Classification of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

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SWIM 2013 - Small Workshop on Intervals Methods - Brest
http://www.ensta-bretagne.fr/swim13/

Outline

Objects

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Invariant

The set of eigenvalues is an invariant because

$$A \sim B \Rightarrow sp(A) = sp(B)$$

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$$\exists A, B \in \mathcal{M}_n, A \not\sim B \text{ and } sp(A) = sp(B)$$

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Example

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Invariant

The set of eigenvalues is *not a strong enough* invariant since

$$\exists A, B \in \mathcal{M}_n, A \nsim B \text{ and } sp(A) = sp(B)$$

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A really strong invariant

Let us call by J the Jordan method, we have

$$A \sim B \Leftrightarrow J(A) = J(B)$$

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Smooth maps	$f \sim f'$ if there exist diffeomorphic changes of variables (g, h) on X and Y such that $f = g \circ f' \circ h$	?

Global picture

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Definition - Equivalence

Let f and f' be two smooth maps. Then $f \sim f'$ if there exists diffeomorphisms $g : X \to X'$ and $h : Y' \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \uparrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

commutes.

Example

$$\begin{tikzcd} \mathbb{R} \arrow[r, "2x+6"] \arrow[d, "g"] & \mathbb{R} \\ \mathbb{R} \arrow[r, "x+1"] & \mathbb{R} \arrow[u, "h"'] \end{tikzcd}$$

Example

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{2x+6} & \mathbb{R} \\ \downarrow{\scriptstyle x+2} & & \uparrow{\scriptstyle 2y} \\ \mathbb{R} & \xrightarrow{x+1} & \mathbb{R} \end{array}$$

Examples

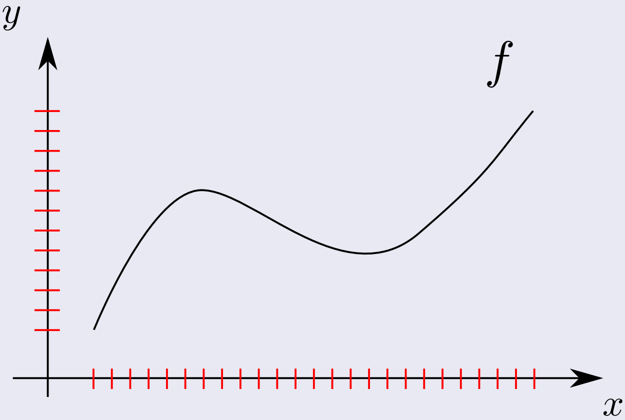
1. $f_1(x) = x^2,\ f_2(x) = ax^2 + bx + c,\ a \neq 0$

$$f_1 \sim f_2$$

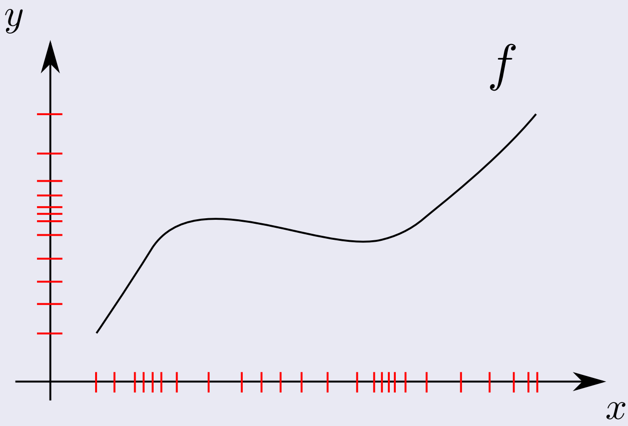
2. $f_1(x) = x^2 + 1,\ f_2(x) = x + 1,$

$$f_1 \nsim f_2$$

Examples



Examples



Proposition

Suppose that $f \sim f'$ with

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & y_1 \\ \downarrow g & & \uparrow h \\ x_2 & \xrightarrow{f'} & y_2 \end{array}$$

then $f^{-1}(\{y_1\})$ is homeomorphic to $f'^{-1}(\{y_2\})$.

Proposition

Suppose that $f \sim f'$ with

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & y_1 \\ \downarrow g & & \uparrow h \\ x_2 & \xrightarrow{f'} & y_2 \end{array}$$

then $\operatorname{rank} df_{x_1} = \operatorname{rank} df'_{x_2}$.

Proof

Chain rule, $df = dh \cdot df' \cdot dg$

Definition

Let us defined by S_f the set of critical points of f :

$$S_f = \{x \in X \mid df(x) \text{ is singular } \}.$$

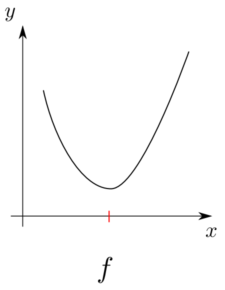
Corollary

$$f \sim f' \Rightarrow S_f \simeq S_{f'}$$

where $\simeq$ means homeomorphic.
i.e. the topology of the critical points set is an invariant.

This is not a strong enough invariant,
there exists smooth maps $f, f' : [0,1] \to [0,1]$ such that

$$S_f \simeq S_{f'} \text{ and } f \not\sim f'.$$



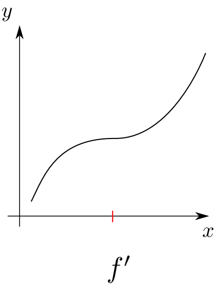


FIGURE: Singularity theory.

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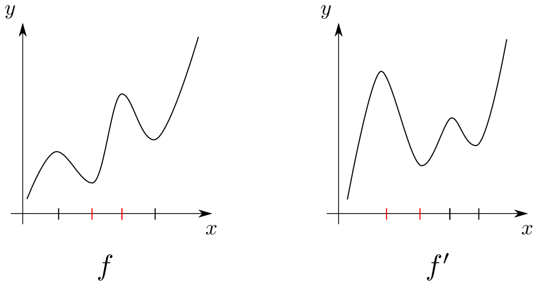
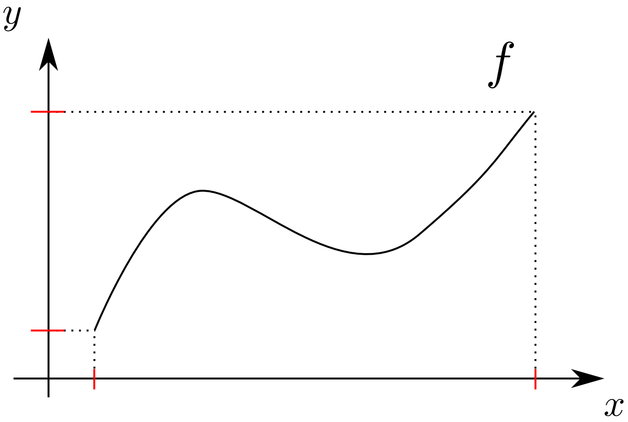


FIGURE: Topology of X.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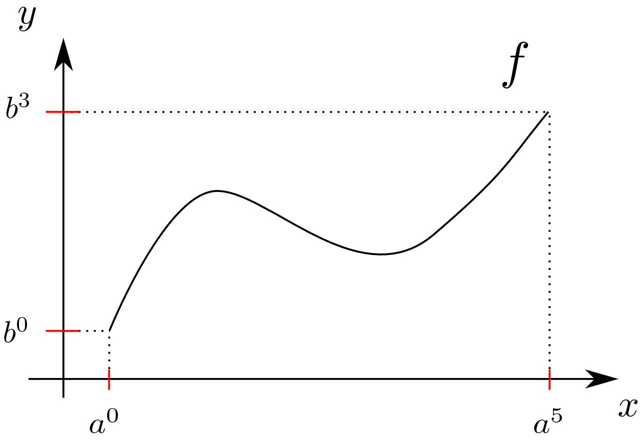
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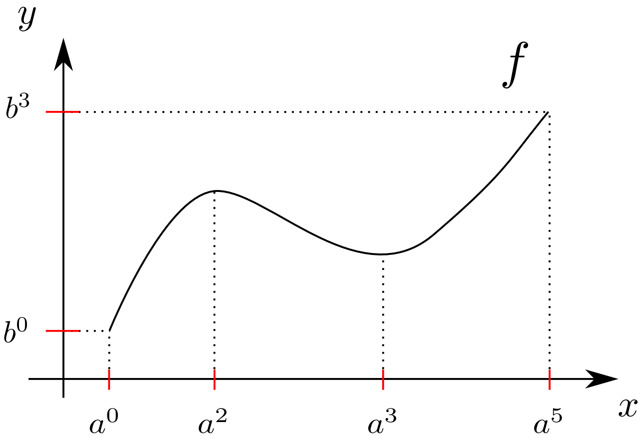
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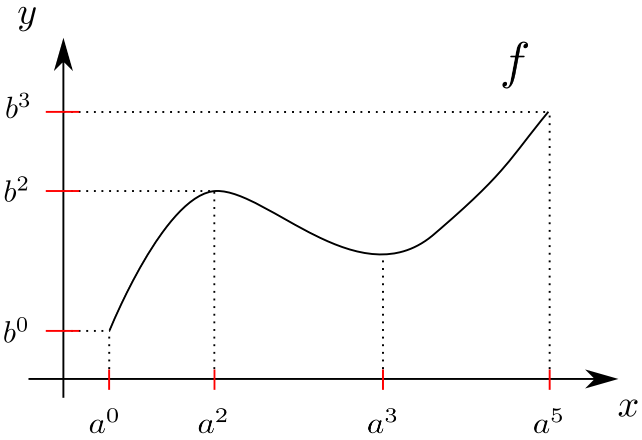
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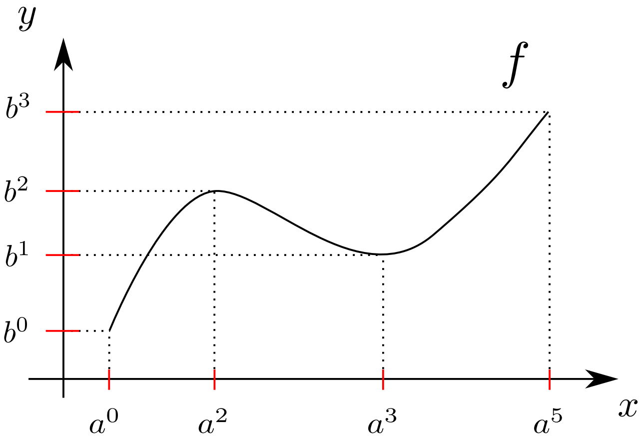
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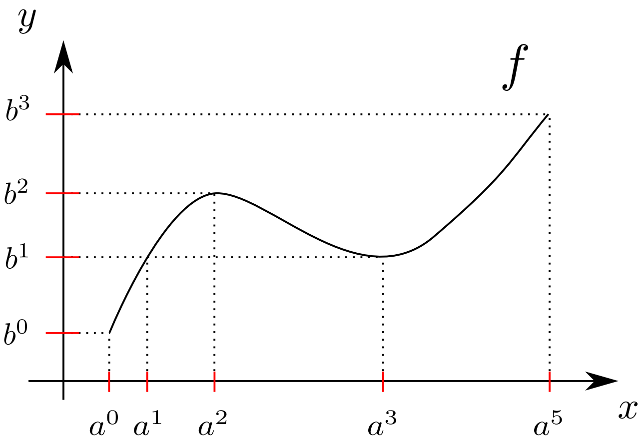
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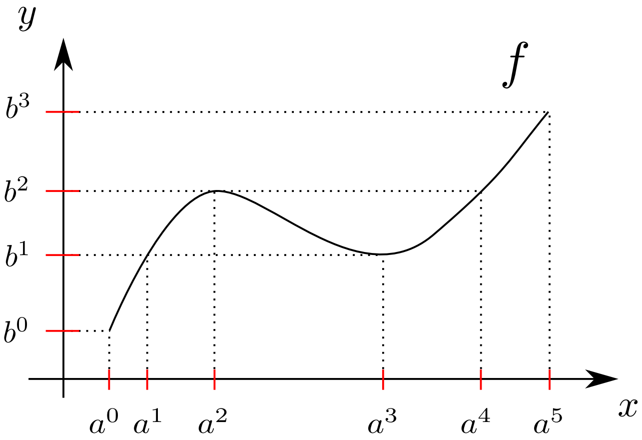
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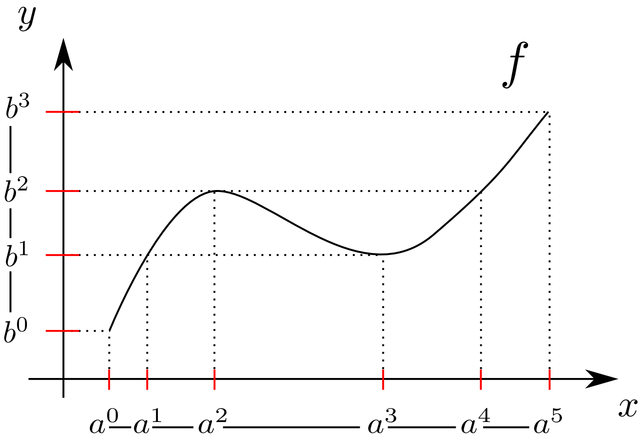
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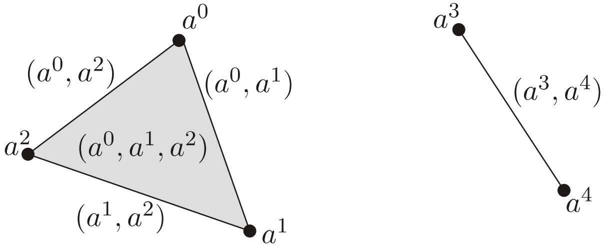
a^0 a^1 a^2 a^3 a^4 a^5

Definition - Abstract simplicial complex

Let $\mathcal{N}$ be a finite set of symbols $\{(a^0),(a^1),\ldots,(a^n)\}$
An abstract simplicial complex $\mathcal{K}$ is a subset of the powerset of $\mathcal{N}$ satisfying : $\sigma \in \mathcal{K} \Rightarrow \forall \sigma_0 \subset \sigma, \sigma_0 \in \mathcal{K}$

$$\begin{aligned}\mathcal{K} \;=\; & \{(a^0),(a^1),(a^2),(a^3),(a^4),\\ & (a^0,a^1),(a^1,a^2),(a^0,a^2),(a^3,a^4),\\ & (a^0,a^1,a^2)\}\end{aligned}$$

This will be denoted by $a^0a^1a^2 + a^3a^4$



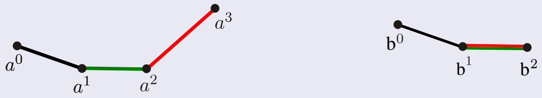
Definition

Given abstract simplicial complexes $\mathcal{K}$ and $\mathcal{L}$, a *simplicial map* $F : \mathcal{K}^0 \to \mathcal{L}^0$ is a map with the following property :

$$(a^0, a^1, \ldots, a^n) \in \mathcal{K} \Rightarrow (F(a^0), F(a^1), \ldots, F(a^n)) \in \mathcal{L}.$$

Example - Simplicial map

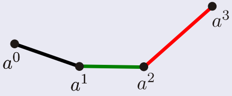
$$\mathcal{K} = a_0 a_1 + a_1 a_2 + a_2 a_3, \quad \mathcal{L} = b_0 b_1 + b_1 b_2$$

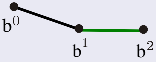


$$F : \begin{array}{ll} a^0 & \mapsto b^0 \\ a^1 & \mapsto b^1 \\ a^2 & \mapsto b^2 \\ a^3 & \mapsto b^1 \end{array}$$

Example - NOT a Simplicial map

$$\mathcal{K} = a_0 a_1 + a_1 a_2 + a_2 a_3, \quad \mathcal{L} = b_0 b_1 + b_1 b_2$$





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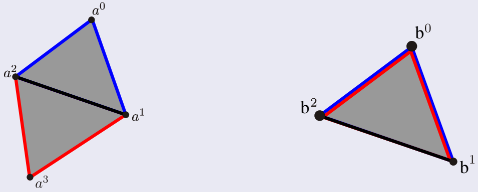
Example - Simplicial map

$$\mathcal{K} = a_0 a_1 a_2 + a_1 a_2 a_3, \quad \mathcal{L} = b_0 b_1 b_2$$



$$F \quad : \quad \begin{array}{ll} a^0 & \mapsto b^0 \\ a^1 & \mapsto b^1 \\ a^2 & \mapsto b^2 \\ a^3 & \mapsto b^0 \end{array}$$

Definition

Let f and f' be continous maps. Then f and f' are *topologically conjugate* if there exists homeomorphism $g : X \to X'$ and $h : Y \to Y'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \uparrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

commutes.

Proposition

$$f \sim f' \Rightarrow f \sim_0 f'$$

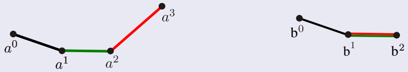
Definition

Let f be a smooth map and F a simplicial map, F is a *portrait* of f if

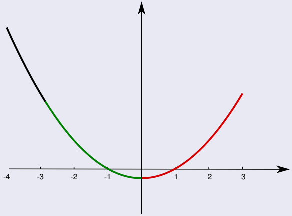
$$f \sim_0 F$$

Example - Simplicial map

The simplicial map



is a portrait of $[-4, 3] \ni x \mapsto x^2 - 1 \in \mathbb{R}$



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For every closed subset A of $\mathbb{R}^n$, there exists a smooth real valued function f such that

$$A = f^{-1}(\{0\})$$

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We are not going to consider all cases ...

Definition

Let f be a smooth map, f is *stable* if their exists a nbrd N_f such that

$$\forall f' \in N_f, f' \sim f$$

Examples

1. $g : x \mapsto x^2$ is stable,
2. $f_0 : x \mapsto x^3$ is not stable, since with $f_\epsilon : x \mapsto x(x^2 - \epsilon)$,

$$\epsilon \neq 0 \Rightarrow f_\epsilon \not\sim f_0.$$

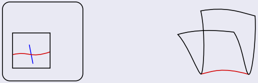
Withney theorem

Let X and Y be 2-dimentional manifolds and f be generic. The critical point set S_f is a regular curve. With $p \in S_f$, one has

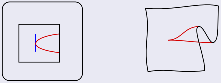
$$T_pS_f \oplus \ker df_p = T_pX \text{ or } T_pS_f = \ker df_p$$

Geometric representation

1. if $T_pS_f \oplus \ker df_p = T_pX$,

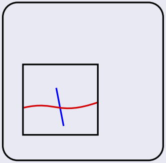


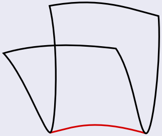
2. if $T_pS_f = \ker df_p$,



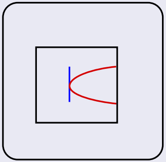
Geometric representation

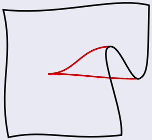
1. $T_pS_f \oplus \ker df_p = T_pX$: fold point





2. $T_pS_p = \ker df_p$: cusp point





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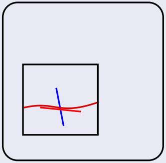

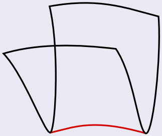

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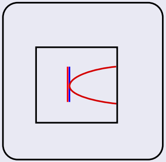
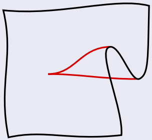




Normal forms

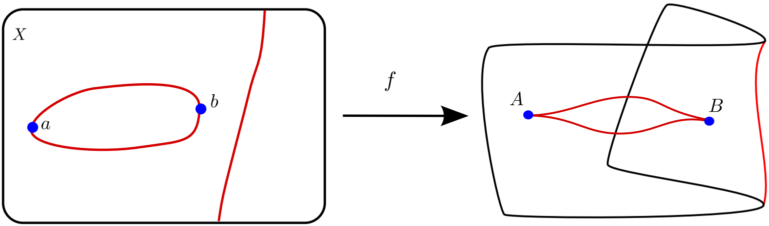
1. If $T_pS_f \oplus \ker df_p = T_pX$, then there exists a nbrd N_p such that
$$f|N_p \sim (x,y) \mapsto (x, y^2).$$
2. If $T_pS_f = \ker df_p$, then there exists a nbrd N_p such that
$$f|N_p \sim (x,y) \mapsto (x, xy + y^3).$$

Definition

Let f a smooth map from $X \to \mathbb{R}^2$ with X a simply connected compact subset of $\mathbb{R}^2$ with smooth boundary ∂X. The *apparent contour* of f is

$$f(S_f \cup \partial X)$$

The topology of the Apparent contour is an invariant.



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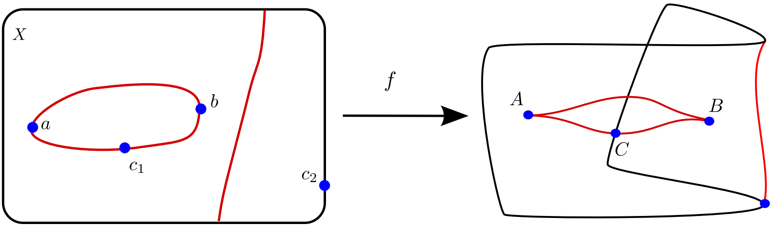

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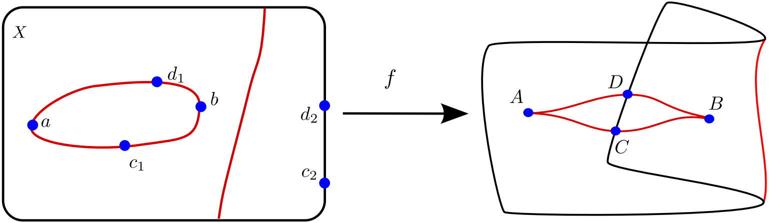

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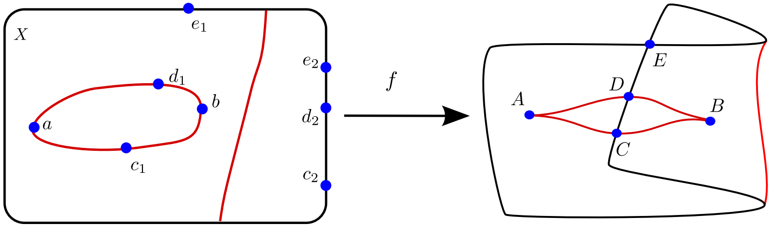

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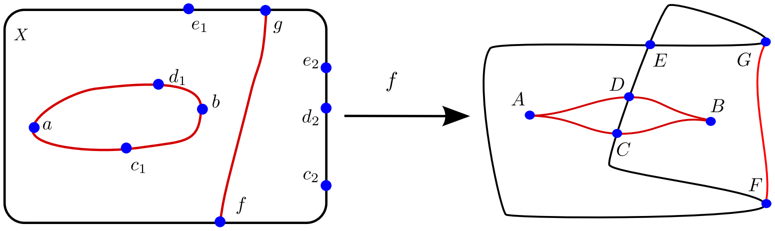
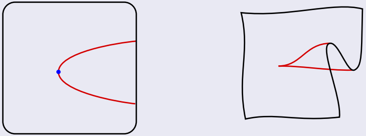


Theorem (Global properties of generic maps)

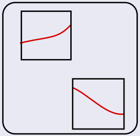
Let X be a compact simply connected domain of $\mathbb{R}^2$ with $\partial X = \Gamma^{-1}(\{0\})$. A generic smooth map f from X to $\mathbb{R}^2$ has the following properties :

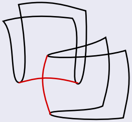
1. *S_f is regular curve. Moreover, elements of S are folds and cusp. The set of cusp is discrete.*



Theorem

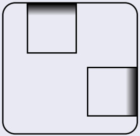
3. *3 singular points do not have the same image,*
4. *2 singular points having the same image are folds points and they have normal crossing.*

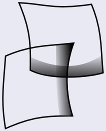




Theorem

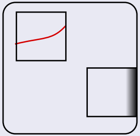
5. *3 boundary points do not have the same image,*
6. *2 boundary points having the same image cross normally.*

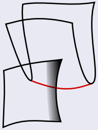




Theorem

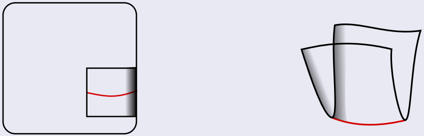
7. *3 different points belonging to $S_f \cup \partial X$ do not have the same image,*
8. *If a point on the singularity curve and a boundary have the same image, the singular point is a fold and they have normal crossing.*

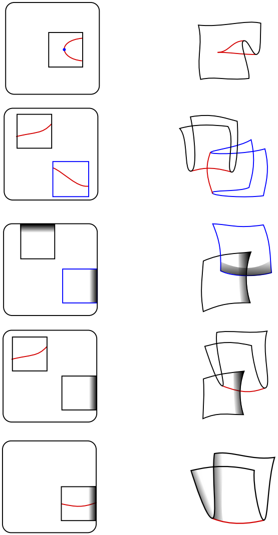


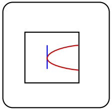


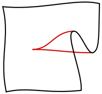
Theorem

9. *if the singularity curve intersects the boundary, then this point is a fold,*
10. *moreover tangents to the singularity curve and boundary curve are different.*









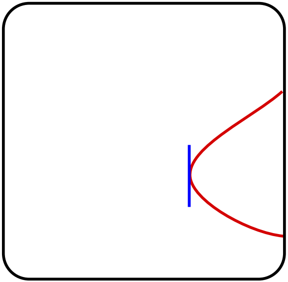
Proposition

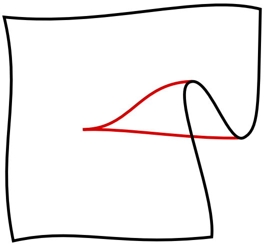
Let f be a smooth generic map from X to $\mathbb{R}^2$, let us denote by c the map defined by :

$$\begin{array}{cccc} c & : & X & \rightarrow & \mathbb{R}^2 \\ & & p & \mapsto & df_p \xi_p \end{array} \quad (1)$$

where ξ is the vector field defined by $\xi_p = \begin{pmatrix} \partial_2 \det df_p \\ -\partial_1 \det df_p \end{pmatrix}$.

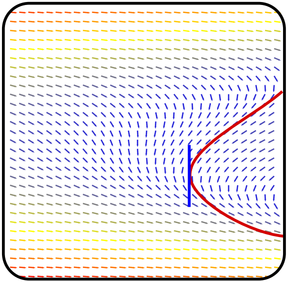
If $c(p) = 0$ and dc_p is invertible then p is a simple cusp. This sufficient condition is locally necessary.

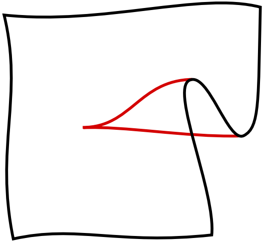




Interval Newton method

$$\begin{array}{rcccc} c & : & X & \to & \mathbb{R}^2 \\ & & p & \mapsto & df_p\xi_p \end{array} \tag{2}$$

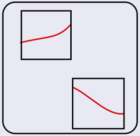


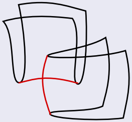


Interval Newton method

$$\begin{array}{cccc} c & : & X & \to & \mathbb{R}^2 \\ & & p & \mapsto & df_p\xi_p \end{array} \quad (3)$$

2 different folds





$$S^{\Delta 2} = \{(x_1, x_2) \in S \times S - \Delta(S) \mid f(x_1) = f(x_2)\} / \simeq$$

where $\simeq$ is the relation defined by
$(x_1, x_2) \simeq (x_1', x_2') \Leftrightarrow (x_1, x_2) = (x_2', x_1')$.

Method

Bisection scheme on $X \times X$.

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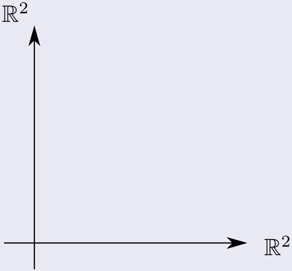

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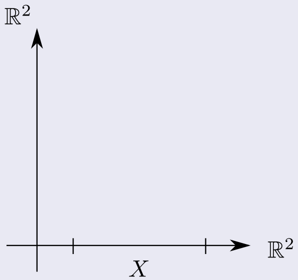

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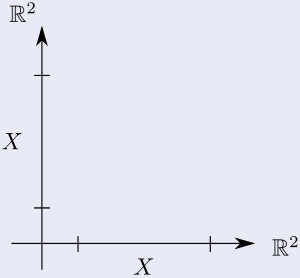

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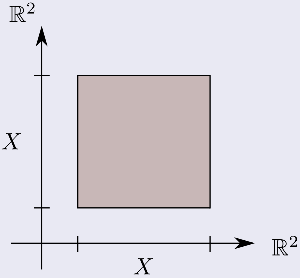

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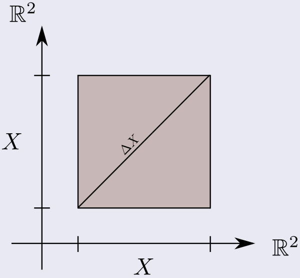

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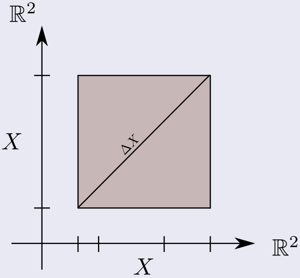

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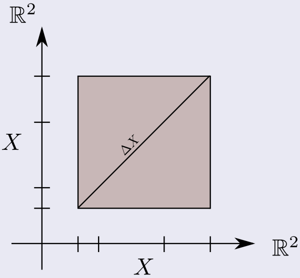

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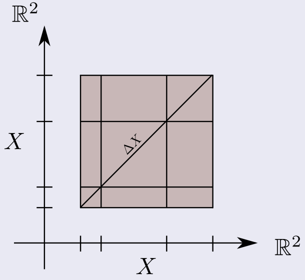

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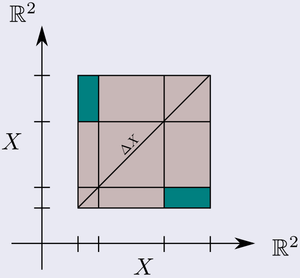

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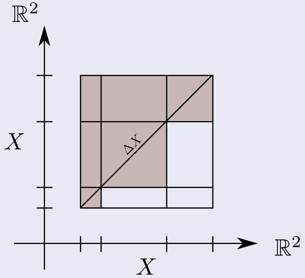

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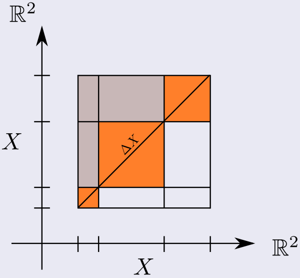
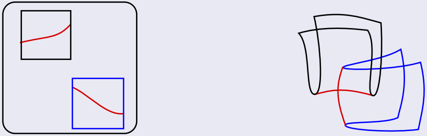
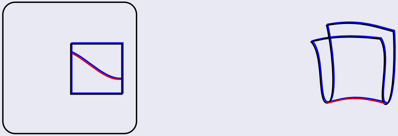




$$[x_1] \neq [x_2]$$



$$[x_1] = [x_2]$$

Let us define the map *folds* by

$$\begin{array}{lccc} folds & : & X \times X & \to & \mathbb{R}^4 \\ & & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \det df(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array}$$

One has

$$S^{\Delta 2} = folds^{-1}(\{0\}) - \Delta S / \simeq .$$

For any (α, α) in ΔS, the d *folds* is conjugate to

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{pmatrix}$$

which is not invertible since $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det df(\alpha) = 0$. In other words, as any box of the form $[x_1] \times [x_1]$ contains ΔS, the interval Newton method will fail.

One needs a method to prove that $f|S \cap [x_1]$ is an embedding.

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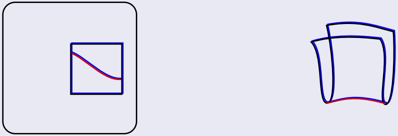

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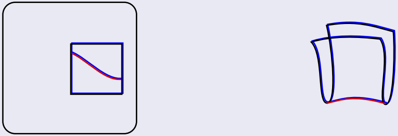
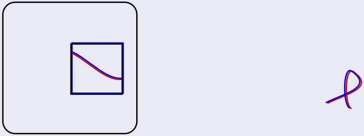


$$[x_1] = [x_2]$$

Not in this case ...



Corollary

Let $f : X \to \mathbb{R}^2$ be a smooth map and X a compact subset of $\mathbb{R}^2$. Let $\Gamma : X \to \mathbb{R}$ be a submersion such that the curve $S = \{x \in X \mid \Gamma(x) = 0\}$ is contractible. If

$$\forall J \in \tilde{d}f(X) \cdot \begin{pmatrix} \partial_2\Gamma(X) \\ -\partial_1\Gamma(X) \end{pmatrix}, \text{rank } J = 1$$

then $f|S$ is an embedding.

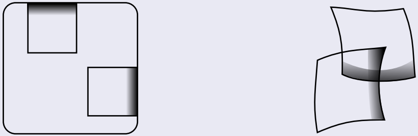
The last condition is not satisfiable if $[x_1]$ contains a cusp ...

Proposition

Suppose that there exists a unique simple cusp p_0 in the interior of X. Let $\alpha \in \mathbb{R}^{2*}$, s.t. $\alpha \cdot \mathrm{Im}\, df_{p_0} = 0$, and ξ a non vanashing vector field such that $\forall p \in S, \xi_p \in T_pS$ (S contractible).
If $g = \sum \alpha_i \xi^3 f_i : X \to \mathbb{R}$ is a nonvanishing function then $f|S$ is injective. This condition is locally necessary.

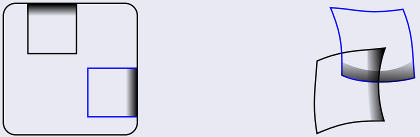
Here the vector field ξ is seen as the derivation of $\mathcal{C}^\infty(X)$ defined by

$$\xi = \sum \xi_i \frac{\partial}{\partial x_i}.$$



$$\partial X^{\Delta 2} = \{(x_1, x_2) \in \partial X \times \partial X - \Delta(\partial X) \mid f(x_1) = f(x_2)\}/ \simeq$$

$[x_1] \neq [x_2]$



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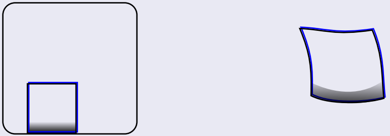


Let us define the map *boundaries* by

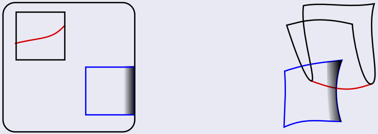
$$\begin{array}{rcccc} \textit{boundaries} & : & X \times X & \to & \mathbb{R}^4 \\ & & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \Gamma(x_1, y_1) \\ \Gamma(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array}$$

One has

$$\partial X^{\Delta 2} = \textit{boundaries}^{-1}(\{0\}) - \Delta \partial X / \simeq .$$

$$BF = \{(x_1, x_2) \in \partial X \times S \mid f(x_1) = f(x_2)\}$$

$[x_1] \neq [x_2]$



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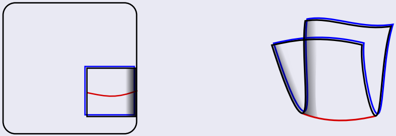

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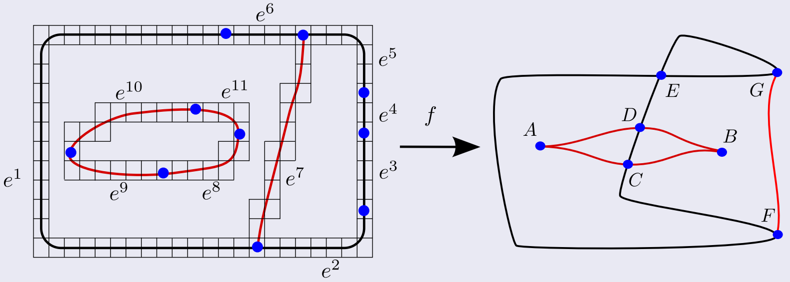
$$\begin{array}{ccc} X \times X & \to & \mathbb{R}^4 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \gamma(x_2, y_2) \\ f_1(x_1, y_1) - f_1(x_2, y_2) \\ f_2(x_1, y_1) - f_2(x_2, y_2) \end{pmatrix} \end{array}$$

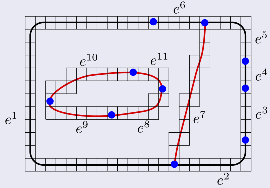
$[x_1] = [x_2]$

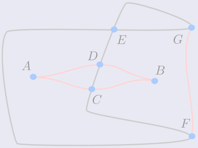
$$\begin{array}{ccc} X & \to & \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & \mapsto & \begin{pmatrix} \det df(x_1, y_1) \\ \gamma(x_1, y_1) \end{pmatrix} \end{array}$$

Analysis



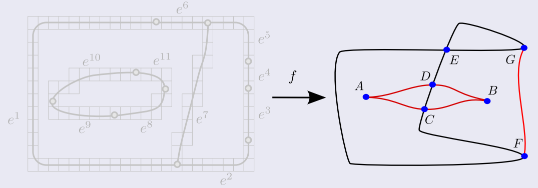
Synthesis





$$
\mathcal{X} = \begin{array}{c} \\ a \\ b \\ c_1 \\ c_2 \\ d_1 \\ d_2 \\ e_1 \\ e_2 \\ f \\ g \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} e^1 & e^2 & e^3 & e^4 & e^5 & e^6 & e^7 & e^8 & e^9 & e^{10} & e^{11} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Synthesis



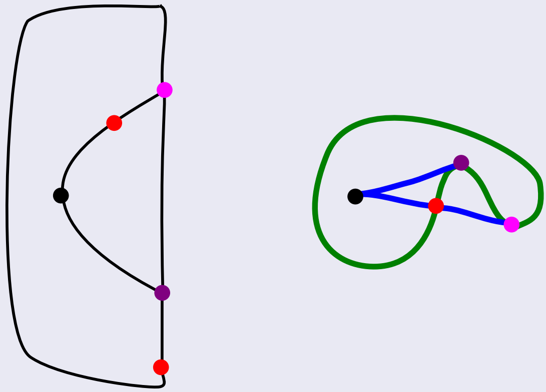
$$\mathcal{X}/f = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccccccc} e^1 & e^2 & e^3 & e^4 & e^5 & e^6 & e^7 & e^8 & e^9 & e^{10} & e^{11} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Theorem

For every portrait F of f, the 1-skeleton of ImF contains a subgraph that is an expansion of $\mathcal{X}/f$.

Conjecture

From $\mathcal{X}/f$ and its right embedding in $\mathbb{R}^2$ it is possible to create a portrait for f.



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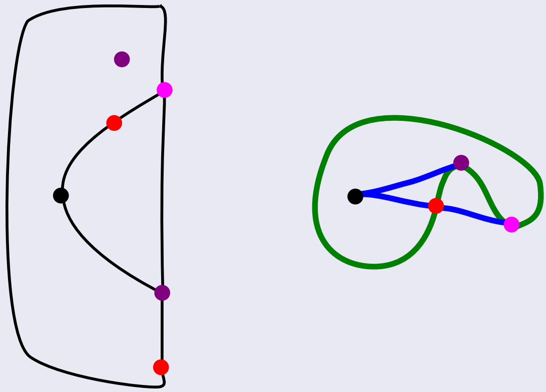

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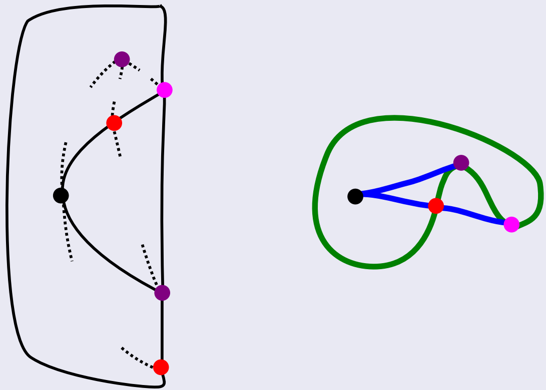

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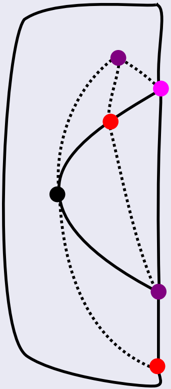

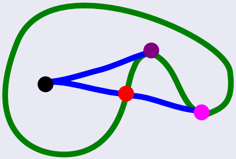

Source code is available on my webpage.

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Merci pour votre attention.