Linear Relaxations in Global Optimization: Gradient-based Method and Affine Reformulation Technique.

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Introduction

We consider global optimization of Mixed Integer Non Linear Programming problems in a deterministic and reliable way.

Problem

$$\left\{\begin{array}{cl} \min\limits_{x,y\in X\times Y\subset\mathbb{R}^n\times\mathbb{Z}^m} & f(x,y) \\ s.t. & g_l(x,y)\leq 0\ ,\ \forall l\in\{1,\ldots,p\}, \\ & h_k(x,y)=0\ ,\ \forall k\in\{1,\ldots,q\}. \end{array}\right.$$

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- Comparison and Combining different kinds of reliable linear relaxation method.

$\Longrightarrow$ Accelerate resolution of a Branch and Bound Algorithm based on Interval Analysis

Branch and Bound Algorithm based on Interval Analysis

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- Reliable Linear Program $\Rightarrow$ Computing the residual of the dual by Interval Arithmetic

  (*A. Neumaier, O. Shcherbina, Math. Program., Ser. A 99: 283–296, 2004*)

Contents

Inclusion Functions based on Taylor's Expansions

Let f be a univariate differentiable function, and x, y and ξ, 3 variables of X an interval of $\mathbb{R}$.

$$f(x) = f(y) + (x-y)f'(y) + \frac{(x-y)^2}{2} f''(y) + \ldots + \frac{(x-y)^n}{n!} f^{(n)}(\xi)$$

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Hence,

$$f(x) \in f(y) + (x-y)f'(y) + \frac{(x-y)^2}{2}f''(y) + \ldots + \frac{(x-y)^n}{n!}F^{(n)}(X), \forall y \in X,$$

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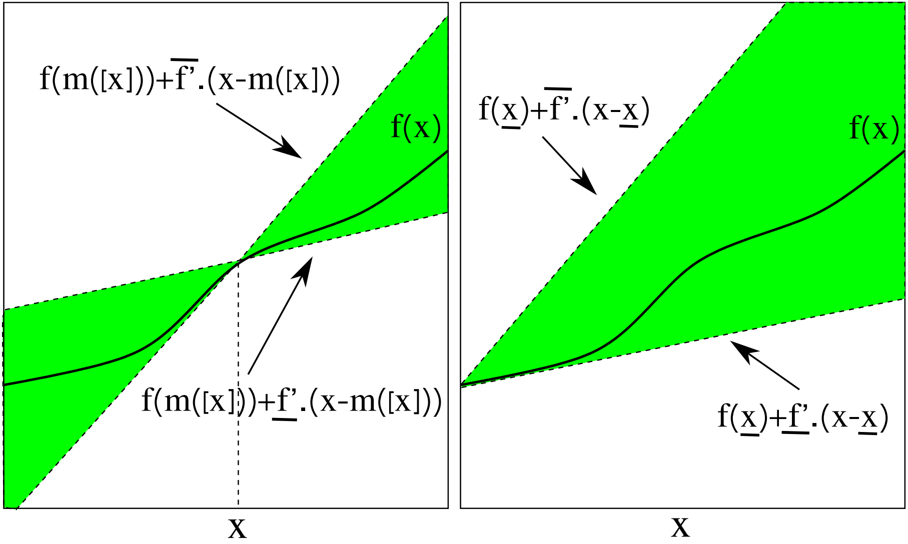
$\Longrightarrow$ Inclusion functions:

$$T_1(y, X) = f(y) + (X-y)F'(X)$$

Representation of the Taylor Inclusion Function



X-Newton Method: I.Araya, G. Trombetonni, B.Neveu

- Choose several point among the 2^n corner of the hypercube:
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- Validate the result with the Neumaier-Shcherbina's criteria.

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Affine Arithmetic: J.L.D. Comba, J. Stolfi (1993)

Definition

Each quantity is represented by an affine form $\widehat{x}$

$$\widehat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i,$$
$$\text{with } \forall i \in [0, n], x_i \in \mathbb{R} \text{ and } \epsilon_i = [-1, 1].$$

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- Example: $A = [1, 3]$ and $B = [-2, 0]$,

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$$\begin{array}{rcl} \widehat{A} & \rightarrow & 2 + \epsilon_1, \\ \widehat{B} & \rightarrow & -1 + \epsilon_2, \\ 1 + \widehat{A} & = & 3 + \epsilon_1, \\ 5 \times \widehat{B} & = & -5 + 5\epsilon_2, \\ \widehat{A} + \widehat{B} & = & 1 + \epsilon_1 + \epsilon_2. \end{array}$$

Non-Affine Operator

Multiplication

$$
\begin{aligned}
\hat{x} \times \hat{y} &= (x_0 + \sum_{i=1}^{n+1} x_i\epsilon_i) \times (y_0 + \sum_{i=1}^{n+1} y_i\epsilon_i), \\
&= x_0y_0 + \sum_{i=1}^{n} (x_0y_i + x_iy_0)\epsilon_i + \\
&\left(x_0y_{n+1} + x_{n+1}y_0 + \left(\sum_{i=1}^{n+1} \mid x_i \mid \times \sum_{i=1}^{n+1} \mid y_i \mid \right) \right) \epsilon_{\pm}.
\end{aligned}
$$

Log, exp, $\sqrt{\ }$, power, cos,...

$$
\widehat{f}(\widehat{x}) = \zeta + \alpha\widehat{x} + \delta\epsilon_{\pm},
$$

$$
\text{with } \alpha, \delta, \zeta \in \mathbb{R} \text{ and } \widehat{x} = x_0 + \sum_{i=1}^{n} x_i\epsilon_i.
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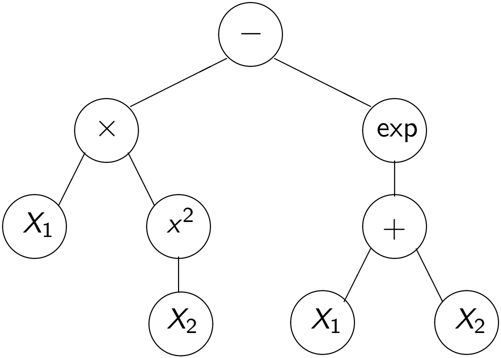
Visualization of AA by expression tree

$\forall x \in [1,2]\times[2,6], f(x) = x_1 x_2^2 - \exp(x_1 + x_2)$



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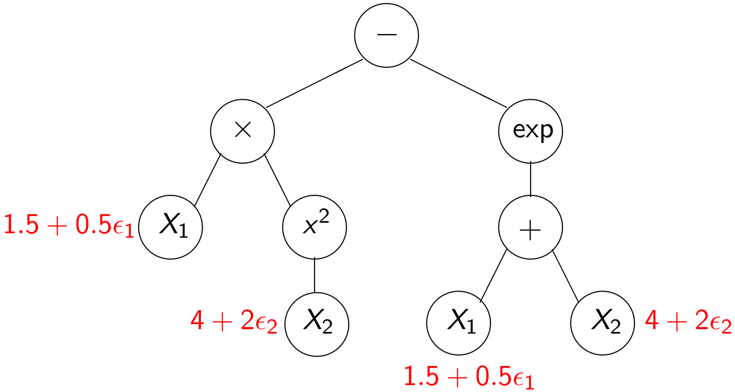

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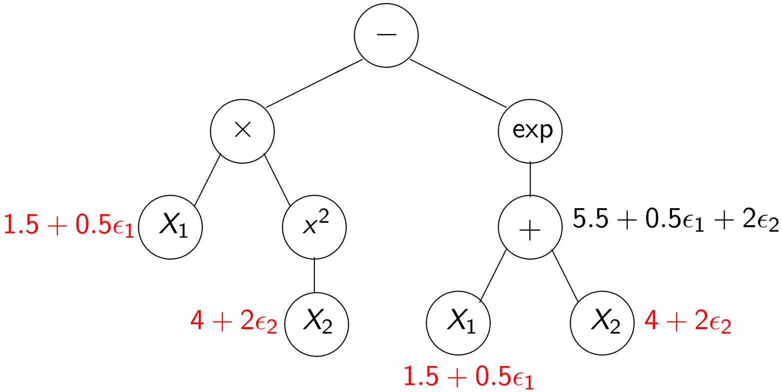

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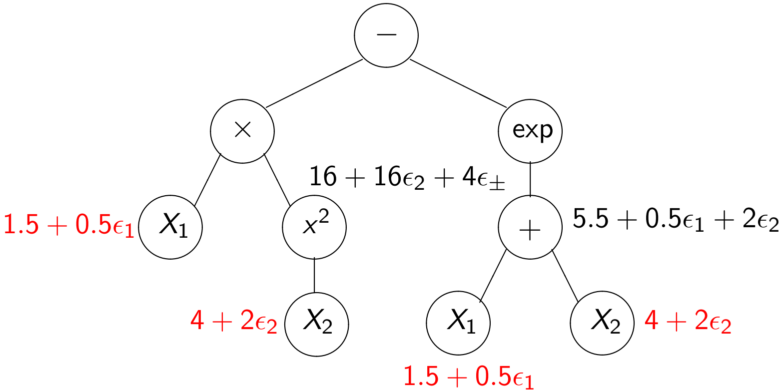

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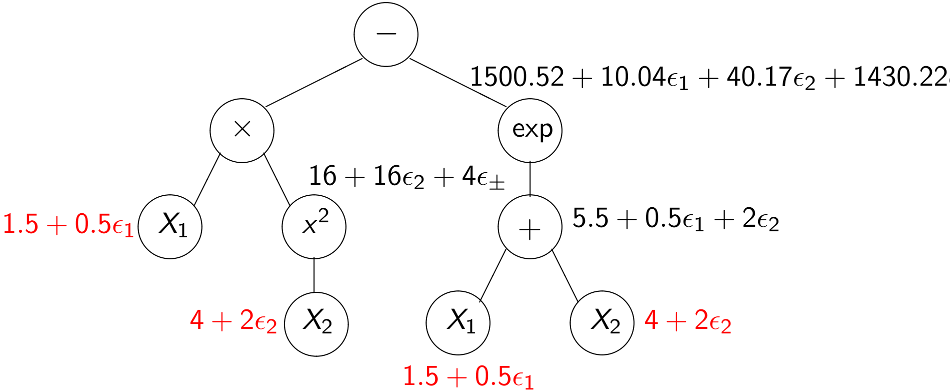

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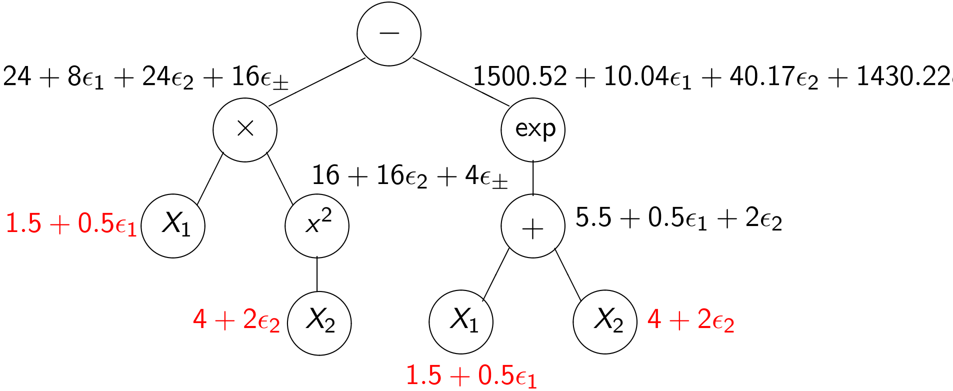

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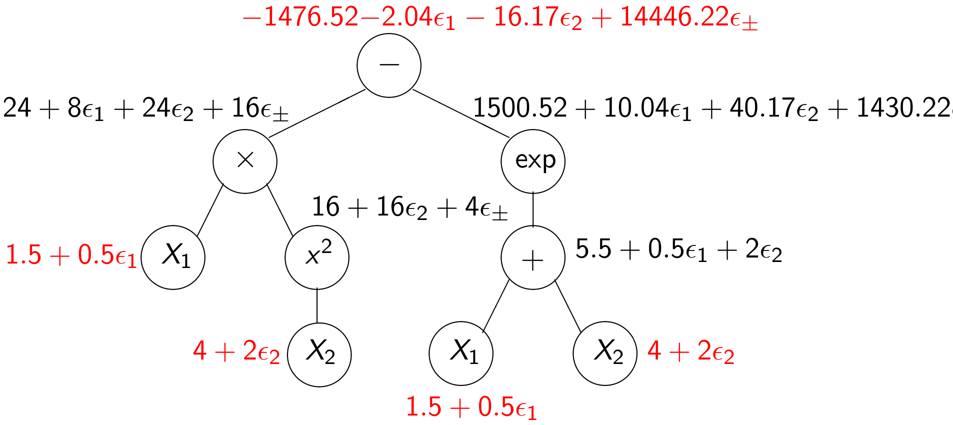

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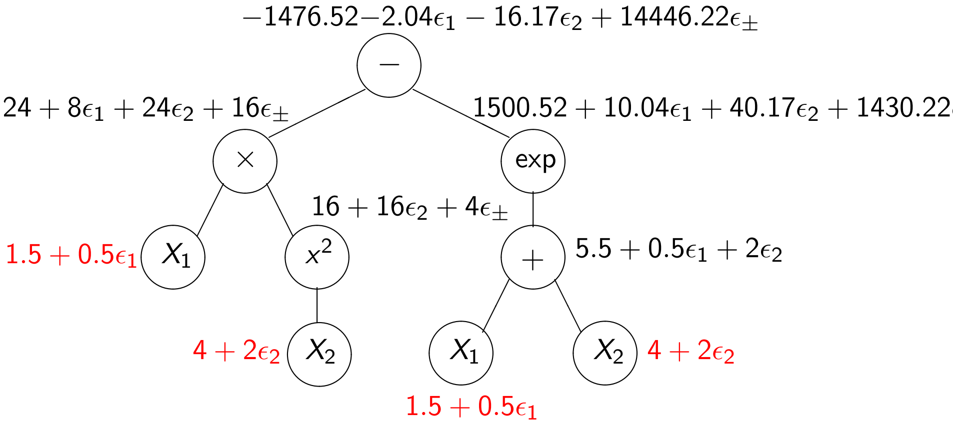
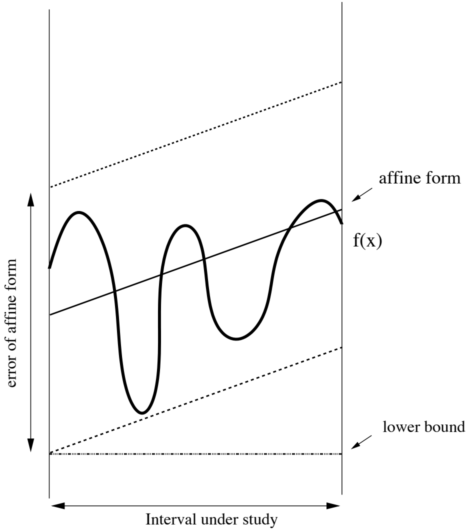
$$\forall x \in [1,2]\times[2,6],\ f(x) = x_1 x_2^2 - \exp(x_1 + x_2) \in [-2940.9579, -12.0855]$$



Graphical Representation



Affine Reformulation Technique: J.Ninin, F.Messine, P.Hansen

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$$\forall x \in X, z = \mathcal{T}(x), f(x) - \sum_{i=1}^{n} f_i z_i \in [f_0 - f_{\pm}, f_0 + f_{\pm}]$$

Reformulation of a NLP problem

$$\left\{\begin{array}{ll} \min\limits_{x\in X} & f(x) \\ s.\ t. & g_l(x) \leq 0\ ,\ \forall l \in \{1,...,p\}, \\ & h_k(x) = 0\ ,\ \forall k \in \{1,...,q\}. \end{array}\right.$$

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$$\begin{cases} \min\limits_{z \in [-1,1]^n} & \sum\limits_{i=1}^{n} f_i z_i & \\ s.\ t. & \sum_{i=1}^{n} (g_l)_i z_i \leq (g_l)_{\pm} - (g_l)_0\ , & \forall l \in \{1, ..., p\}, \\ & \sum_{i=1}^{n} (h_k)_i z_i \leq (h_k)_{\pm} - (h_k)_0\ , & \forall k \in \{1, ..., q\}, \\ & -\sum_{i=1}^{n} (h_k)_i z_i \leq (h_k)_{\pm} + (h_k)_0\ , & \forall k \in \{1, ..., q\}. \end{cases}$$

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Reformulation-Linearization-Techniques

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1. E. Smith and C. Pantelides. A symbolic reformulation/spatial branch-and-bound algorithm for the global optimisation of nonconvex MINLPs. *Computers & Chemical Engineering*, 23:457–478, 1999.

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2. W.P. Adams and H.D. Sherali, A reformulation-linearization technique for solving discrete and continuous nonconvex problems, *Nonconvex Optimization and Its Applications* , Vol. 31, Kluwer Academic Publishers, 1998.

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Integration in IBEX: G.Chabert et al.

IBEX is a library containing a deterministic global optimization algorithm based on Interval Arithmetic.

- Compare XNewton reformulation, ART and a combination,
- Improve only the lower bound or Contract the domain of each variable.

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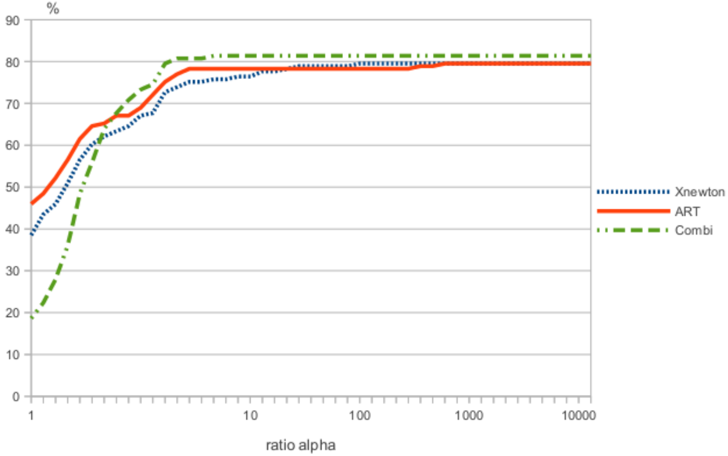
161 problems from the COCONUT database
(a library of global optimization test problems)

less than 50 variables

Comparison: Contracting the box

	Nb of success	Nb Success only by	Time	Time only if success
ART	128	3	140.67 s	24.44 s
XNewton	128	3	143.06 s	28.35 s
Combining	131	-	132.03 s	29.16 s

Performance Profiles



Conclusion

Preliminary results:

- Gradient-based Method and Affine Arithmetic-based method seem to be equivalent.
- The combination slows down the performance, but we need to test the merge of the two linearizations into one LP.

IBEX

http://www.emn.fr/z-info/ibex/