On Continuation Methods for Non-Linear Multi-Objective Optimization

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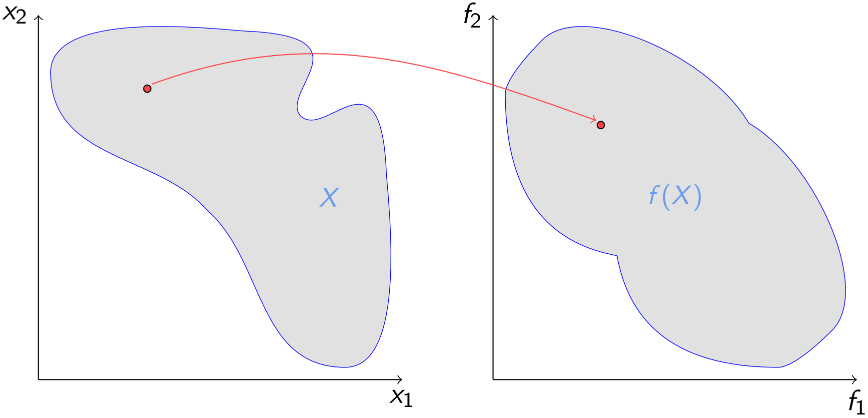
Non-Linear Multi-Objective Optimization



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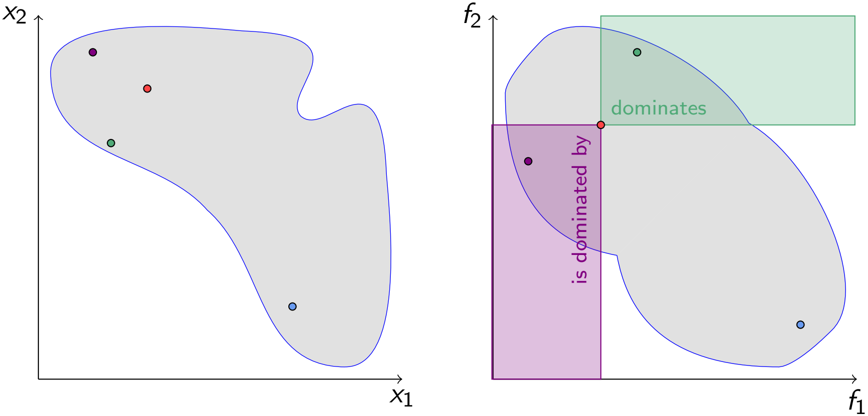
$$x \in X \subseteq \mathbb{R}^n$$

Non-Linear Multi-Objective Optimization



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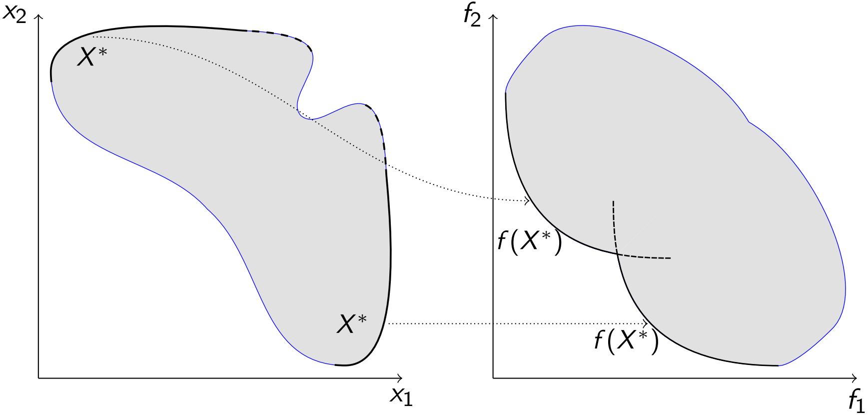

X^* set of non-dominated solutions: Pareto solutions (plain lines)
$f(X^*)$ set of non-dominated outcomes: Pareto set (plain lines)

Non-Linear Multi-Objective Optimization

General Non-Linear Multi-Objective Optimization (NLMOO) problem:

$$\left[\begin{array}{ll} \min & f(x) \\ \text{s.t} & g(x) \leq 0 \\ & h(x) = 0 \\ & x \in \mathbb{R}^n \end{array}\right] \quad (1)$$

Let $X = \{x \in \mathbb{R}^n | g(x) \leq 0, h(x) = 0\}$.

- Objective functions: $f : \mathbb{R}^n \to \mathbb{R}^k$,
- Inequality constraints: $g : \mathbb{R}^n \to \mathbb{R}^p$,
- Equality constraints: $h : \mathbb{R}^n \to \mathbb{R}^q$.

Functions may be non-linear.

What is continuation ?

Unformal definition

Local approximation/coverage of a manifold of solutions.

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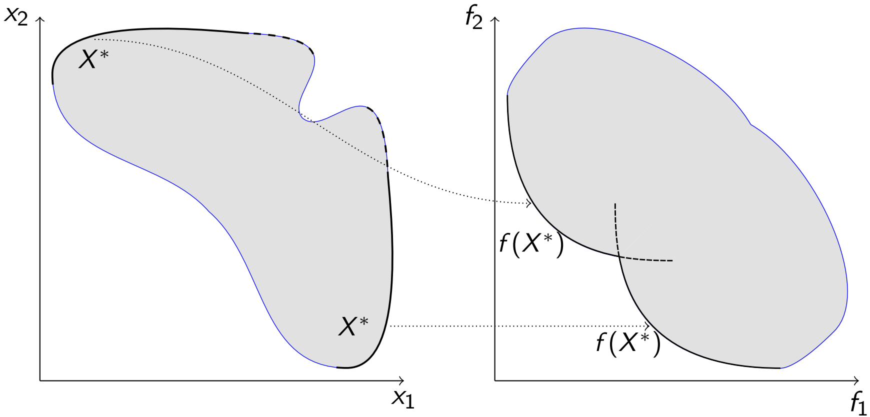
Unformal definition

Local approximation/coverage of a manifold of solutions.

- Local mean the use of local informations/observations,
- Solutions: of a system of equations, an optimization problem, . . . ; inducing (implicit) parameters,
- In NLMOO, when regular:
  - Two objectives → Manifold of dimension 1 (curves of solutions),
  - Three objectives → Manifold of dimension 2 (surfaces of solutions),
  - . . .

Continuation in Non-Linear Multi-Objective Optimization



X^* manifold of non-dominated solutions (plain lines)

$f(X^*)$ manifold of non-dominated outcomes (plain lines)

Scalarizing Methods

Non-Linear Multi-Objective Optimization problem

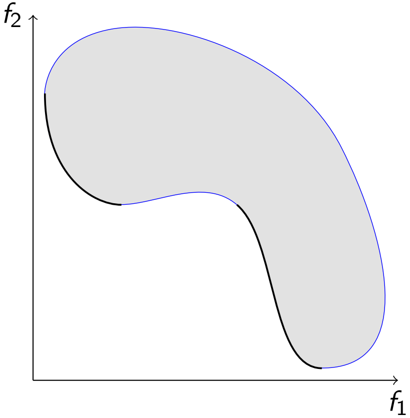
$$\begin{bmatrix} min & f(x) = (f_1(x), \dots, f_k(x)) \\ s.t & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

Scalarizing $\downarrow$

$$\begin{bmatrix} min & \hat{f}(x, v) \\ s.t & \hat{g}(x, v) \leq 0 \\ & \hat{h}(x, v) = 0 \\ & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

Sequence of Mono-objective problems, $v \in \{v_1, v_2, \dots\}$

Scalarizing Methods: Examples



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- ϵ-Constraint:
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- Normal Boundary Intersection:
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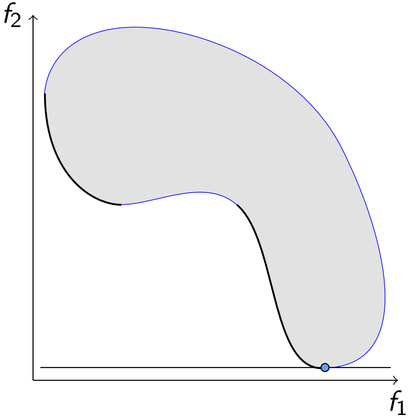

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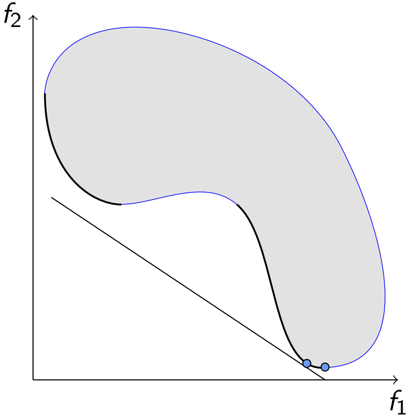

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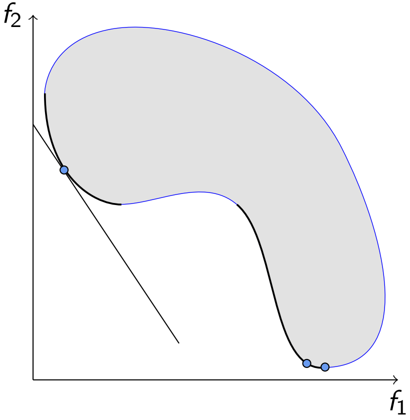

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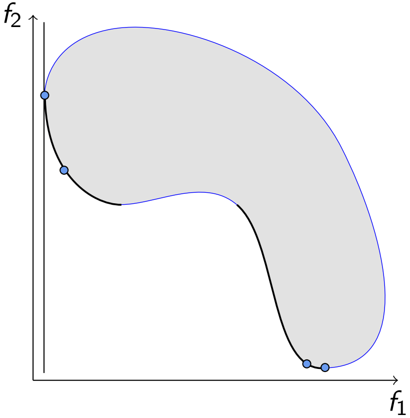

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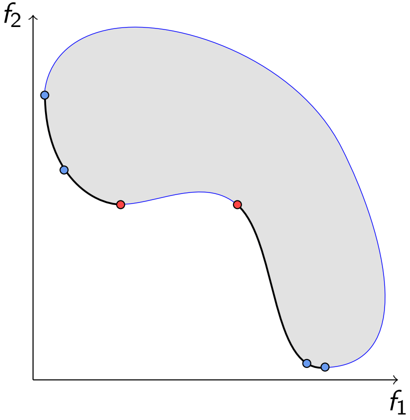

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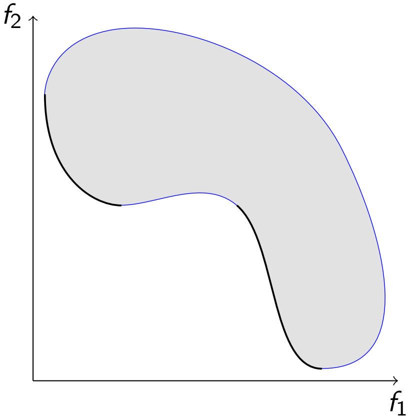

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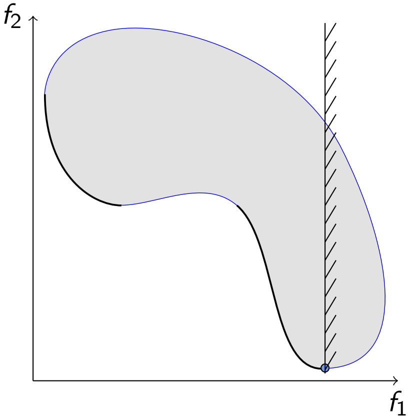

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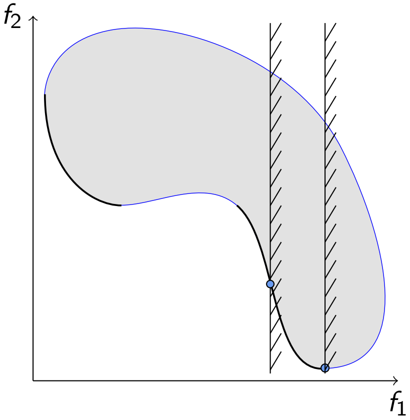

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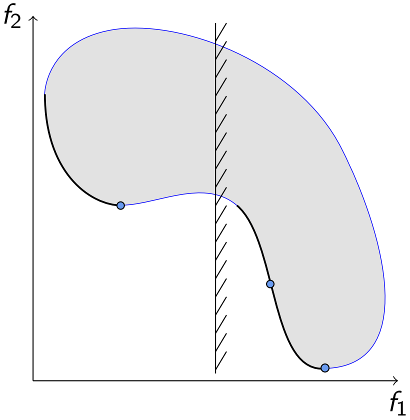

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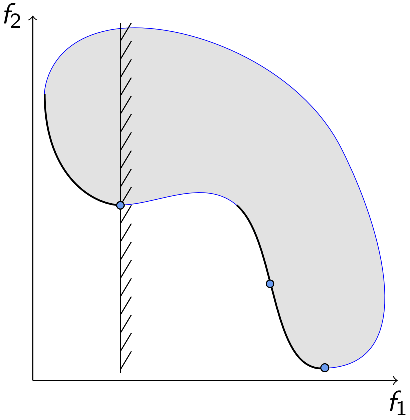

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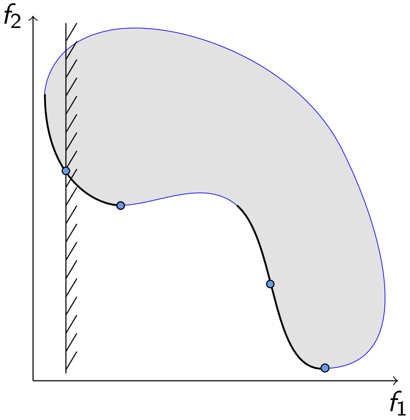

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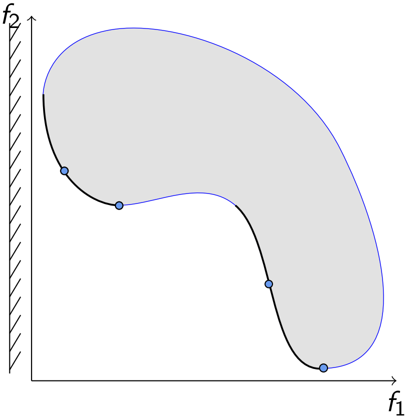

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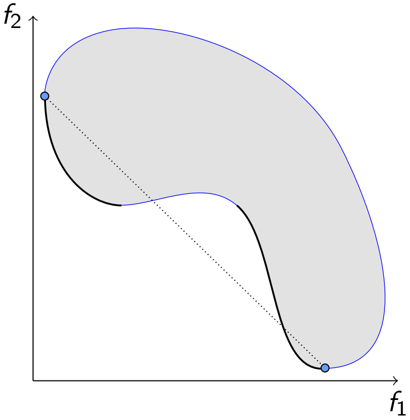

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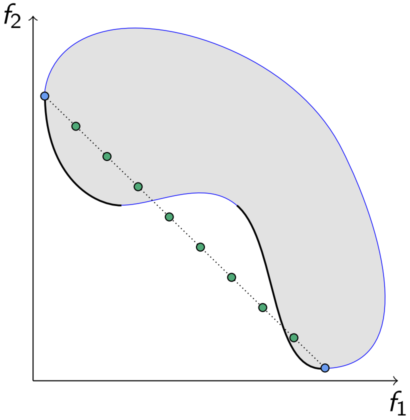

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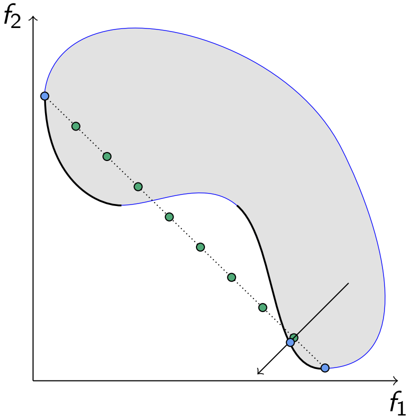

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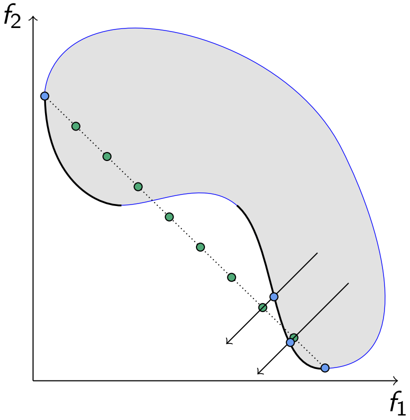

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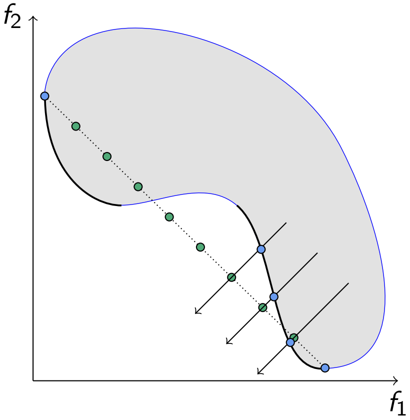

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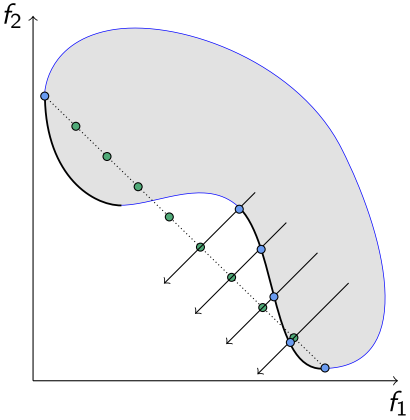

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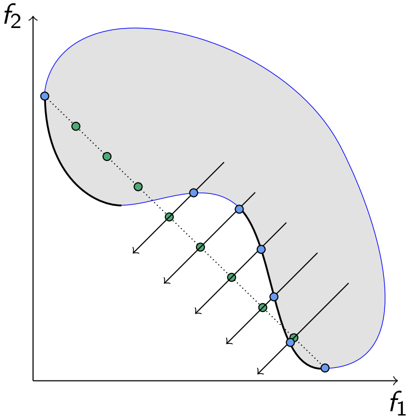

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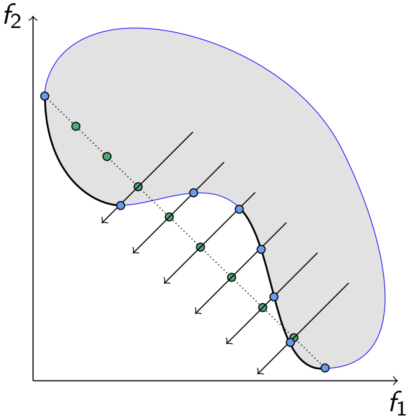

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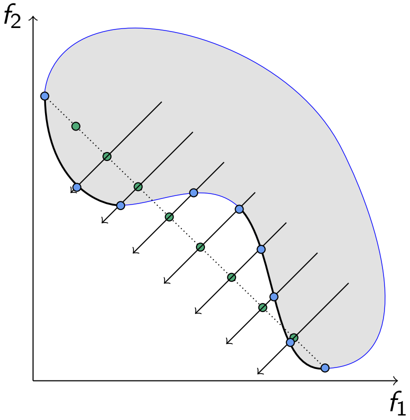
Scalarizing Methods: Examples



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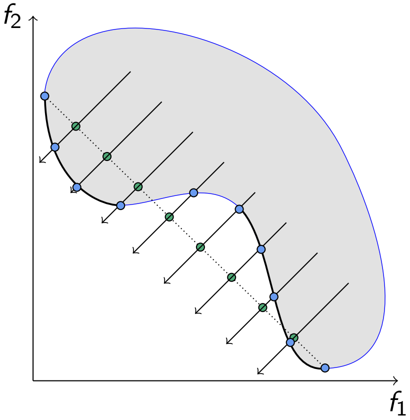

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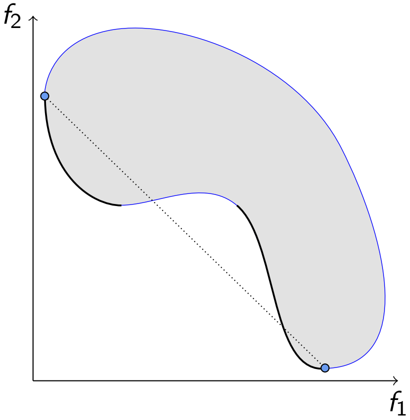

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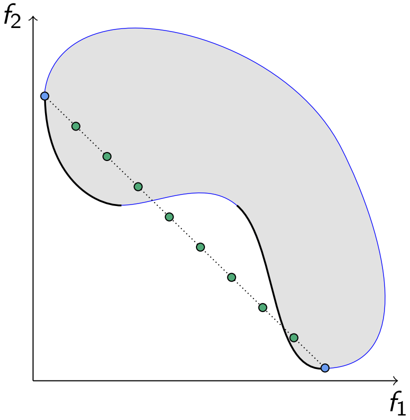

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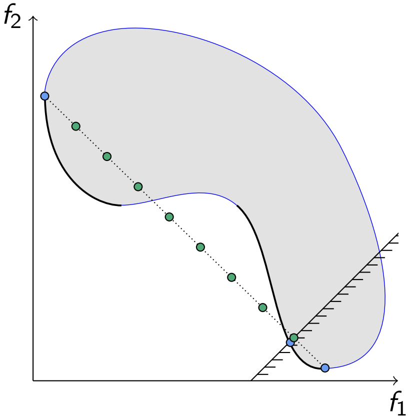

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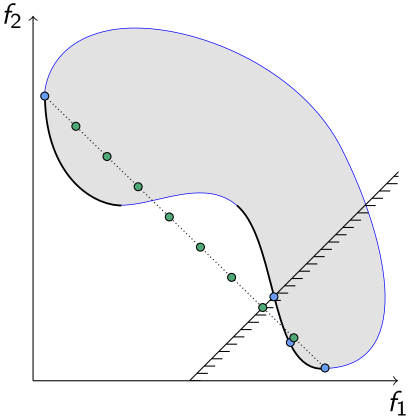

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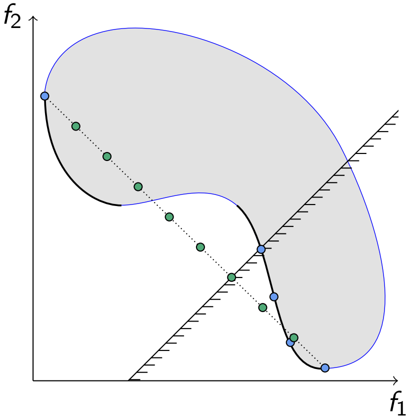

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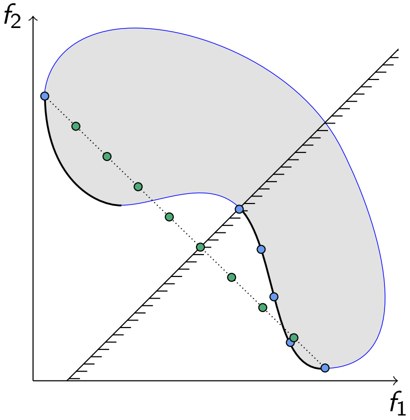

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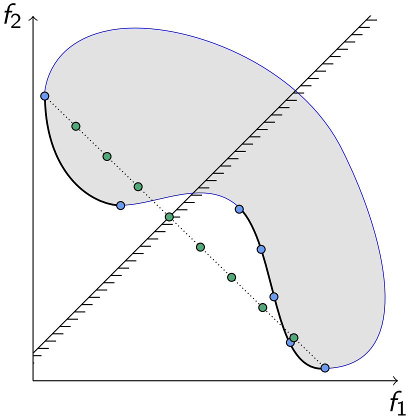

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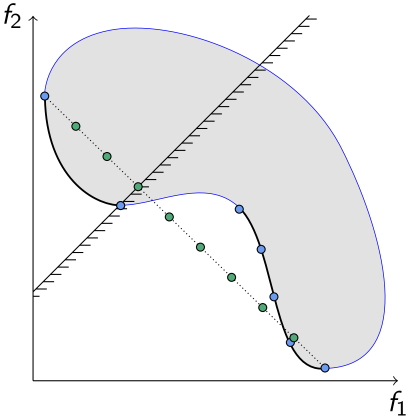

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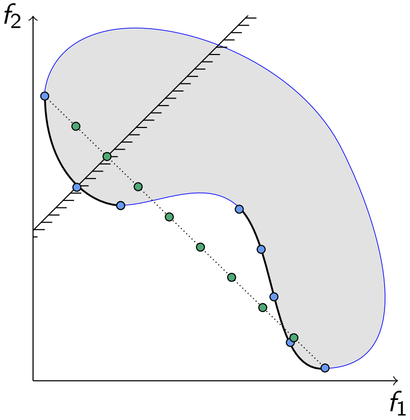

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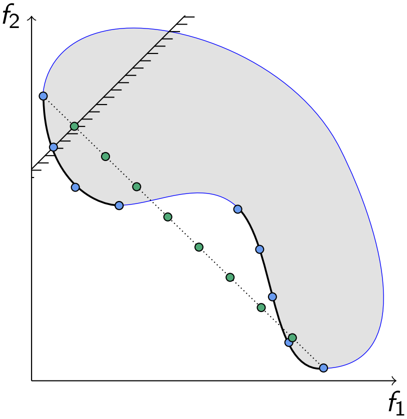

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Scalarizing Methods: Examples

$$\begin{bmatrix} min & f(x) \\ s.t & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

Scalarizing $\downarrow$

$$\begin{bmatrix} min & \hat{f}(x, v) \\ s.t & \hat{g}(x, v) \leq 0 \\ & \hat{h}(x, v) = 0 \\ & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

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Parametric Optimization

Parametric Optimization problem:

$$\left[\begin{array}{ll} \min & f(x,v) \\ \text{s.c} & h(x,v) = 0 \\ & g(x,v) \leq 0 \\ & x \in \mathbb{R}^n \end{array}\right]$$

$f : \mathbb{R}^{n+m} \to \mathbb{R}$ and $v \in V \subseteq \mathbb{R}^m$ vector of parameters.
Continuation is natural in such applications:

- Parameters are explicit.
- Use of local informations interesting.
- Optimal solutions are usually computed as solutions to first order optimality conditions.

Parametric Optimization

First order conditions:

$$\begin{aligned} \nabla_x f(x,v)\lambda + \nabla_x g(x,v)r + \nabla_x h(x,v)s &= 0 \\ (\forall i = 1,\dots,p)\ g_i(x,v)r_i &= 0 \\ (\forall i = 1,\dots,q)\ h_i(x,v) &= 0 \\ \lambda^T\lambda + r^T r + s^T s - 1 &= 0 \end{aligned}$$

With $x \in X \subseteq \mathbb{R}^n$, $v \in \mathbb{R}^m$, $\lambda \in \mathbb{R}_+$, $r \in \mathbb{R}_+^p$ and $s \in \mathbb{R}^q$.
System of $n+m+1+p+q$ variables with $n+p+q+1$ equations: m-dimensional manifold of solutions.

State of the art

Literature on Parametric Optimization:

- Algorithms [Rao and Papalambros, 1989, Rakowska et al., 1991].
- Singularity detections [Lundberg and Poore, 1993].
- Multi-Parametric [Domínguez et al., 2010].

Towards Multi-Objective Optimization:

- Tackling Multi(Bi)-Objective optimization [Rakowska et al., 1993].

State of the art

First order optimality conditions:
Parametric problem based on Weighted Sum

$$\begin{aligned}
\nabla_x f_1(x)\lambda_1 + \nabla_x f_2(x)\lambda_2 + \nabla_x g(x) r + \nabla_x h(x) s &= 0 \\
(\forall i = 1, \dots, p)\ g_i(x) r_i &= 0 \\
(\forall i = 1, \dots, q)\ h_i(x) &= 0 \\
\lambda^T \lambda + r^T r + s^T s - 1 &= 0
\end{aligned}$$

NLMOO first order conditions

$$\begin{aligned}
\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla g(x) r + \nabla h(x) s &= 0 \\
(\forall i = 1, \dots, p)\ g_i(x) r_i &= 0 \\
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\lambda^T \lambda + r^T r + s^T s - 1 &= 0
\end{aligned}$$

Continuation methods and applications

Continuation methods used to solve underconstrained systems of equations.

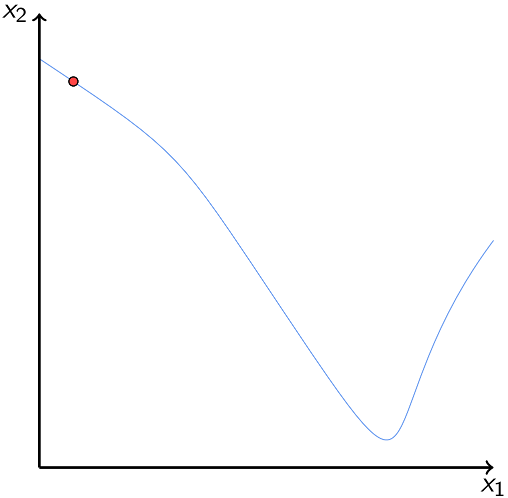
General problem

$$F(x) = 0, \; F : \mathbb{R}^{n+m} \to \mathbb{R}^n$$

Solutions form a *m*-dimensional manifold.
Appears in:

- Study of parameters in differential equations;
- Homotopy for solving polynomial systems;
- Non-Linear Optimization (interior-point methods, **parametric optimization**);
- ...

Continuation methods example: Predictor/Corrector



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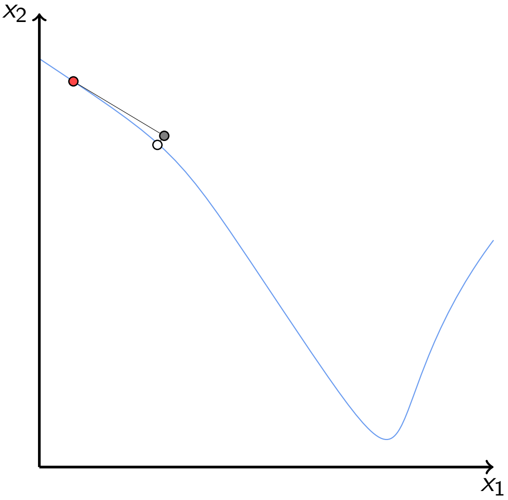

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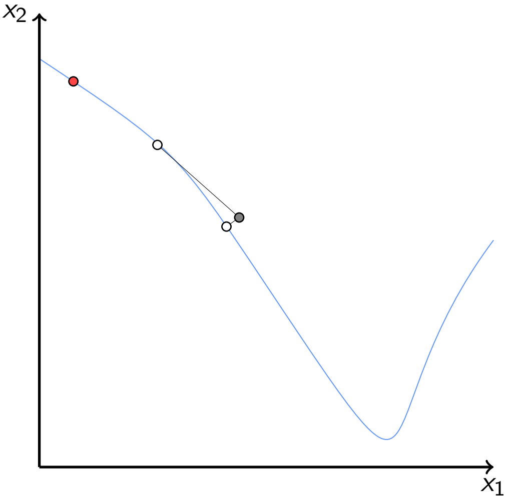

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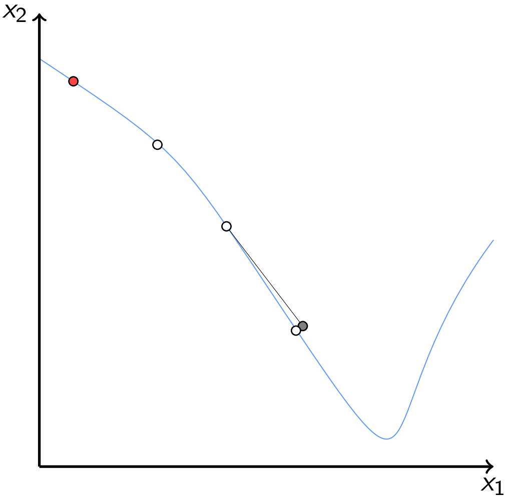

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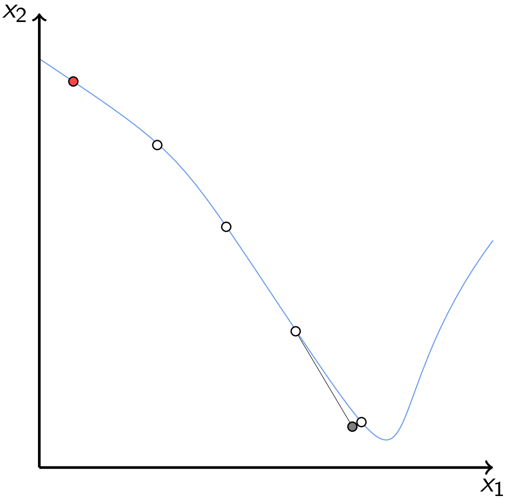

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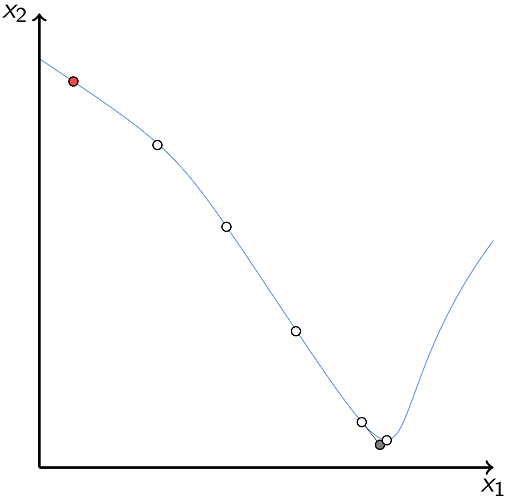

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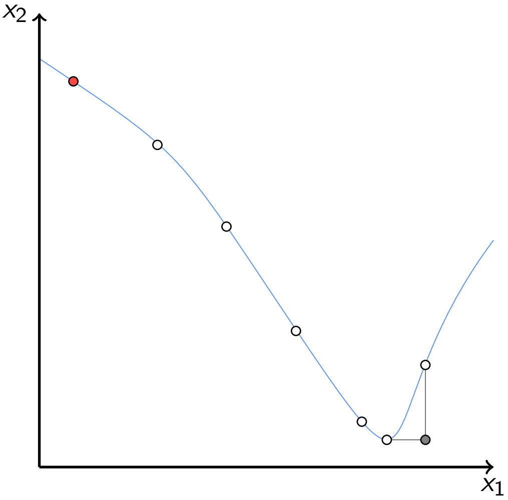

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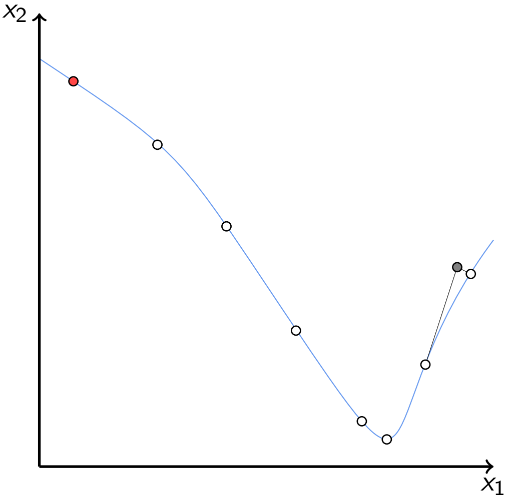

NLMOO hybridized with Continuation Methods

Continuation methods:

- For first order conditions of NLMOO [Hillermeier, 2001].

Applications in Metaheuristics:

- Curve-based Genetic Algorithm [Harada et al., 2007]
- PSO and continuation [Schütze et al., 2008]
- Steepest Descent (HCS) as continuation [Schütze et al., 2009]

Applications in Global methods:

- Recovering algorithm [Schütze et al., 2005]
- Bi-objective method inspired by NBI [Pereyra, 2009, Pereyra et al., 2013]
- Global Search [Lovison, 2011, Lovison, 2012]

Summary

- Continuation methods + NLMOO promising,
- Help for both metaheuristics and global algorithms,

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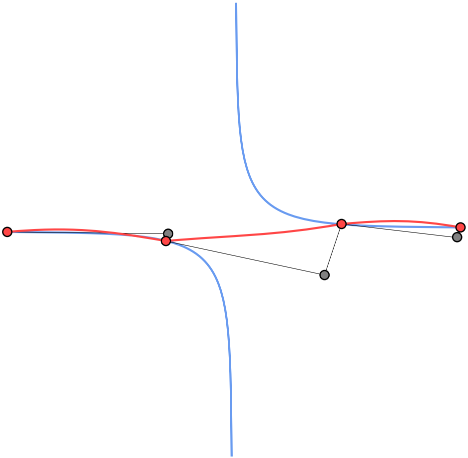
- ⊕ Continuation methods + NLMOO promising,
- ⊕ Help for both metaheuristics and global algorithms,
- ⊖ Few actually consider inequality constraints,
- ⊖ Few gives certification of the continuity.
  - False representation of the manifold.
  - Loss of solutions.

Summary



Summary

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- ⊕ Help for both metaheuristics and global algorithms,
- ⊖ Few actually consider inequality constraints,
- ⊖ Few gives certification of the continuity.

Certification can be (numerically) achieved:

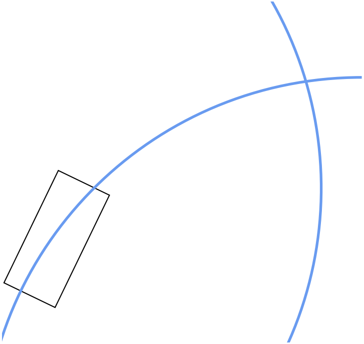
- Smale α-theory or Kantorovich theorem $\rightarrow$ maximal step [Beltrán and Leykin, 2012, Faudot and Michelucci, 2007],
- Interval Analysis and parametric Interval Newton operators [Kearfott and Xing, 1994, Martin et al., 2012].

Goal

Towards a certified and rigorous continuation method for (inequality) constrained NLMOO.

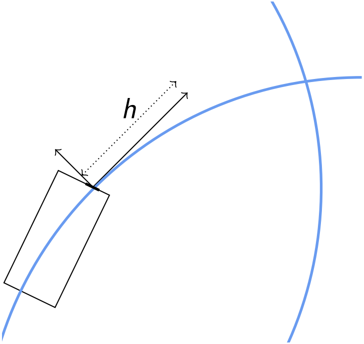
Here, restricted to the Bi-Objective case.

ParCont: Certified Continuation with Parallelotopes



- Parallelotopes and parametric Interval Newton [Goldsztejn and Granvilliers, 2010]
  - Used in Constraint Programming,
  - Based on interval analysis,
  - Spouse the shape of the manifold.
- ParCont: Parallelotope-based Continuation [Martin et al., 2012]
- Singularities

ParCont: Certified Continuation with Parallelotopes



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- ParCont: Parallelotope-based Continuation [Martin et al., 2012]
  - Equivalent PC method,
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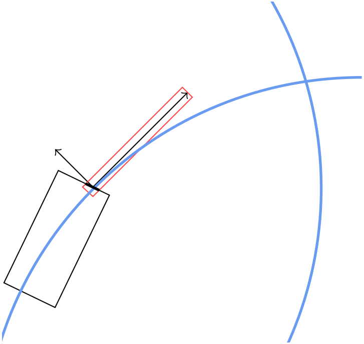
ParCont: Certified Continuation with Parallelotopes



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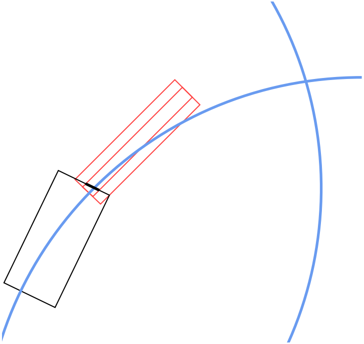

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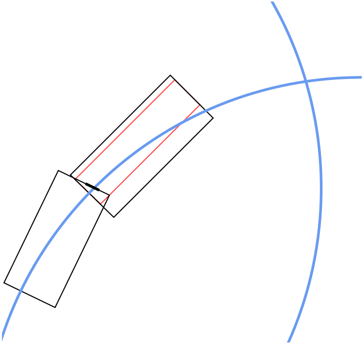

- Parallelotopes and parametric Interval Newton [Goldsztejn and Granvilliers, 2010]
- ParCont: Parallelotope-based Continuation [Martin et al., 2012]
  - Equivalent PC method,
  - Builds locally new parallelotopes along the manifold,
  - Connects two consecutive parallelotopes.
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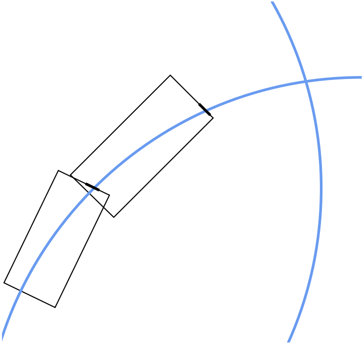

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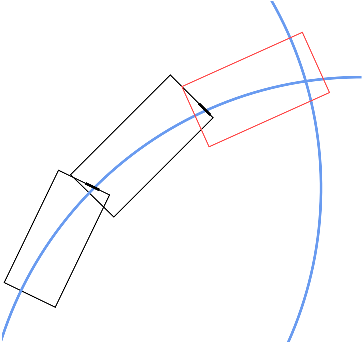

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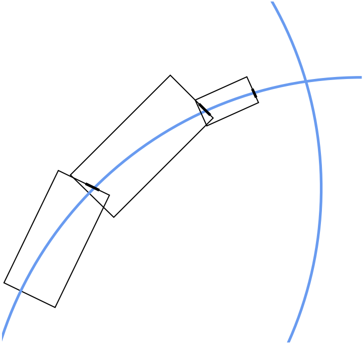

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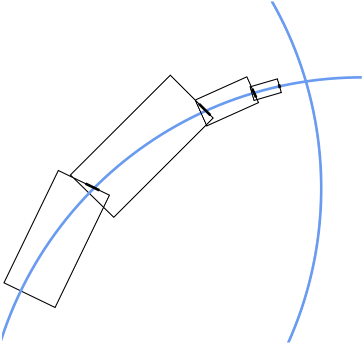

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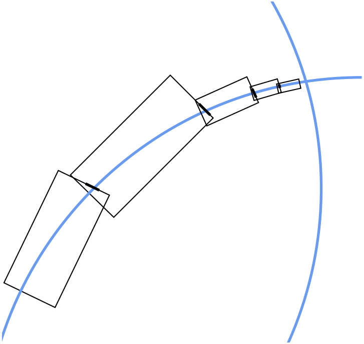

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Optimality conditions

Let the system of first order optimality conditions:

$$\begin{aligned}
\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla g(x)r + \nabla h(x)s &= 0 \\
(\forall i = 1, \dots, p)\ g_i(x)r_i &= 0 \\
(\forall i = 1, \dots, q)\ h_i(x) &= 0 \\
\lambda^T\lambda + r^T r + s^T s - 1 &= 0
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$$

Singularity when there exists i with $r_i = 0$ and $g_i(x) = 0$: change in the set of active constraints.

Problem

ParCont can not handle inequality constraints.

Towards a certified active set management strategy

Dealing with inequalities [Rakowska et al., 1993]

Definition

Let $\bar{\mathcal{A}} \subseteq \{1, 2, \ldots, p\}$ be the set of active constraints at a feasible solution x. Let $\bar{g}$ and $\bar{r}$ be the induced inequality vector and weights.

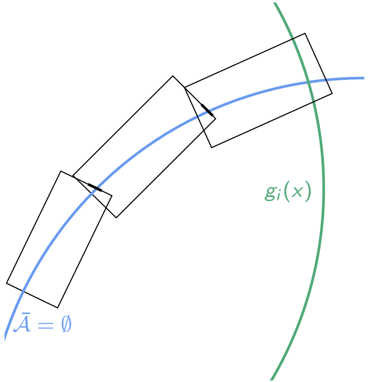
To deal with singularities from change in the active constraint set:

- Solve the system:

$$\begin{aligned}
\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla \bar{g}(x)\bar{r} + \nabla h(x)s &= 0 \\
(\forall i \in \bar{\mathcal{A}})\ g_i(x) &= 0 \\
(\forall i = 1, \ldots, q)\ h(x) &= 0 \\
\lambda^T\lambda + r^T r + s^T s - 1 &= 0
\end{aligned} \tag{2}$$

- Change the set $\bar{\mathcal{A}}$ when activating/disactivating a constraint.

Detecting change in the active set



Example:

- Detection of a possible activation ($g_i(x) = 0$),
- Certify the activation: Interval Newton,
- Change $\bar{\mathcal{A}}$, isolate the activation, orient the continuation ($r_i > 0$),
- Restart the Continuation.

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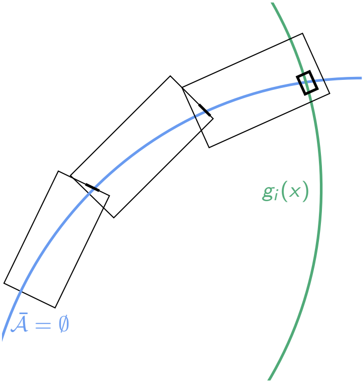

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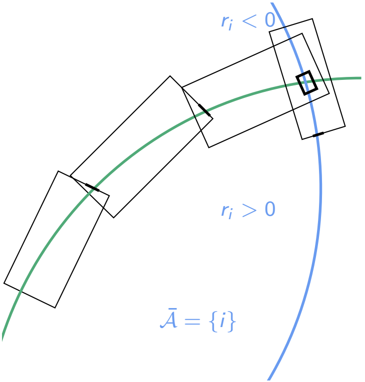

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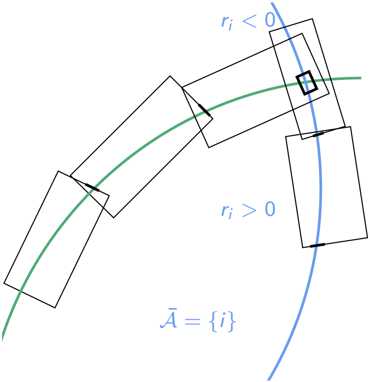

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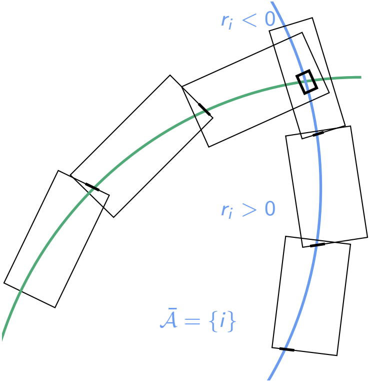

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Implementation

ParCont is implemented in C++, with:

- RealPaver API [Granvilliers and Benhamou, 2006],
- Gaol interval arithmetic library [Goualard, 2006],
- Lapack linear algebra library [Anderson et al., 1999],
- Crlibm verified rounding library.

Towards using Certified Continuation as a post-process of a metaheuristic (NSGAII [Deb et al., 2002]):

- As suggested in [Harada et al., 2007].
- Certify local optimality (and feasibility),
- Comparison of the efforts of the two methods.

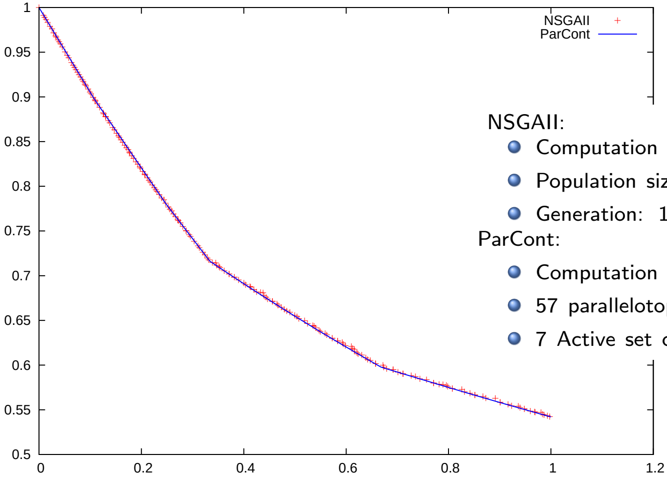
Experiments

CTP1 [Deb et al., 2001]: Standard bi-objective problem with 2 variables.

$$\left[\begin{array}{ll} \min & f_1(x) = x_1 \\ \min & f_2(x) = (1 + x_2)\exp(-x_1/(1 + x_2)) \\ \text{s.t} & g_1(x) = 1 - f_2(x)/(0.858\exp(-0.541 f_1(x))) \leq 0 \\ & g_2(x) = 1 - f_2(x)/(0.728\exp(-0.295 f_1(x))) \leq 0 \\ & x_1, x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}\right]$$

Start ParCont at f_2^*.

Experiments



NSGAII:

- Computation time: 0.25 s,
- Population size: 200,
- Generation: 100.

ParCont:

- Computation time: 0.0925 s,
- 57 parallelotopes,
- 7 Active set changes.

Experiments

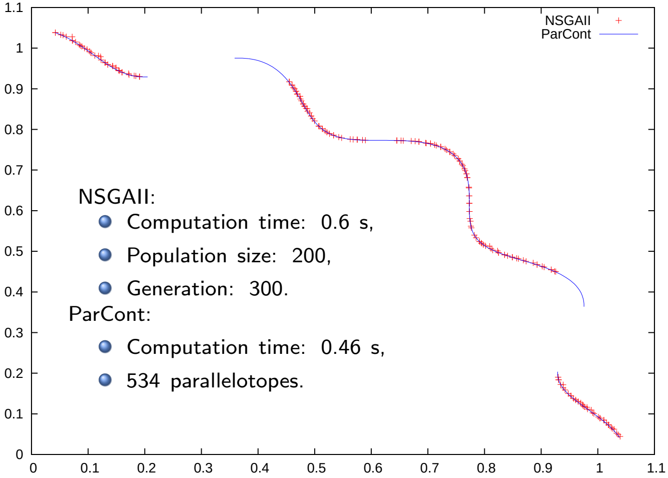
Tanaka [Tanaka et al., 1995]: Bi-objective problem with 2 variables.

$$\left[\begin{array}{ll} \min & f_1(x) = x_1 \\ \min & f_2(x) = x_2 \\ \text{s.t} & g_1(x) = -x_1^2 - x_2^2 + 1 + 0.1\cos(16\arctan(x_1/x_2)) \leq 0 \\ & g_2(x) = 2x_1^2 + 2x_2^2 - 1 \leq 0 \\ & x_1, x_2 \leq \pi \\ & x_1, x_2 \geq 0 \end{array}\right]$$

Disconnected Pareto-front. ParCont started at:

$$x^A = \begin{pmatrix} 0.042 \\ 1.038 \end{pmatrix},\ x^B = \begin{pmatrix} 0.586 \\ 0.774 \end{pmatrix},\ x^C = \begin{pmatrix} 1.039 \\ 0.043 \end{pmatrix}$$

Experiments



NSGAII:

- Computation time: 0.6 s,
- Population size: 200,
- Generation: 300.

ParCont:

- Computation time: 0.46 s,
- 534 parallelotopes.

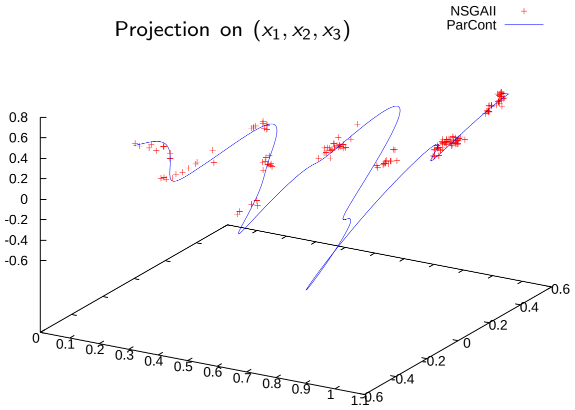
Experiments

LZ3 [Li and Zhang, 2009] Modified: n-dimensional bi-objective problem.

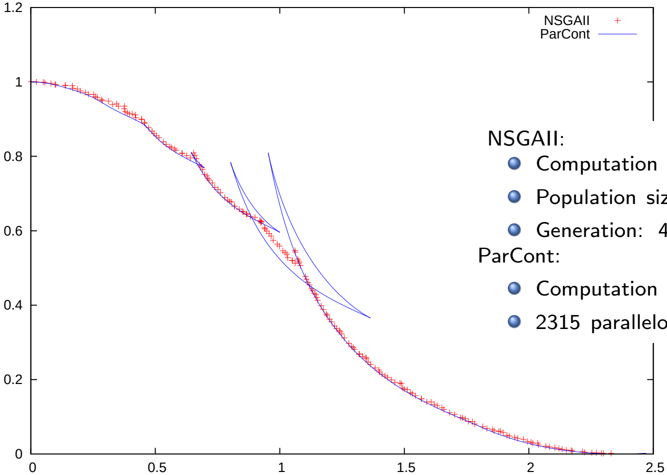
$$\left[\begin{array}{ll} \min & f_1(x) = x_1 + \frac{2}{n}\sum_{i=2}^{n}(x_i - 0.8x_1\cos(6\pi x_1 + \frac{i\pi}{n}))^2 \\ \min & f_2(x) = 1 - x_1^2 + \frac{2}{n}\sum_{i=2}^{n}(x_i - 0.8x_1\sin(6\pi x_1 + \frac{i\pi}{n}))^2 \\ \text{s.t} & x_1 \leq 1 \\ & x_1 \geq 0 \\ & x_i \leq 1, i = 2, \ldots, n \\ & x_i \geq -1, i = 2, \ldots, n \end{array}\right]$$

ParCont start at f_2^* with $n = 10$.

Experiments



Experiments



NSGAII:

- Computation time: 1.3 s,
- Population size: 200,
- Generation: 400.

ParCont:

- Computation time: 11.2 s,
- 2315 parallelotopes.

Summary

Informations:

- Each step of ParCont has $O(n^3)$ time complexity,
- Compared to non-certified methods, it has to use a smaller step length.

Pros and Cons:

- ⊕ ParCont ables to produce certified enclosures of Pareto-optimal solutions,
- ⊕ Local optimality is proven for each enclosure,
- ⊕ Use only local informations.
- ⊖ Required twice continuously differentiable objectives and constraints,
- ⊖ Some singularities not handled,
- ⊖ Limited to 1-dimensional manifolds (bi-objective problems).

Summary

We have seen that:

- Many state of art approaches attempt to parameterize Pareto-Optimal solutions,
- Continuation methods and NLMOO promising in different applications,
- A certified continuation method ParCont for bi-objective problems, dealing with change in active set of constraints, is proposed.

Next ?

- Integration of ParCont in a global method: only one point per connected components is required,

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Next ?

- Integration of ParCont in a global method: only one point per connected components is required,
- Adaptation of ParCont to 3-Objectives.

On Continuation Methods for Non-Linear Multi-Objective Optimization

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SWIM 2013

Small Workshop on Interval Methods

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