Optimal separator for an hyperbola
Application to localization

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Abstract. This paper proposes a minimal contractor and a minimal separator for an area delimited by an hyperbola of the plane. The task is facilitated using actions induced by the hyperoctahedral group of symmetries. An application related to the localization of an object using a TDOA (Time Differential Of Arrival) technique is proposed.

1 Introduction

Consider the quadratic function

$$f(\mathbf{q},\mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \tag{1}$$

where $\mathbf{q} = (q_0, \dots, q_5)$ is the parameter vector and $\mathbf{x} = (x_1, x_2)$ is the vector of variables. Equivalently, we can write the function in a matrix form:

$$f(\mathbf{q},\mathbf{x}) = \mathbf{x}^{\mathrm{T}} \cdot \underbrace{\begin{pmatrix} q_3 & \frac{1}{2}q_4 \\ \frac{1}{2}q_4 & q_5 \end{pmatrix}}_{\mathbf{Q}} \cdot \mathbf{x} + \begin{pmatrix} q_1 & q_2 \end{pmatrix} \cdot \mathbf{x} + q_0. \tag{2}$$

The zeros of this quadratic function is a conic section (a circle or other ellipse, a parabola, or a hyperbola). The characteristic polynomial of the matrix $\mathbf{Q}$ is

$$\begin{array}{rcl} P(s) & = & (s-q_3)(s-q_5) - \frac{1}{4}q_4^2 \\ & = & s^2 - (q_3+q_5)s + q_3 q_5 - \frac{1}{4}q_4^2 \end{array}$$

Its discriminant is

$$\begin{array}{rcl} \Delta & = & (q_3+q_5)^2 - 4q_3q_5 + q_4^2 \\ & = & q_3^2 + q_5^2 - 2q_3q_5 + q_4^2 \\ & = & (q_3-q_5)^2 + q_4^2 \end{array}$$

which is always positive. Which means that the matrix $\mathbf{Q}$ has two real values (this is not a surprise since $\mathbf{Q}$ is symmetric). We will assume here that $\mathbf{Q}$ has eigen values with different signs. It means that

$$
\begin{array}{rc}
 & \left(q_3+q_5-\sqrt{\Delta}\right)\left(q_3+q_5+\sqrt{\Delta}\right)<0 \\
\Leftrightarrow & \left(q_3+q_5\right)^2-\Delta<0 \\
\Leftrightarrow & \left(q_3+q_5\right)^2-((q_3-q_5)^2+q_4^2)<0 \\
\Leftrightarrow & q_3^2+q_5^2+2q_3q_5-(q_3^2+q_5^2-2q_3q_5+q_4^2)<0 \\
\Leftrightarrow & 4q_3q_5-q_4^2<0 \\
\Leftrightarrow & \det\mathbf{Q}<0
\end{array}
$$

Define the set

$$\mathbb{X}=\{(x_1,x_2|f(\mathbf{q},\mathbf{x})\leq 0\}\,. \tag{3}$$

In our case $\mathbb{X}$ has a boundary which is an hyperbola. We wee call it an *hyperbolic area*. In this paper, we propose an interval-based method [15] to generate an optimal separator [11] for the set $\mathbb{X}$. The technique is similar to that proposed in [13] for ellipses. This separator will be used to generate an inner and an outer approximations for $\mathbb{X}$. As an application, we will consider the problem of the localization of an object using a TDOA technique.

This paper is organized as follows. Section 2 introduces the notion of symmetries that will be used in the construction of the separators. Section **??** builds the separator for the hyperbolic area. Section 5 illustrates the use of the separator to approximate the set of position for an object. Section 6 concludes the paper.

2 Symmetries

2.1 Conjugate pair

Define an equation of the form

$$f(\mathbf{q},\mathbf{x})=0. \tag{4}$$

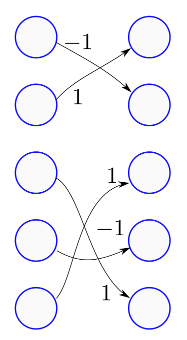
The pair of transformations (σ,γ) is *conjugate* with respect to f if

$$f(\gamma(\mathbf{q}),\sigma(\mathbf{x}))=0\Leftrightarrow f(\mathbf{q},\mathbf{x})=0. \tag{5}$$

2.2 Hyperoctahedral group

Transformations that will be consider are limited to the *hyperoctahedral group* B_n [4] which is the group of symmetries of the hypercube $[-1,1]^n$ of $\mathbb{R}^n$. The group B_n corresponds to the group of $n\times n$ orthogonal matrices whose entries are integers. Each line and each column of a matrix should contain one and only one non zero entry which should be either 1 or -1. Figure 1 shows different notations usually considered to represent a symmetry σ of B_5. We will prefer the Cauchy one line notation [20] which is shorter. We should understand the symmetry σ of the figure as the function:

$$\sigma(x_1,x_2,x_3,x_4,x_5)=(-x_2,x_1,x_5,-x_4,x_3). \tag{6}$$



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 5 & -4 & 3 \end{pmatrix}$$

Fig. 1: Different representations of an element σ of B_5. Left: graph; Top right: Matrix notation; Bottom right: Cauchy one line notation

Even if the matrix representation looks more intuitive, for efficiency reasons, we use the Cauchy one line representation to compose the symmetries.

In the plane, the group B_2 has eight elements. If we use the matrix form, the elements of B_2 are

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\} \tag{7}$$

or equivalently with the Cauchy notation

$$\{(1,2), (-1,2), (2,1), (-1,-2), (1,-2), (-2,1), (2,-1), (-2,-1)\}$$

Therefore, a symmetry of B_2 can also be written as

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = (\sigma_{11} + 2\sigma_{21}, \sigma_{12} + 2\sigma_{22})$$

with $\sigma_{ij}^2 \in \{0,1\}$, $\sigma_{i1}^2 + \sigma_{i2}^2 = 1$, $\sigma_{1j}^2 + \sigma_{1j}^2 = 1$.

2.3 Hyperbolic symmetry

Proposition 1. *Take a point* $\mathbf{x} = (x_1, x_2)$ *such*

$$f(\mathbf{q}, \mathbf{x}) \overset{(1)}{=} q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 = 0$$

and a symmetry

$$\sigma = (\sigma_{11} + 2\sigma_{21}, \sigma_{12} + 2\sigma_{22}) \in B_2$$

Define

$$\gamma = \left(q_0, \sigma_{11}q_1 + \sigma_{21}q_2, \sigma_{12}q_1 + \sigma_{22}q_2, \sigma_{11}^2 q_3 + \sigma_{21}^2 q_5, \sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21})q_4, \sigma_{12}^2 q_3 + \sigma_{22}^2 q_5\right)$$

The pair (σ^{-1}, γ) is conjugate with respect to f.

Proof. Define

$$\begin{aligned} x_1 &= \sigma_{11} \cdot y_1 + \sigma_{12} \cdot y_2 \\ x_2 &= \sigma_{21} \cdot y_1 + \sigma_{22} \cdot y_2 \end{aligned}$$

$$\begin{aligned} f(\mathbf{q}, \mathbf{x}) &= q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \\ &= q_0 + q_1(\sigma_{11}y_1 + \sigma_{12}y_2) + q_2(\sigma_{21}y_1 + \sigma_{22}y_2) + q_3(\sigma_{11}y_1 + \sigma_{12}y_2)^2 + q_4(\sigma_{11}y_1 + \sigma_{12}y_2)(\sigma_{21}y_1 + \sigma_{2} \\ &= q_0 + (\sigma_{11}q_1 + \sigma_{21}q_2)\, y_1 + (\sigma_{12}q_1 + \sigma_{22}q_2)\, y_2 + (\sigma_{11}^2 q_3 + \sigma_{21}^2 q_5) y_1^2 + (\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}) q_4 y_1 y_2 + (\sigma \end{aligned}$$

Thus

$$\begin{aligned} & f(\mathbf{q}, \mathbf{x}) = 0 \\ \Leftrightarrow\quad & f(\gamma(\mathbf{q}), \mathbf{y}) = 0 \\ \Leftrightarrow\quad & f(\gamma(\mathbf{q}), \sigma^{-1}(\mathbf{x})) = 0 \end{aligned}$$

□

2.4 Choice function

We thus get the choice function ψ [10]:

$$\psi_\sigma(\mathbf{q}) = \left(q_0, \alpha_{11}q_1 + \alpha_{21}q_2, \alpha_{12}q_1 + \alpha_{22}q_2, \alpha_{11}^2 q_3 + \alpha_{21}^2 q_5, \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})q_4, \alpha_{12}^2 q_3 + \alpha_{22}^2 q_5\right) \tag{8}$$

where $\alpha = \sigma^{-1}$.

Given a symmetry σ, this choice function allows us to get a symmetry γ such that (σ, γ) is a conjugate pair.

Note that in our implementation, the symmetries σ that are considered are involutive (i.e. $\alpha = \sigma$). This is why α does not appear.

3 Cardinal functions

3.1 Some definitions

Definition 1. A cardinal vector of $\mathbb{R}^n$ is a vector

$$\mathbf{e} = (e_1, \ldots, e_n)^{\mathrm{T}}$$

such that $\|\mathbf{e}\| = 1$ and $e_i \in \{-1, 0, 1\}$.

For instance $\mathbf{e}_3 = (0, 0, 1, 0)^{\mathrm{T}}$ and $\mathbf{e}_{-2} = (0, -1, 0, 0)$ are two cardinal vectors if $\mathbb{R}^4$. For use the notation $\mathbf{e}_i$ where $i \in I = \{-n, \ldots, -1, 1, \ldots, n\}$ to specify the cardinal vector. For instance $\mathbf{e}_{-2}$ is the vector parallel to the 2 axis and with a negative direction.

Definition 2. Given a closed set $\mathbb{X}$ of $\mathbb{R}^n$. A cardinal function φ_i with $I = \{-n, \ldots, -1, 1, \ldots, n\}$ is defined by

$$\varphi_i(x_1, \ldots, x_{|i|-1}, x_{|i|+1}, \ldots, x_n) = \max\left\{\mathbf{x}^{\mathrm{T}} \cdot \mathbf{e}_i \,|\, \mathbf{x} = (x_1, \ldots, x_{|i|-1}, x_i, x_{|i|+1}, \ldots, x_n) \in \mathbb{X}\right\} \tag{9}$$

Figure 2 shows in case of $n = 2$, a representation of the functions $\varphi_1(x_2)$ (red) and $\varphi_{-1}(x_2)$ (blue). The small squares correspond to cardinal points (East in red and West in blue).

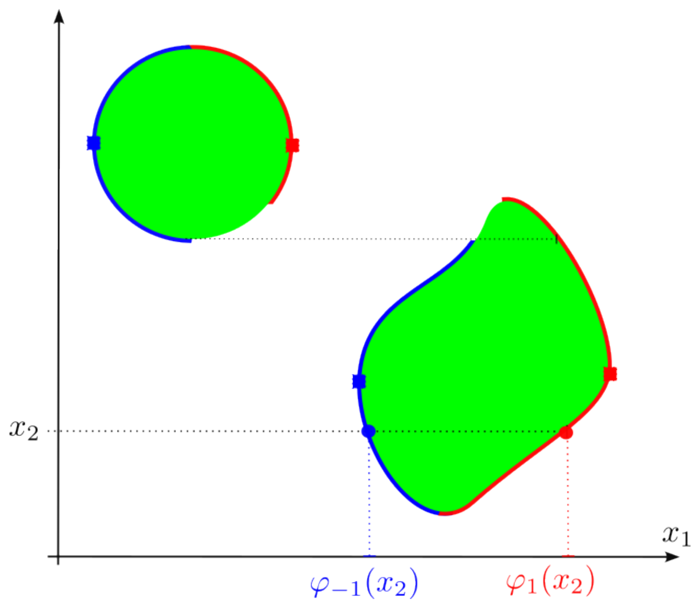


Fig. 2: Graphs of the functions $\varphi_1(x_2)$ (red) $\varphi_{-1}(x_2)$ (blue)

Figure 3 is a representation of the functions $\varphi_2(x_1)$ (black) and $\varphi_{-2}(x_1)$ (orange). The small squares correspond to cardinal points (North in black and South in orange).

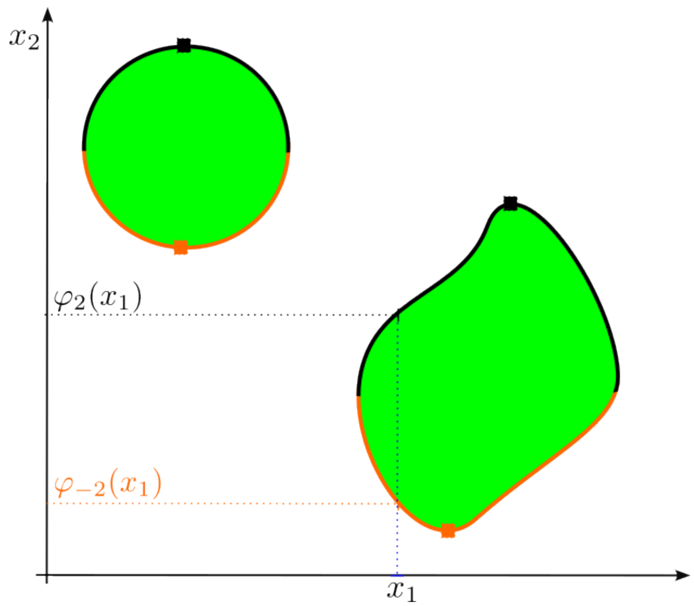


Fig. 3: Graphs of the functions $\varphi_2(x_1)$ (black) $\varphi_{-1}(x_2)$ (orange)

In Figure 2, we observe that graphs of the function φ_1 and φ_{-1} do not cover the boundary of $\mathbb{X}$. This is due to the fact that $\mathbb{X}$ is not row convex. We define the notion of row convexity (similar to the definition in [19])

Definition 3. A set $\mathbb{X} \subset \mathbb{R}^n$ is said to be row convex the boundary of $\mathbb{X}$ corresponds to the union of the graphs of its cardinal functions, *i.e.*,

$$\partial\mathbb{X} = \cup_i \text{graph}(\varphi_i)$$

3.2 Case of the hyperbola

For the hyperbola defined by

$$f(\mathbf{q}, \mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 = 0.$$

We have four cardinal functions $\varphi_i, i \in \{-2, -1, 1, 2\}$.

To find φ_1, we fix x_2 and we search for the maximal value for x_1 we yields the following theorem. Other cardinal functions will be obtained by symmetries.

Proposition 2. *Take a point $x = (x_1, x_2)$ such that $f(\mathbf{q}, \mathbf{x}) = 0$. Given x_2, the largest x_1 such that $f(\mathbf{x}) = 0$ is given by*

$$\begin{array}{rcl} x_1 & = & \varphi_1(\mathbf{q}, x_2) \\ & = & \frac{-(q_1+q_4x_2)+sign(q_3)\cdot\sqrt{(q_1+q_4x_2)^2-4q_1(q_0+q_2x_2+q_5x_2^2)}}{2q_3} \end{array} \quad (10)$$

Proof. Given x_2, let us compute the possible values for x_1. Since

$$f(\mathbf{q},\mathbf{x}) = q_3 x_1^2 + (q_1 + q_4 x_2)\, x_1 + q_2 x_2 + q_0 + q_5 x_2^2, \tag{11}$$

we get the following discriminant:

$$\Delta_1 = b_1^2 - 4a_1 c_1 \tag{12}$$

where

$$a_1 = q_3,\ b_1 = q_1 + q_4 x_2,\ c_1 = q_0 + q_2 x_2 + q_5 x_2^2 \tag{13}$$

The largest solution is

$$x_1 = \frac{-b_1 + \mathrm{sign}(a_1)\cdot\sqrt{\Delta_1}}{2a_1}. \tag{14}$$

We have thus proved (10) . □

Definition 4. The cardinal points are the (x_1, x_2) which belong to each graph of three functions φ_i, $i \in \{-2,-1,1,2\}$.

For instance a North belongs to the graphs of $\varphi_1, \varphi_{-1}, \varphi_2$ and a East belongs to the graphs of $\varphi_2, \varphi_{-2}, \varphi_1$. For our hyperbola we easily find that there exist four cardinal points. Of course, the cardinal points depend on $\mathbf{q}$.

Proposition 3. *Define the interval function*

$$\rho(\mathbf{q}) = \frac{-2q_3q_2 + q_1q_4 + [-1,1]\cdot\sqrt{(2q_3q_2 - q_1q_4)^2 - (4q_3q_5 - q_4^2)(4q_3q_0 - q_1^2)}}{4q_3q_5 - q_4^2} \tag{15}$$

If we set $[x_2] = \rho(\mathbf{q})$, *then the North is* $(x_2^-, \varphi_1(x_2^-))$ *and the South is* $(x_2^+, \varphi_1(x_2^+))$.

Note that is the square root is not defined, then there is no cardinal points.

Proof. A value for x_2 yields a feasible x_1 if $\Delta_1 \geq 0$ (see (3)), *i.e.*,

$$\begin{array}{lll} & b_1^2 - 4a_1c_1 & \geq 0 \\ \Leftrightarrow & -(q_1 + q_4x_2)^2 + 4q_3(q_0 + q_2x_2 + q_5x_2^2) & \geq 0 \\ \Leftrightarrow & (4q_3q_5 - q_4^2)x_2^2 + (4q_3q_2 - 2q_1q_4)x_2 + 4q_3q_0 - q_1^2 & \leq 0 \end{array}$$

which is quadratic in x_2. The discriminant is

$$\Delta_2 = b_2^2 - 4a_2c_2 \tag{16}$$

where

$$\begin{array}{lcc} a_2 & = & 4q_3q_5 - q_4^2 \\ b_2 & = & 4q_3q_2 - 2q_1q_4 \\ c_2 & = & 4q_3q_0 - q_1^2 \end{array} \tag{17}$$

The corresponding values for x_2 is

$$x_2 = \frac{-b_2 \pm \sqrt{(\Delta_2}}{2a_2}.$$

and the North corresponds to the smallest one and the South to the largest. □

Corollary. Take symmetry $\sigma = (1, 3, 2, 6, 5, 4)$ and set $[x_1] = \rho(\sigma(\mathbf{q}))$, the East is $(x_2^-, \varphi_1(x_2^-))$ and the West is $(x_2^+, \varphi_1(x_2^+))$.

Proof. The symmetry σ permutes x_1 and x_2. The East become the North and the West becomes the South. □

4 Separator for the hyperbola

4.1 Interval extension of the cardinal function

Let us assume that $\mathbf{q}$ is fixed. The dependency with respect to the parameter vector $\mathbf{q}$ will omitted for simplicity. As defined in the book of Moore [15], the interval extension function of $\varphi_1(x_2)$ is

$$[\varphi_1]([x_2]) = [\{x_1 | \exists x_2 \in [x_2], x_1 = \varphi_1(x_2)\}]$$

which returns the smallest interval which contains the set $\varphi_1([x_2])$. The same definition applies for other direction to get $[\varphi_{-1}]([x_2]), [\varphi_2]([x_1])$ and $[\varphi_{-2}]([x_1])$.

Due to the monotonicity of φ_1 between the cardinal points, we have

$$[\varphi_1]([x_2]) = [\varphi_1(\{x_2^-, x_2^+, c_2(1), c_2(2), \dots\})]$$

where $\mathbf{c}(1), \mathbf{c}(2), \dots$ are the cardinal points inside the box $[-\infty, \infty] \times [x_2]$.

Take for instance $\mathbf{q} = (-1, 5, 2, -2, 30, -2)$, *i.e.*,

$$f(x_1, x_2) = -1 + 5x_1 + 2x_2 - 2x_1^2 + 30x_1x_2 - 2x_2^2.$$

For an interval sampling $[x_2] = \frac{1}{5} \cdot [k, k+1]$, $k \in \mathbb{N}$, the function $[\varphi_1]([x_2])$ generates the red boxes of Figure 4. If we do the same for $[\varphi_{-1}]([x_2])$, we get the blue boxes. The small black square corresponds to the North and small orange square corresponds to the South.

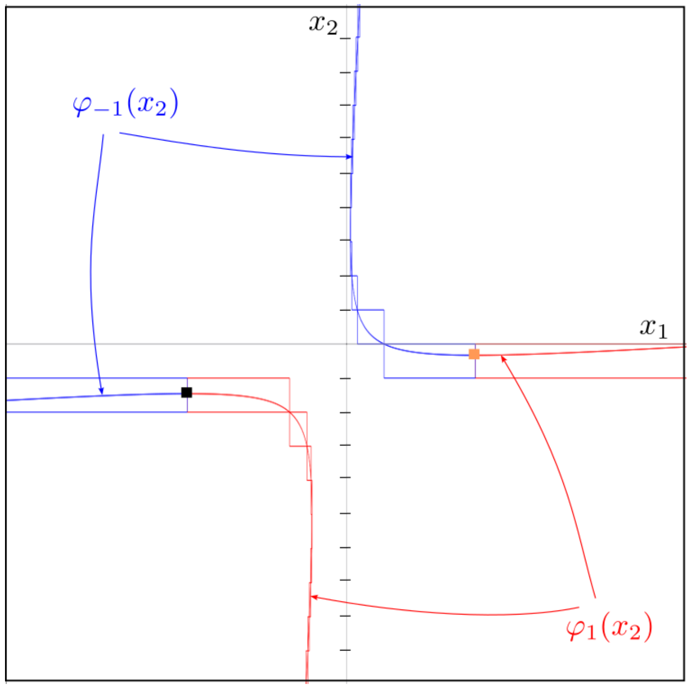


Fig. 4: Minimal inclusion for $\varphi_1([x_2])$ (red) and $\varphi_{-1}([x_2])$ (blue). The frame box in $[-2, 2]$

For a similar sampling along x_1, Figure 5 represents $[\varphi_2]([x_1])$ (black) and $[\varphi_{-2}]([x_1])$ (orange).

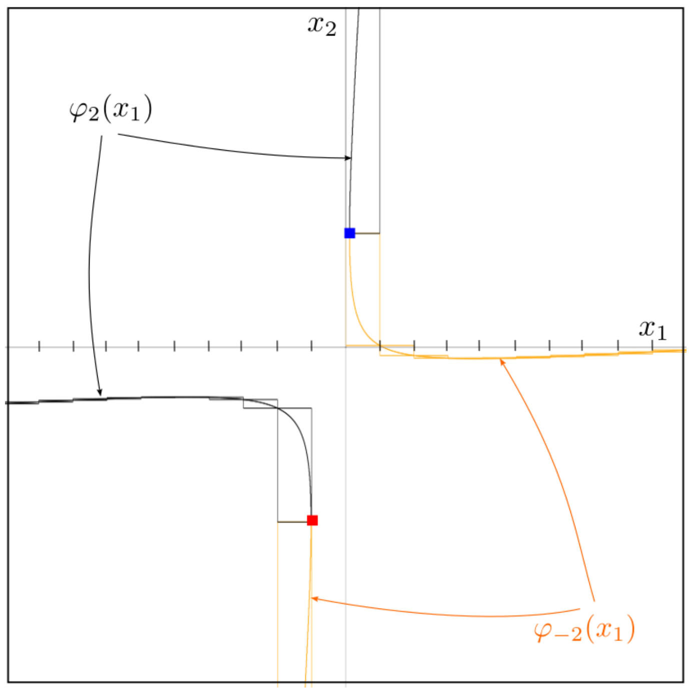


Fig. 5: Minimal inclusion for $\varphi_2([x_1])$ (black) and $\varphi_{-2}([x_1])$ (orange)

4.2 Seed contractor

From the interval evaluation, we can build a contractor for the set $x_1 = \varphi_1(x_2)$. It is is given by

$$C_0 : [\mathbf{x}] \to [x_1] \times [\varphi_1]([x_2]). \tag{18}$$

This contractor will be called a *seed contractor* because it will be used to construct all other contractors using symmetries. The contractor (18) is not minimal. It is only minimal with respect to x_1. Since this contractor depends on $\mathbf{q}$, we will write $C_0^{\mathbf{q}}$.

We understand that $C_0^{\mathbf{q}}$ correspond to a small portion of the hyperbola. The main challenge is now to build the the separator for the whole hyperbola using a single parametric contractor $C_0^{\mathbf{q}}$ and symmetries. Of course, we could add some other seed contractors, but our idea is to factorize the implementation as much as possible to avoid bugs and make the code adaptable to other type of sets.

4.3 Contractor for the hyperbola

We have a contractor $C_0^{\mathbf{q}}$ which is minimal in the direction of x_2. Recall that $C_0^{\mathbf{q}}([\mathbf{x}])$ contracts the box $[\mathbf{x}]$ with respect to a small portion of the hyperbola.

Using the notion of contractor action [8], we show how we can extend this contractor $C_0^{\mathbf{q}}$ to other portions. We recall that the action of a symmetry σ to the contractor C is defined by

$$\sigma \bullet C([\mathbf{x}]) = \sigma \circ C \circ \sigma^{-1}([\mathbf{x}]).$$

This means that $\sigma \bullet C$ is a contractor that has been built from the contractor C as follows:

- Apply to the box $[\mathbf{x}]$ the symmetry σ^{-1}
- Apply the contractor C
- Apply to the resulting box $C \circ \sigma^{-1}([\mathbf{x}])$ the symmetry σ.

For the hyperbola, we can make a partition of curve into four portions :

- North-East : $\mathbb{X}^{(1,2)} = \{(x_1, x_2 | x_1 = \varphi_1(x_2) \text{ and } x_2 = \varphi_2(x_1)\}$
- North-West : $\mathbb{X}^{(1,-2)} = \{(x_1, x_2 | x_1 = \varphi_1(x_2) \text{ and } x_2 = \varphi_{-2}(x_1)\}$
- South-East : $\mathbb{X}^{(-1,2)} = \{(x_1, x_2 | x_1 = \varphi_{-1}(x_2) \text{ and } x_2 = \varphi_2(x_1)\}$
- South-West : $\mathbb{X}^{(-1,2)} = \{(x_1, x_2 | x_1 = \varphi_{-1}(x_2) \text{ and } x_2 = \varphi_{-2}(x_1)\}$

If we consider the pair (σ, γ) conjugate with respect to the hyperbola, the contractor $\sigma \bullet C_0^{\psi_\sigma(\mathbf{q})}$ is associated to another part of the hyperbola. The selection of the symmetries (σ, γ) to be selected is made using the choice function (8). These symmetries can be understood geometrically but can also be computed automatically as shown in [8].

To understand the construction, consider the symmetry $\sigma = (2, 1) \in B_2$. The contractor associated to $\mathbb{X}^{(1,2)}$:

$$C_1^{\mathbf{q}}([\mathbf{x}]) = \left(\sigma \bullet C_0^{\psi_\sigma(\mathbf{q})} \cap C_0^{\mathbf{q}}\right)([\mathbf{x}])$$

It is minimal with respect to both directions x_1 and x_2 as illustrated by Figure 6. Note that the North-East portion is delimited by the two cardinal points North (black square) and East (red square).

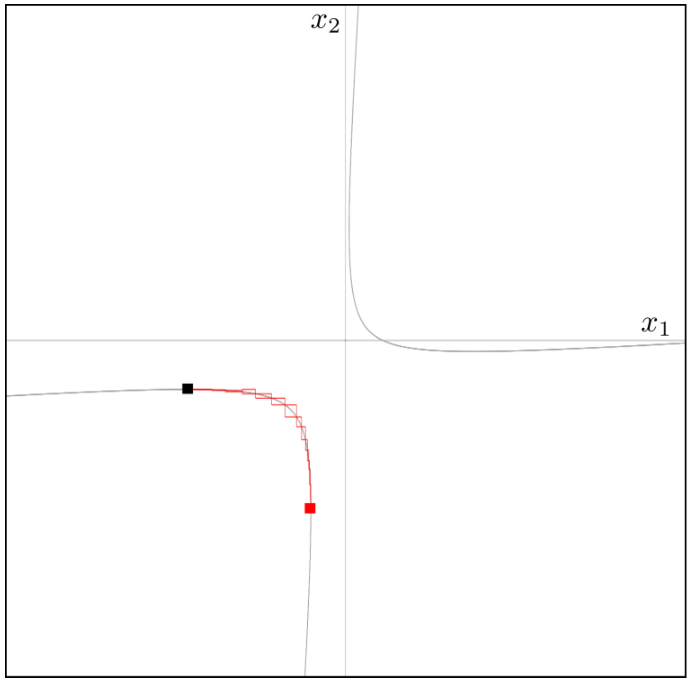


Fig. 6: Approximation of the North-East portion of the hyperbola using $(2,1)\bullet C_0^{\psi_{(2,1)}(\mathbf{q})}\cap C_0^{\mathbf{q}}$

The following proposition shows that the contractor for the hyperbola can be expressed by a simple formula involving symmetries and the seed contractor $C_0^{\psi_\sigma(\mathbf{q})}$. Getting such a formula will ease the implementation of the contractor.

Proposition 4. *A minimal contractor associated to* $\mathbf{f}(\mathbf{q},\mathbf{x})=0$ *is*

$$\bigcup_{\sigma\in\{(1,2),(1,-2),(-1,2),(-1,-2)\}}\sigma\bullet\left((2,1)\bullet C_0^{\psi_{(2,1)}\cdot\psi_\sigma(\mathbf{q})}\cap C_0^{\psi_\sigma(\mathbf{q})}\right)$$

Proof. The minimal contractor for the North-East portion $\mathbb{X}^{(1,2)}$ is

$$C_1^{\mathbf{q}}=(2,1)\bullet C_0^{\psi_{(2,1)}(\mathbf{q})}\cap C_0^{\mathbf{q}}. \tag{19}$$

The three other portions can be defined by applying symmetries in $\{(1,-2),(-1,2),(-1,-2)\}$. Since, the union of contractors is minimal, we get

$$\bigcup_{\sigma\in\{(1,2),(1,-2),(-1,2),(-1,-2)\}}\sigma\bullet C_1^{\psi_\sigma(\mathbf{q})}$$

is a minimal contractor for $\mathbf{f}(\mathbf{q},\mathbf{x})=0$. Combining with (19), we get the minimal contractor with respect to the seed contractor $C_0^{\mathbf{q}}$. $\square$

Figure 7 illustrates the minimality of the contractor for the hyperbola.

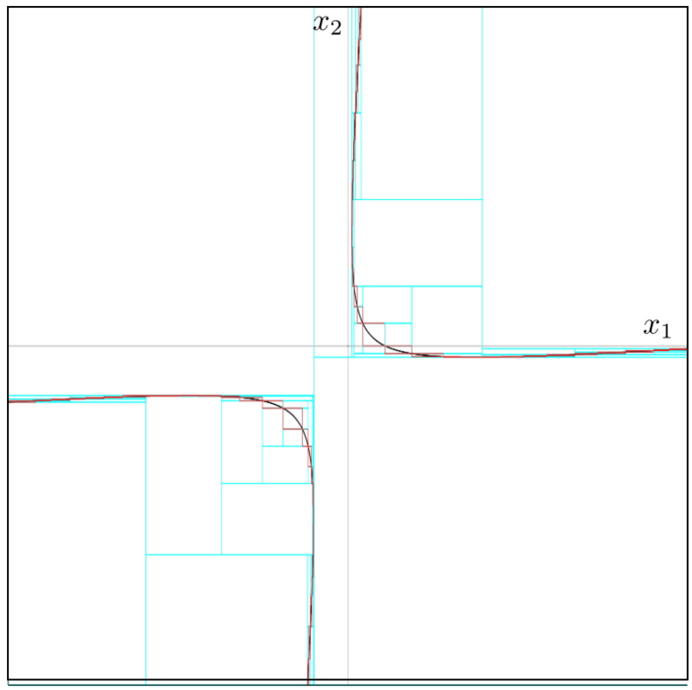


Fig. 7: Illustration of the minimality of the contractor for the hyperbola

4.4 Minimal separator for the hyperbola area

This section proposes an optimal separator for an hyperbola area defined by

$$\mathbb{X} = \{\mathbf{x} | q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \leq 0\}. \tag{20}$$

This separator is then used by a paver to compute boxes that are completely inside or outside the solution set.

As shown in [13], from a contractor on the boundary of a set $\mathbb{X}$ and a test for $\mathbb{X}$, we can obtain a separator. As a consequence, we can get an inner and an outer approximations for $\mathbb{X}$ as illustrated by Figure 8 for $\mathbf{q} = (-1, 5, 2, -2, 30, -2)^T$. The magenta boxes are proved to be inside $\mathbb{X}$ and the blue is outside $\mathbb{X}$. The accuracy is taken as $\varepsilon = 0.1$ and corresponds to the size of the small uncertain boxes (yellow). The cardinal points (North, South, West, East) are represented by the small squares (black, orange, blue, red).

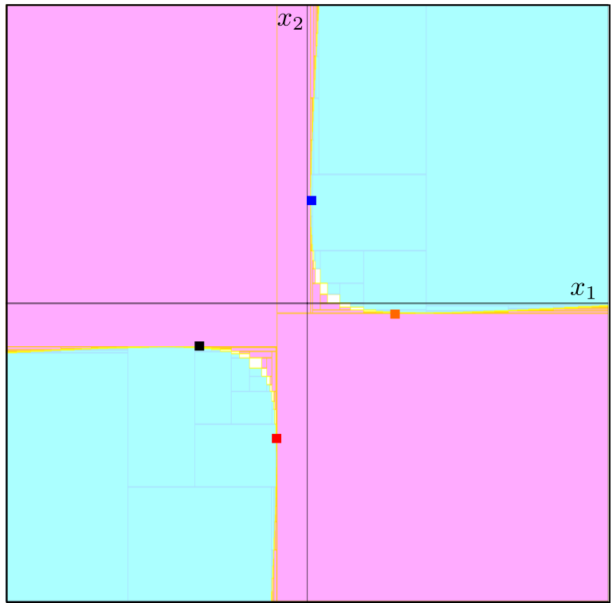


Fig. 8: Approximation of the hyperbola area obtained by our minimal separator for the hyperbola set

Figure 9 corresponds to the approximation obtained with the same accuracy with a classical forward-backward contractor. The benefice of our method seems small, but we will see later, that the improvement can become significant.

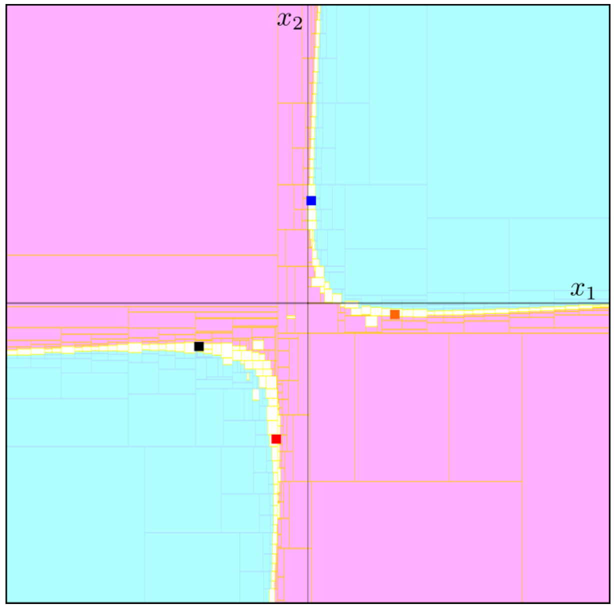


Fig. 9: Hyperbola area computed using a classical forward-backward contractor

For $\mathbf{q} = (-1, 1, 1, 3, 30, -2)$ we only have two cardinal points (West and Eats). The formula provided by Proposition 4 is still valid and we are able to generate Figure 10. This shows the ability of the symmetries to consider different situations easily and in elegantly.

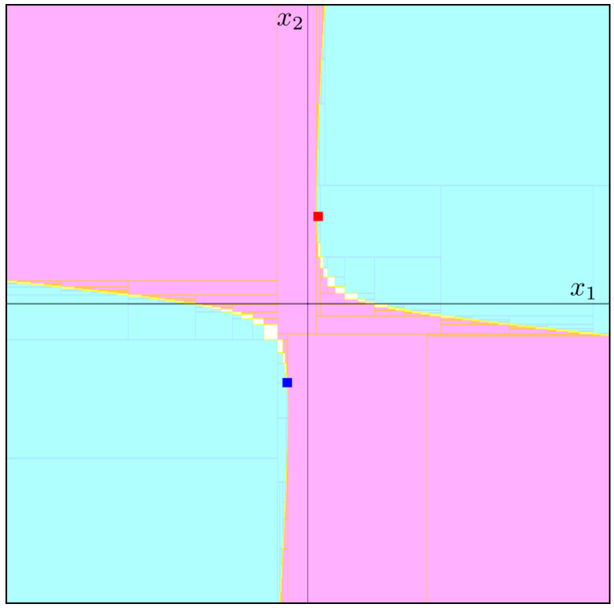


Fig. 10: Illustration of the application of the separator for the hyperbola set

If we compare with a classical forward-backward contractor [2] (see **??**) of other contractors such as [1] our contractor yields a more accurate approximation.

Remark. We have assumed that we had no uncertainties on $\mathbf{q}$. In case of interval uncertainty, the set to be characterized becomes

$$\mathbb{X} = \{\mathbf{x} | \exists \mathbf{q} \in [\mathbf{q}], q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \leq 0\}. \tag{21}$$

The resolution is still possible as shown in [10].

5 Application

Interval methods have been used for localization of robots for several decades [12][18][3][5]. This section proposes to deal with a specific localization problem where the sum of distances are measured.

5.1 Hyperbola from foci

Proposition 5. *Consider two points* $\mathbf{a}, \mathbf{b}$ *of the plane. The set* $\mathbb{X}$ *of all points such that*

$$\|\mathbf{x} - \mathbf{a}\| - \|\mathbf{x} - \mathbf{b}\| \leq \ell \tag{22}$$

is an ellipse with foci points $\mathbf{a}, \mathbf{b}$. *The set* $\mathbb{X}$ *is defined by the inequality*

$$\mathbf{f}_{\mathbf{a},\mathbf{b},\ell}(\mathbf{x}) \leq 0 \tag{23}$$

where

$$\mathbf{f}_{\mathbf{a},\mathbf{b},\ell}(\mathbf{x}) = q_0 + q_1 x_1 + q_2 x_2 + q_3 x_1^2 + q_4 x_1 x_2 + q_5 x_2^2 \tag{24}$$

with

$$\begin{array}{lcl}
q_0 & =- & a_1^4 - 2a_1^2 a_2^2 + 2a_1^2 b_1^2 + 2a_1^2 b_2^2 + 2a_1^2 \ell^2 \\
 & & -a_2^4 + 2a_2^2 b_1^2 + 2a_2^2 b_2^2 \\
 & & +2a_2^2 \ell^2 - b_1^4 - 2b_1^2 b_2^2 + 2b_1^2 \ell^2 - b_2^4 + 2b_2^2 \ell^2 - \ell^4 \\
q_1 & = & 4a_1^3 - 4a_1^2 b_1 + 4a_1 a_2^2 - 4a_1 b_1^2 - 4a_1 b_2^2 \\
 & & -4a_1 \ell^2 - 4a_2^2 b_1 + 4b_1^3 + 4b_1 b_2^2 - 4b_1 \ell^2 \\
q_2 & = & 4a_1^2 a_2 - 4a_1^2 b_2 + 4a_2^3 - 4a_2^2 b_2 - 4a_2 b_1^2 \\
 & & -4a_2 b_2^2 - 4a_2 \ell^2 + 4b_1^2 b_2 + 4b_2^3 - 4b_2 \ell^2 \\
q_3 & = & -4a_1^2 + 8a_1 b_1 - 4b_1^2 + 4\ell^2 \\
q_4 & = & -8a_1 a_2 + 8a_1 b_2 + 8a_2 b_1 - 8b_1 b_2 \\
q_5 & = & -4a_2^2 + 8a_2 b_2 - 4b_2^2 + 4\ell^2
\end{array}$$

Proof. We have

$$\begin{array}{ll}
 & \|\mathbf{x} - \mathbf{a}\| - \|\mathbf{x} - \mathbf{b}\| = \ell \\
\Rightarrow & \left(\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} - \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2}\right)^2 = \ell^2 \\
\Leftrightarrow & (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_1 - b_1)^2 + (x_2 - b_2)^2 - 2\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}\sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2} \\
\Leftrightarrow & (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_1 - b_1)^2 + (x_2 - b_2)^2 - \ell^2 = 2\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}\sqrt{(x_1 - b_1)^2 + (x_2 - \\
\Leftrightarrow & \left((x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_1 - b_1)^2 + (x_2 - b_2)^2 - \ell^2\right)^2 - 4\left((x_1 - a_1)^2 + (x_2 - a_2)^2\right)\left((x_1 - b_1)^2 + (x_2
\end{array} \tag{25}$$

i.e.

$$\left((x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_1 - b_1)^2 + (x_2 - b_2)^2 - \ell^2\right)^2 = 4\left((x_1 - a_1)^2 + (x_2 - a_2)^2\right)\left((x_1 - b_1)^2 + (x_2 - b_2)^2\right.$$

After some trivial symbolic calculus, we get to get rid of the square root to get

$$\begin{array}{rcl}
4\left((x_1 - a_1)^2 + (x_2 - a_2)^2\right)\left((x_1 - b_1)^2 + (x_2 - b_2)^2\right) & & \\
-\left(\ell^2 - (x_1 - a_1)^2 - (x_2 - a_2)^2 - (x_1 - b_1)^2 - (x_2 - b_2)^2\right)^2 & = & 0
\end{array} \tag{26}$$

We can develop the expression to get the coefficients of the proposition. □

5.2 Localization

We consider an example taken from [7] related to localization which can be seen as special case of interval data fitting problem [14]. Consider a robot which emits a sound at an unknown time t_0. This sound is received with a delay by three microphones located points $\mathbf{a} : (13, 7), \mathbf{b} : (4, 6), \mathbf{c} : (16, 10)$ of the plane (see Figure 11). From the time of flight of the sound we want to estimate the position of the object.

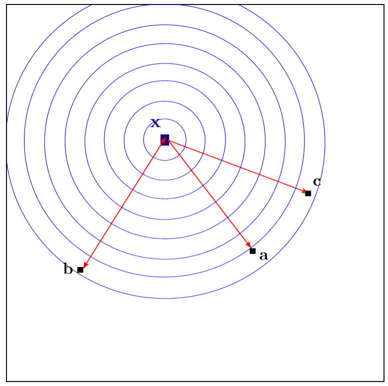


Fig. 11: The robot at position $\mathbf{x}$ emits a sound received later by three microphones $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

We assume that we were able to collect two distance intervals such that $\ell_b \in [7.9, 8.1]$ and $\ell_c \in [3.9, 4.1]$. The solution set $\mathbb{X}$ is defined by

$$\begin{array}{lrcl} \text{(i)} & \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| & = & \ell_b \in [7.9, 8.1] \\ \text{(ii)} & \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{c}\| & = & \ell_c \in [3.9, 4.1] \end{array} \quad (27)$$

From Proposition 5, we get that $\mathbb{X}$ is defined by

$$\mathbb{X} : \begin{cases} \mathbf{f}_{\mathbf{a},\mathbf{b},6}(\mathbf{x}) \leq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{b},4}(\mathbf{x}) \geq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{c},9}(\mathbf{x}) \leq 0 \\ \mathbf{f}_{\mathbf{a},\mathbf{c},7}(\mathbf{x}) \geq 0 \end{cases} \quad (28)$$

Using a paver, we are thus able to get in inner and an outer approximations for the set of $\mathbb{X}$ (see Figure **??**).The frame box is $[-7, 7] \times [-7, 7]$. Figure 12 represents the inequality (12,i) and Figure 13 correspond to the inequality (12,ii). All results are guaranteed since outward rounding is implemented [16].

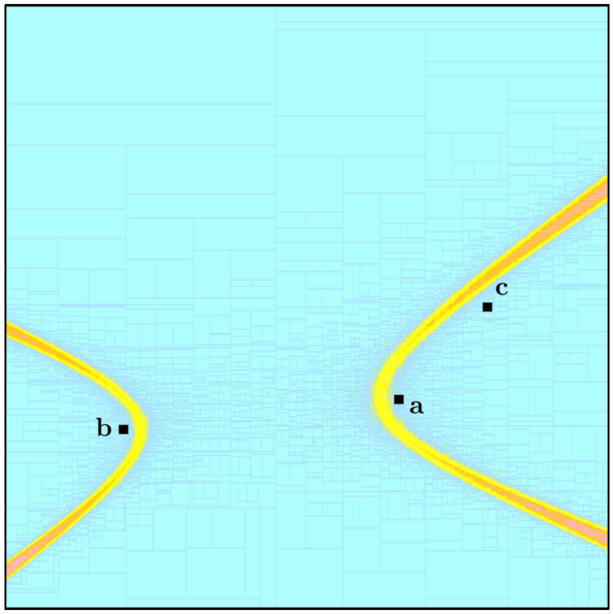


Fig. 12: Set of positions consistent with the path $\mathbf{a}, \mathbf{b}$

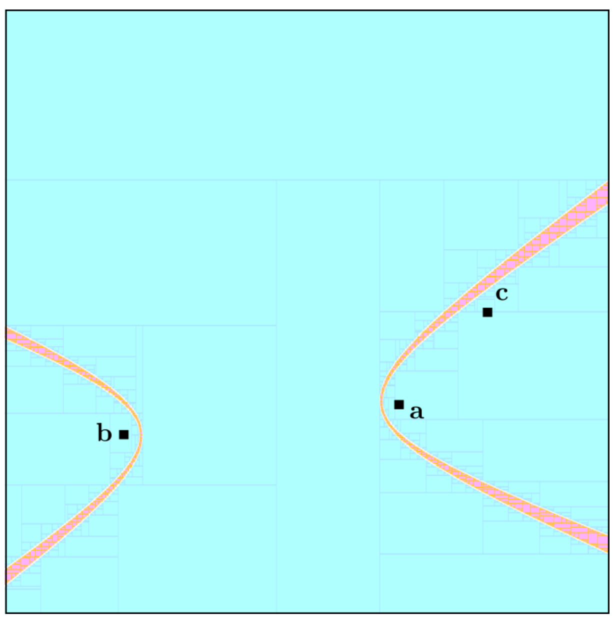


Fig. 13: Set of positions consistent with 2 $\mathbf{a}, \mathbf{c}$

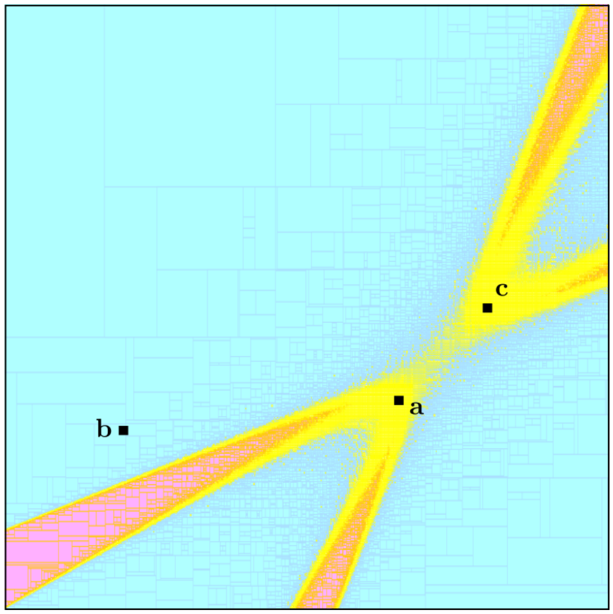


Fig. 14: Set of positions consistent with 2 $\mathbf{a}, \mathbf{c}$

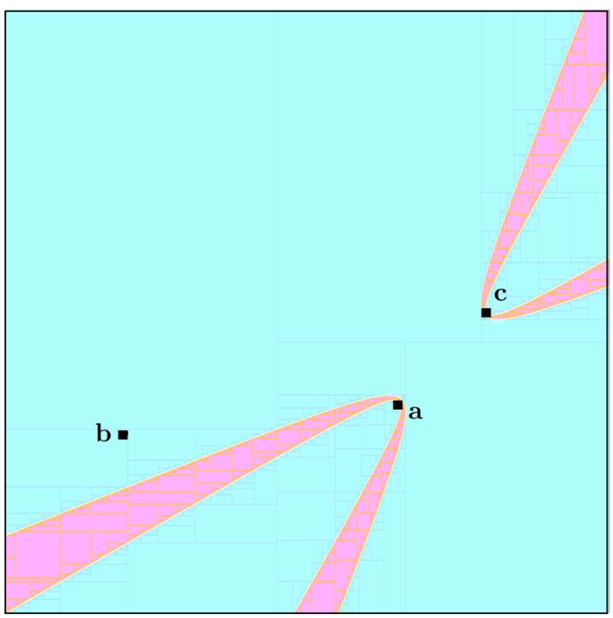


Fig. 15: Set of positions consistent with 2 $\mathbf{a}, \mathbf{c}$

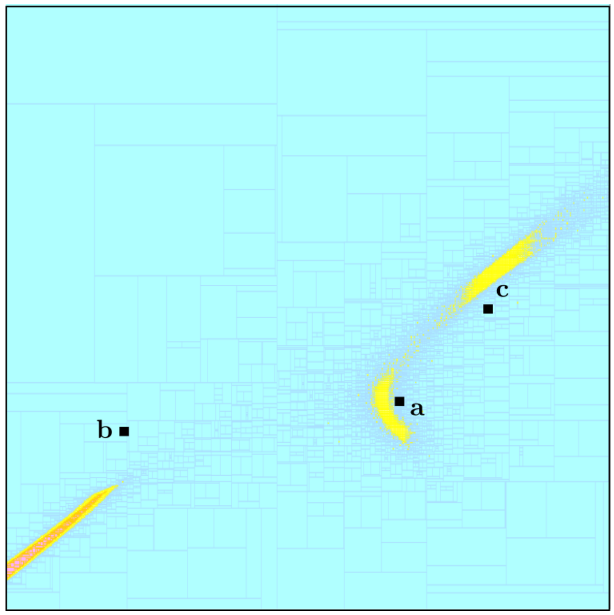
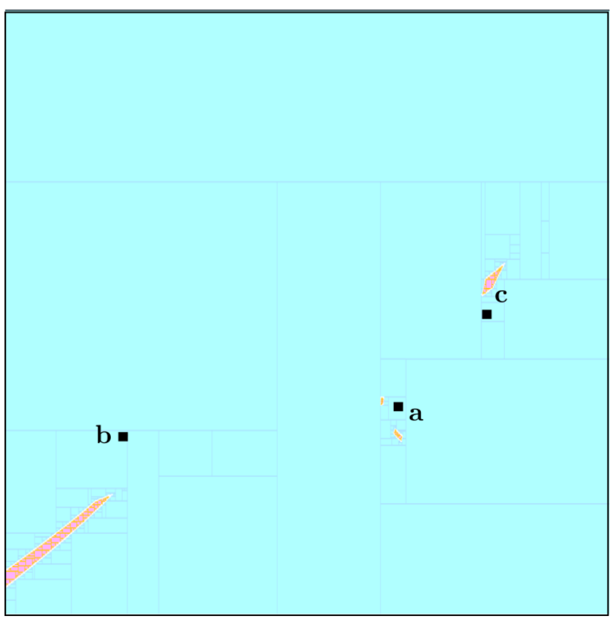


Fig. 16: Set of positions consistent 6



(a) Set of positions consistent 6

6 Conclusion

This paper has proposed a minimal contractor and a minimal separator for an hyperbola area of the plane. The notion of actions derived from hyperoctahedral symmetries allowed us to limit the analysis in on part of the constraint where the monotonicity can be assumed. The symmetries was used to extend the analysis to the whole plane.

The goal of this paper was to provide a simple example which illustrates how to use the hyperoctahedral symmetries in order to build minimal separators. Now, as shown in [10], the use of these symmetries is more interesting when we deal with projection problems where quantifier elimination is needed. This type of projection problem is indeed much more difficult to solve with classical interval approaches [6].

When we build an optimal contractor for a set $\mathbb{X}$ using symmetries, the main difficulty is to find the portion of the set that can be used to reconstruction $\mathbb{X}$ using a copy-paste process yield to the actions of the symmetries. For the hyperbola, the pattern is a cardinal function and for the ellipse, it was a quarter of the ellipse. But there is no general procedure to find the right pattern.

The Python code based on Codac [17] is given in[9].

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