Chapitre 2

Petri nets

2.1 Principle

A *Petri net* is a graphical modeling language that can be used for the description of distributed systems or robots. A Petri net is a directed bipartite graph with two types of nodes : the *transitions* and *places*. The transitions are signified by bars and places are represented by circles. Arcs run either from a place to a transition or from a transition to a place. Places may contain *tokens*. A distribution of tokens over the places is called a *marking*. A transition may be *fired* is there is a token at the start of all its upstream place. When a transition fires one token is taken at each upstream place and one token is added to each downstream place.

Example. Consider a mission involving two sailboat robots. At initial time, both boats are in the harbor. In the way to reach the ocean, there is a channel with a strong current that changes with tides every 6 hours. It takes one hour for each boat the reach the channel from the harbor and two hours to reach the ocean from the entry of the channel, when the current is favorable. When the current is not favorable, the boats have to wait at the entrance of the channel. The situation can be represented by the Petri net of Figure 2.1. The meaning of places and transitions are given below

Transitions	**Events**
t_1	one boat leaves the port
t_2	one boat enters the channel
t_3	one boat reaches the ocean
t_4	low tide
t_5	high tide

Places	**Tokens**
p_1	boats in the harbor
p_2	Boats moving toward the channel
p_3	Boats inside the channel
p_4	favorable current
p_5	unfavorable current
p_6	allowed to enter the channel

Formal definition. A Petri net is a 3-tuple $(\mathcal{P}, \mathcal{T}, w)$ where $\mathcal{P}$ is a finite set of places, $\mathcal{T}$ is a finite set of transitions, $w : (\mathcal{P} \times \mathcal{T}) \cup (\mathcal{T} \times \mathcal{P}) \to \{0, 1\}$ is a set of arcs.

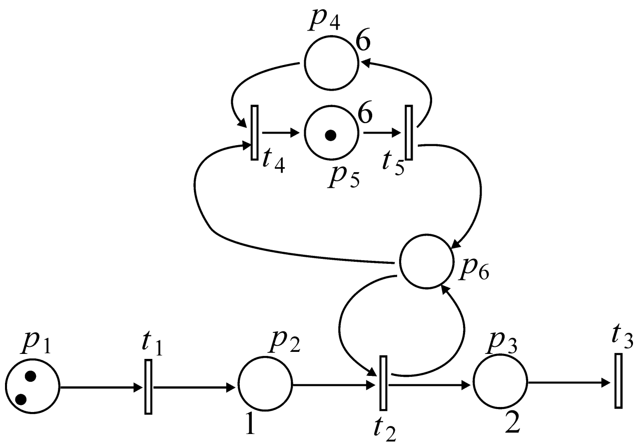

Figure 2.1 – Petri net representing a mission involving two robots

We define the *preset* of a transition t as the set of all its upstream places

$$\text{preset}\,(t) = \{p \in \mathcal{P}, w\,(p,t) = 1\}\,.$$

We define the *postset* of a transition t as the set of all the downsteam places

$$\text{postset}\,(t) = \{p \in \mathcal{P}, w\,(t,p) = 1\}\,.$$

For instance, $\text{preset}(t_4) = \{p_4, p_6\}$ and $\text{postset}(t_4) = \{p_5\}$.

Incidence matrix

We define the *forward incidence matrix* $\mathbf{W}^-$ and the *backward incidence matrix* $\mathbf{W}^+$, as the matrices with entries

$$w_{ij}^- = w\,(p_i, t_j) \text{ and } w_{ij}^+ = w\,(t_j, p_i)\,.$$

For our example, we have

$$\mathbf{W}^- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } \mathbf{W}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The *incidence matrix* is defined by

$$\mathbf{W} = \mathbf{W}^+ - \mathbf{W}^-.$$

For our example, we have

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

A *marking* is vector of integers which represents the number of tokens that are assigned to each place. It corresponds to the state vector of the system. For our example, the initial marking is

$$\mathbf{m}_0 = (2 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0)^{\mathrm{T}}\,.$$

Denote by $\mathbf{s}$ the vector the ith entry of which represents the numbers of firing of transition t_i from the beginning. Then, the marking is

$$\mathbf{m} = \mathbf{m}_0 + \mathbf{W}.\mathbf{s}.$$

For instance, if the following sequence of transition t_1, t_5, t_2 has been fired, then we have

$$\mathbf{m} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Which means that one boat is still in the harbor and one is in the channel.

2.2 Max-plus algebra

2.2.1 Dioid

Definition 1 (Monoid) *$(\mathcal{M}, \oplus, \varepsilon)$ is a monoid if $\oplus$ is a closed law, associative, and having a neutral element denoted ε ($\forall a \in \mathcal{M}, a \oplus \varepsilon = \varepsilon \oplus a = a$). If law $\oplus$ is commutative, the monoid is said to be commutative. The monoid is idempotent if $\forall a \in \mathcal{M}, a \oplus a = a$.*

For instance, $(\mathbb{R}, \max)$ is an idempotent commutative monoid.

Definition 2 (Semiring, dioid) *$(\mathcal{D}, \oplus, \otimes)$ is an idempotent semiring, also called dioid, if*

- *$(\mathcal{D}, \oplus, \varepsilon)$ is an idempotent commutative monoid , $\forall a \in \mathcal{D}, a \oplus a = a$,*
- *$(\mathcal{D}, \otimes, e)$ is a monoid,*
- *law $\otimes$ distributes over law $\oplus$, $(a \otimes (b \oplus c) = (a \otimes b \oplus a \otimes c))$*
- *ε is absorbing for law $\otimes$, $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.*

If $(\mathcal{D}, \otimes, e)$ is a commutative monoid, the idempotent semiring $(\mathcal{D}, \oplus, \otimes)$ is said to be commutative.

For instance, $(\overline{\mathbb{R}}, \max, +)$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is a dioid with $\varepsilon = -\infty$, $e = 0$.

Theorem. In a dioid $(\mathcal{D}, \oplus, \otimes)$, a solution of the implicit equation

$$x = a \otimes x \oplus b$$

is $x = a^* \otimes b$ where

$$a^* = \bigoplus_{k\geq 0} a^k = e \oplus a \oplus a^2 \oplus a^3 \oplus \dots$$

Proof. If $x = a^* \otimes b$, we have

$$\begin{aligned} a \otimes x \oplus b &= \underbrace{a \otimes a^*}_{\substack{= a \otimes (e \oplus a \oplus a^2 \oplus \dots) \\ = (a \oplus a^2 \oplus a^3 \oplus \dots)}} \otimes b \oplus \underbrace{b}_{=e \otimes b} \\ &= \left(e \oplus a \oplus a^2 \oplus a^3 \oplus \dots\right) \otimes b \\ &= a^* \otimes b = x. \end{aligned}$$

Example. Consider the equation

$$\begin{aligned} x &= -2 \otimes x \oplus 3 \\ &= \max(x-2, 3). \end{aligned}$$

in the dioid $(\overline{\mathbb{R}}, \max, +) = (\overline{\mathbb{R}}, \oplus, \otimes)$. One solution is $a^* \otimes b$ with $a = -2$ and $b = 3$. Thus

$$a^* \otimes b = \underbrace{\left(e \oplus a \oplus a^2 \oplus a^3 \oplus \dots\right)}_{=\max(0,-2,-4,\dots)} \otimes b = 0 \otimes 3 = 3.$$

2.2.2 Matrices in dioids

We shall now consider matrices in $(\overline{\mathbb{R}}, \max, +)$. The matrix sum is defined componentwise. For instance

$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \oplus \begin{pmatrix} e & 8 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & 7 \end{pmatrix}.$$

If $\mathbf{A} \in \mathbb{R}^{m\times p}$, $\mathbf{B} \in \mathbb{R}^{p\times n}$, the product $\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{m\times n}$ is the matrix with entries

$$c_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes a_{kj}.$$

The null matrix, denoted by ε, is the matrix whose entries are equal to ε. In the same manner the identity matrix, denoted by $\mathbf{I}$, is the matrix whose entries are all equal to ε excepted the diagonal entries which are equal to e. For example, if

$$\begin{pmatrix} 2 & 5 \\ \varepsilon & 3 \\ 1 & 8 \end{pmatrix} \otimes \begin{pmatrix} e \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}.$$

By extension for $n \in \mathbb{N}$,

$$\mathbf{A}^n = \underbrace{\mathbf{A} \otimes \mathbf{A} \otimes \dots \otimes \mathbf{A}}_{n \text{ times}}$$

with $\mathbf{A}^0 = \mathbf{I}$ the identity matrix. It can easily by shown that the set of matrices equipped with operations $\oplus, \otimes$, is a dioid. Moreover, the vector $\mathbf{x} = \mathbf{A}^*\mathbf{b}$ is a solution of the implicit equation

$$\mathbf{x} = \mathbf{A} \otimes \mathbf{x} \quad \oplus \quad \mathbf{b}. \tag{2.1}$$

2.3 Timed event graphs

Timed event graphs (TEG) constitute a subclass of timed Petri nets. Each place admits one and only one upstream transition and one and only one downstream transition. These dynamical systems can be represented by linear state equations

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned}$$

if the algebraic structure is changed into a (max,+) or a (min,+) algebra . The vector of input transitions is $\mathbf{u}$, the vector of internal transitions is $\mathbf{x}$ and y is the vector of output transitions. To each place is associated a delay which characterizes the minimal time that a token has to stay in a place before to contribute to the firing of the downstream transition. A transition is fired when each upstream place has a valid token, i.e. a token having spent the minimal time specified by the temporization. Two dual approaches exist to model a TEG by state equations : the dater approach and the counter approach.

2.3.1 Dater state equations

To the ith transition we associate the date $x_i(k) \in \mathbb{R}$ of the occurrence of kth firing of the transition. If we denote by $\oplus$ the max operator and by $\otimes$ the operator $+$. It is trivial to describe the dynamic of a given TEG by recurrence equations of the form

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}_0\mathbf{x}(k) \oplus \mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k), \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k). \end{aligned}$$

To get this form, it is sometimes necessarry to enlarge the graph in order to guarantee that each place is initially with at the most one token. The evolution equation has an implicit form :

$$\mathbf{x}(k) = \mathbf{A}_0\mathbf{x}(k) \;\oplus\; \underbrace{\mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k)}_{\mathbf{b}}.$$

Since the system is deterministic, it has a unique solution. From Equation (2.1), the solution of

$$\mathbf{x} = \mathbf{A}_0\mathbf{x} \oplus \mathbf{b}$$

is

$$\mathbf{x} = \mathbf{A}_0^*\mathbf{b}.$$

Thus

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}_0^*\left(\mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k)\right) \\ &= \mathbf{A}_0^*\mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{A}_0^*\mathbf{B}\mathbf{u}(k), \end{aligned}$$

or equivalently

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned}$$

with $\mathbf{A} = \mathbf{A}_0^*\mathbf{A}_1$ and $\mathbf{B} = \mathbf{A}_0^*\mathbf{B}_0$. The (max,+) toolbox of SCILAB, is very efficient to handle this kind of model.

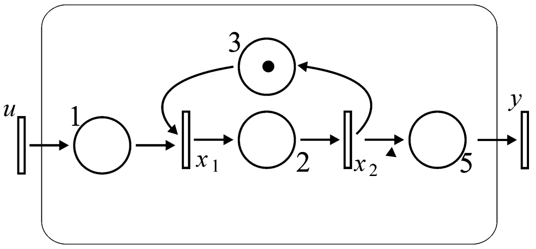

Figure 2.2 – A simple TEG

Example 1. Consider the TEG of Figure 2.2.To the ith transition we associate the date $x_i(k) \in \mathbb{R}$ of the occurrence of kth firing of the transition. We have

$$\begin{array}{lcl} x_1(k) & = & \max(1+u(k), 3+x_2(k-1)) \\ x_2(k) & = & 2+x_1(k) \\ y(k) & = & x_2(k)+5 \end{array}$$

or equivalently

$$\begin{array}{lcl} x_1(k) & = & 1 \otimes u(k) \oplus 3 \otimes x_2(k-1) \\ x_2(k) & = & 2 \otimes x_1(k) \\ y(k) & = & 5 \otimes x_2(k) \end{array}$$

which is linear in the idempotent semiring $(\mathbb{R}, \max, +)$. In a matrix form, we get

$$\left\{ \begin{array}{lcl} \mathbf{x}(k) & = & \underbrace{\begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{pmatrix}}_{\mathbf{A}_0} \mathbf{x}(k) \quad \oplus \quad \underbrace{\begin{pmatrix} \varepsilon & 3 \\ \varepsilon & \varepsilon \end{pmatrix}}_{\mathbf{A}_1} \mathbf{x}(k-1) \oplus \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\mathbf{B}} \mathbf{u}(k) \\ y(k) & = & \begin{pmatrix} \varepsilon & 5 \end{pmatrix} \mathbf{x}(k). \end{array} \right.$$

The evolution equations are

$$\mathbf{x}(k) = \mathbf{A}_0^* \mathbf{A}_1 \mathbf{x}(k-1) \ \oplus \ \mathbf{A}_0^* \mathbf{B} u(k).$$

Now,

$$\mathbf{A}_0^* = \begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{pmatrix}^* = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{pmatrix} \oplus \underbrace{\begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{pmatrix}^2}_{\begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}} \oplus \underbrace{\begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{pmatrix}^3}_{\begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}} + \cdots = \begin{pmatrix} 0 & \varepsilon \\ 2 & 0 \end{pmatrix}.$$

and

$$\mathbf{A}_0^* \mathbf{A}_1 = \begin{pmatrix} 0 & \varepsilon \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon & 3 \\ \varepsilon & \varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon & 3 \\ \varepsilon & 5 \end{pmatrix}.$$

Moreover

$$\mathbf{A}_0^* \mathbf{B} = \begin{pmatrix} 0 & \varepsilon \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

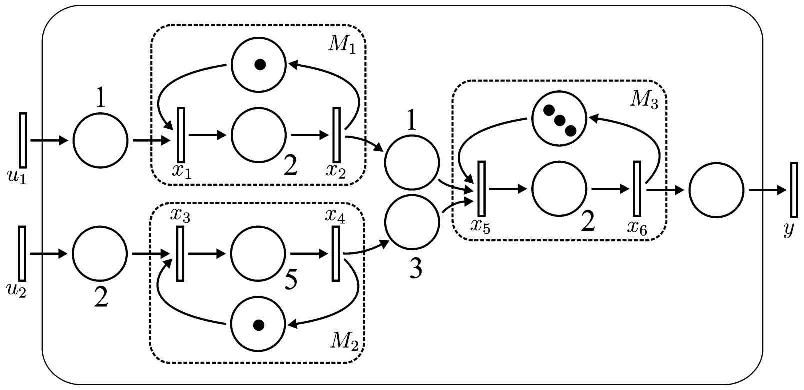

Figure 2.3 – Another timed event graph

The state equations of our TEG are

$$\begin{aligned} \mathbf{x}(k) &= \begin{pmatrix} \varepsilon & 3 \\ \varepsilon & 5 \end{pmatrix} \mathbf{x}(k-1) \oplus \begin{pmatrix} 1 \\ 3 \end{pmatrix} \mathbf{u}(k) \\ \mathbf{y}(k) &= \begin{pmatrix} \varepsilon & 5 \end{pmatrix} \mathbf{x}(k). \end{aligned}$$

Note that for this simple example, the same result could have been obtained directely from the initial equations. Since

$$\begin{aligned} x_1(k) &= \max(1+u(k), 3+x_2(k-1)) \\ x_2(k) &= 2+x_1(k) \\ y(k) &= x_2(k)+5 \end{aligned}$$

we have

$$\begin{aligned} x_1(k) &= \max(1+u(k), 3+x_2(k-1)) \\ x_2(k) &= 2+\max(1+u(k), 3+x_2(k-1)) = \max(3+u(k), 5+x_2(k-1)) \\ y(k) &= x_2(k)+5. \end{aligned}$$

It corresponds to what has been obtained with max-plus tools.

Example 2. Consider the TEG of Figure 2.3.

The input transitions are u_1,u_2, the internal transitions are $x_i, i \in \{1, \ldots, 6\}$ and y is the output transition. This TEG can represent the behavior of an assembly line, constituted of 3 machines M_1, M_2 and M_3. Transition u_1 characterizes the input of raw materials in the system, transition x_1 represents the input of material in machine M_1, it is possible if a token is available in the place located between transitions x_2 and x_1 (*i.e.* the machine has to be available), and transition u_1 has to be fired for one time unit. After 2 time units transition x_2 will be fired (output of machine M_1). Transition x_5 represents the input of machine M_3

which ensures the assembly of products coming from machines M_1 and M_2. For our TEG, we can write

$$\begin{array}{rcl} x_1(k) & = & \max(1+u_1(k), x_2(k-1)) \\ x_2(k) & = & 2+x_1(k) \\ x_3(k) & = & \max(2+u_2(k), x_4(k-1)) \\ x_4(k) & = & 5+x_3(k) \\ x_5(k) & = & \max(3+x_4(k), 1+x_2(k), x_6(k-3)) \\ x_6(k) & = & 2+x_5(k) \\ y(k) & = & x_6(k) \end{array}$$

or equivalently

$$\begin{array}{rcl} x_1(k) & = & 1 \otimes u_1(k) \oplus x_2(k-1) \\ x_2(k) & = & 2 \otimes x_1(k) \\ x_3(k) & = & 2 \otimes u_2(k) \oplus x_4(k-1) \\ x_4(k) & = & 5 \otimes x_3(k) \\ x_5(k) & = & 3 \otimes x_4(k) \oplus 1 \otimes x_2(k) \oplus x_6(k-3) \\ x_6(k) & = & 2 \otimes x_5(k) \\ y(k) & = & x_6(k) \end{array}$$

which is a linear system in the idempotent semiring $(\mathbb{R}, \max, +)$. In a vector form, we get

$$\left\{ \begin{array}{rcl} \mathbf{x}(k) & = & \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix} \mathbf{x}(k) \oplus \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k-1) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k-2) \\ & & \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k-3) \oplus \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & 2 \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \mathbf{u}(k) \\ y(k) & = & \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \mathbf{x}(k). \end{array} \right.$$

An implicit form with a recurrence of order one, can be obtained by enlarging the graph in order to guarantee that each place is initially with at the most one token as illustated by Figure 2.4.

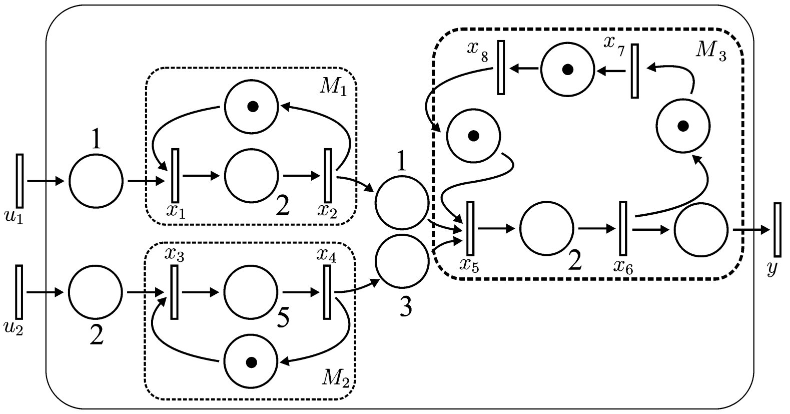

Figure 2.4 – Enlarged TEG

The corresponding equations are

$$
\begin{aligned}
\mathbf{x}(k) &= \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & 1 & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix} \mathbf{x}(k) \oplus \begin{pmatrix}
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon
\end{pmatrix} \mathbf{x}(k-1) \oplus \begin{pmatrix}
1 & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon
\end{pmatrix} \mathbf{u}(k) \\
y(k) &= \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k)
\end{aligned}
$$

For our example this calculus leads to

$$
\mathbf{A} = \mathbf{A}_0^* \mathbf{A}_1 = \begin{pmatrix}
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & 3 & \varepsilon & 8 & \varepsilon & \varepsilon & \varepsilon & e \\
\varepsilon & 5 & \varepsilon & 10 & \varepsilon & \varepsilon & \varepsilon & 2 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon
\end{pmatrix} \text{ and } \mathbf{B} = \mathbf{A}_0^* \mathbf{B} = \begin{pmatrix}
1 & \varepsilon \\
3 & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & 7 \\
4 & 10 \\
6 & 12 \\
\varepsilon & \varepsilon \\
\varepsilon & \varepsilon
\end{pmatrix}.
$$

2.3.2 Counter state equations

From a dual point of view, the behavior of a TEG can be described by considering a dynamic system in the time domain. A counter function x_i is associated to each transition which counts the number of firing of the transition at a time $k \in \mathbb{Z}$. For TEG of Figure 2.3, the following state equations are obtained :

$$\begin{array}{rcl} x_1(k) & = & \min(u_1(k-1), 1+x_2(k)) \\ x_2(k) & = & x_1(k-2) \\ x_3(k) & = & \min(u_2(k-2), 1+x_4(k)) \\ x_4(k) & = & x_3(k-5) \\ x_5(k) & = & \min(x_4(k-3), x_2(k-1), 3+x_6(k)) \\ x_6(k) & = & x_5(k-2) \\ y(k) & = & x_6(k) \end{array}$$

In the dioid $(\mathbb{Z}, \min, +)$, these equations are expressed by :

$$\begin{array}{rcl} x_1(k) & = & u_1(k-1) \oplus 1 \otimes x_2(k) \\ x_2(k) & = & x_1(k-2) \\ x_3(k) & = & u_2(k-2) \oplus 1 \otimes x_4(k) \\ x_4(k) & = & x_3(k-5) \\ x_5(k) & = & x_4(k-3) \oplus x_2(k-1) \oplus 3 \otimes x_6(k) \\ x_6(k) & = & x_5(k-2) \\ y(k) & = & x_6(k). \end{array} \tag{2.3}$$

These dynamic equations are linear in $(\mathbb{Z}, \min, +)$. In a vector form, we have

$$\left\{ \begin{array}{rcl} \mathbf{x}(k) & = & \begin{pmatrix} \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k-1) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \end{pmatrix} \mathbf{x}(k-2) \\ & & \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k-3) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \mathbf{x}(k-5) \\ & & \oplus \begin{pmatrix} e & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \mathbf{u}(k-1) \oplus \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & e \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \mathbf{u}(k-2) \\ y(k) & = & \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix} \mathbf{x}(k) \end{array} \right.$$

In a general manner a TEG can be represented in $(\mathbb{Z}, \min, +)$ by the following equations :

$$\left\{ \begin{array}{rcl} \mathbf{x}(k) & = & \bigoplus_{i=0}^{i_{\max}} \mathbf{A}_i \mathbf{x}(k-i) \oplus \bigoplus_{j=0}^{j_{\max}} \mathbf{B}_j \mathbf{u}(k-j) \\ \mathbf{y}(k) & = & \bigoplus_{\ell=0}^{\ell_{\max}} \mathbf{C}_l \mathbf{x}(k-\ell). \end{array} \right.$$

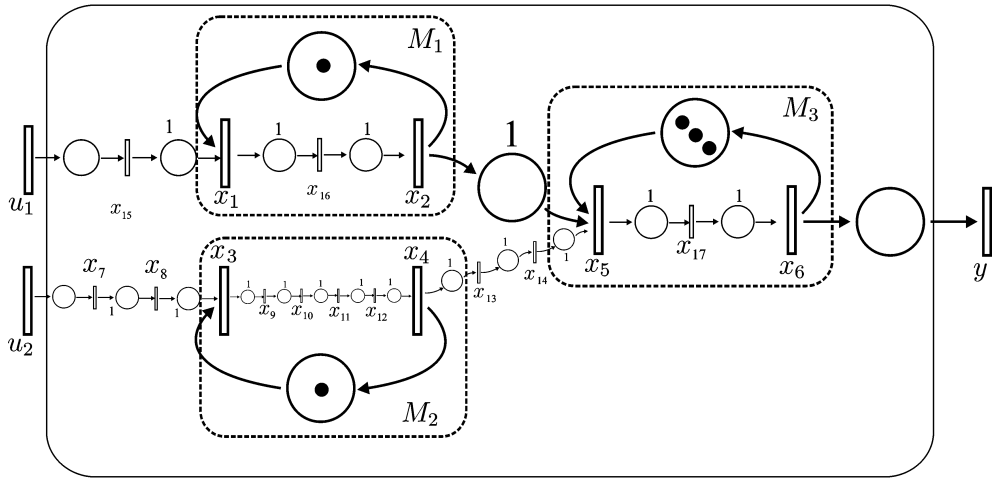

Figure 2.5 – Extended TEG to get one unit temporization at each place

After some extension of the state, it is possible to get a recursive formulation with a delay of one time unit, it consists of increasing the state in order to have only temporization of one time unit on each place. Figure 2.5 yields the corresponding extension of the TEG of Figure 2.3. As a consequence, our TEG can be written under the form

$$\begin{cases} \mathbf{x}(k) &= \mathbf{A}_0\mathbf{x}(k) \oplus \mathbf{A}_1\mathbf{x}(k-1) \oplus \mathbf{B}_0\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}_0\mathbf{x}(k). \end{cases}$$

From Equation (2.1), it is then possible to obtain the following explicit formulation :

$$\begin{cases} \mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k), \end{cases}$$

with $\mathbf{A} = \mathbf{A}_0^*\mathbf{A}_1$ and $\mathbf{B} = \mathbf{A}_0^*\mathbf{B}_0$.