\omega}_M+[\Delta x_{e,\omega_M,2}])+[\alpha_I][\beta_I]u+(\beta+[\Delta\beta_\Delta])\dot{u}\\ 0+[\Delta\alpha_\Delta]\\ 0+[\Delta\beta_\Delta]\\ 0+[\Delta x_{e,z,2}]\end{bmatrix}$$

Interval Sliding-Mode-Observer

$$\dot{\hat{x}}_2=A_2\hat{x}_2+b_2u+H_{p,2}e_{m,2}+P_2^+C_2^T\left(h_{s,2}.*\mathrm{sign}(e_{m,2})\right)$$

$$e_{m,2}=\begin{bmatrix}\omega_M-\hat{\omega}_M\\ \dot{\omega}_M-\dot{\hat{\omega}}_M\\ z-\hat{z}\end{bmatrix}$$

$$[\alpha_I]=(\alpha+[\Delta\alpha_\Delta])$$

$$[\beta_I]=(\beta+[\Delta\beta_\Delta])$$

Estimation of angular velocity, angular acceleration,
system parameters, model error of subsystem 2

Optimal Input Design for Trajectory Planning

- Goal: Improve observability of the system by a suitable excitation of the dynamics
- Reason: Some system parameters are slowly varying (e.g. friction coefficient)
- States (angle, angular velocity etc.) vary faster than parameters

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Pontryagin's Maximum Principle

Optimal Input Design for Trajectory Planning

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Pontryagin's Maximum Principle

- System of ODEs $\dot{\boldsymbol{\eta}} = \begin{bmatrix} \dot{\mathbf{x}}_N & \dot{\mathbf{x}}_D & \dot{\tilde{u}} \end{bmatrix}^T = \begin{bmatrix} \mathbf{f}_N^T(\mathbf{x}_N, u) & \mathbf{f}_D^T(\mathbf{x}_D, u) & v \end{bmatrix}^T$
- State vector of a system $\mathbf{f}_N$ with nominal parameters and states $\mathbf{x}_N$
- State vector of a system $\mathbf{f}_D$ with disturbed parameters and states $\mathbf{x}_D$
- dim$\{\mathbf{x}_N\}$ =dim$\{\mathbf{x}_D\}$
- Integrator $\dot{\tilde{u}} = v$ guarantees smooth, bounded control inputs $u = \bar{u} \cdot \text{tanh}(\tilde{u})$

Optimal Input Design for Trajectory Planning

Pontryagin's Maximum Principle

- Cost function $J = \int f_0 \; dt$
- Integrand $f_0 = \frac{1}{(x_{N1} - x_{D1})^2 + 1} + \gamma_1 \cdot u^2 - \gamma_2 \cdot \left(\tanh\left(\frac{x_{N2}}{\epsilon}\right) - 1\right)$
- Hamiltonian $H = -f_0 + \boldsymbol{\xi}^T \cdot \dot{\boldsymbol{\eta}}$ to be minimized over the interval $t \in [0\; ;\; t_f]$
- Co-state vector $\boldsymbol{\xi}$
- Slope parameter $\epsilon > 0$
- Penalty terms γ_1 (weighting factor for the system input) as well as γ_2 (preventing the velocity from being negative)

Optimal Input Design for Trajectory Planning[4]

Pontryagin's Maximum Principle

- Setting the derivative $\frac{\partial H}{\partial v} = 0$, leads to the optimal input v^*
- canonical equations $\mathbf{g}_{ca}$ with the optimal input v^* are then defined as

$$\mathbf{g}_{ca}\left(v^*\right) = \left[\dot{\boldsymbol{\eta}}^T \ , \quad -\left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)^T\Bigg|_{(v=v^*)}\right]^T =: \left[\dot{\boldsymbol{\eta}}^T \ , \quad \dot{\boldsymbol{\xi}}^T\right]^T$$

- Initial and terminal conditions $\boldsymbol{\eta}(0)$, $\mathbf{x}_N(t_f)$ and $\tilde{u}(t_f)$
- Free terminal conditions $\mathbf{x}_D(t_f)$
- Solving set of canonical equations by MATLAB algorithm `bvp4c`
- Resulting input trajectory u

[4]Senkel, Luise; Rauh, Andreas; Aschemann, Harald: *Optimal Input Design for Online State and Parameter Estimation using Interval Sliding Mode Observers*, 52nd IEEE Conference on Decision and Control CDC 2013, Firenze, Italy, 2013. Under review.