us to Theorem 2. Consider the affine space

$$\mathcal{P} = \{P \in \mathbf{R}^{n\times n},\ P = P^T \quad \text{and} \quad PG = H^T\} \tag{4.6}$$

and define

$$\sigma(P) = \lambda_{\max}\begin{bmatrix} Q^T(PF + F^TP)Q & 0 \\ 0 & -Q^TPQ \end{bmatrix} \tag{4.7}$$

where $\lambda_{\max}$ denotes the largest eigenvalue. If we find a matrix P from $\mathcal{P}$ such that $\sigma(P) \le 0$, then the transfer function matrix $Z(s)$ is positive real. Hence, we have the constraint eigenvalue optimization problem

$$\inf_{P\in\mathcal{P}}\sigma(P). \tag{4.8}$$

The crucial point is to find a handy basis for the affine space $\mathcal{P}$, i.e.,

$$P = P(\gamma) = P_0 + \sum_{j=1}^{m}\gamma_j P_j, \qquad \gamma = (\gamma_1, \cdots, \gamma_m)^T \tag{4.9}$$

where $P_0, P_1, \cdots, P_m$ are symmetric with $P_0G = H^T$ and $P_jG = 0$ with $j = 1(1)m$. This leads to the following unconstrained eigenvalue optimization problem:

$$\inf_{\gamma\in\mathbf{R}^m}\sigma(P(\gamma)). \tag{4.10}$$

There exist iterative methods working very satisfactorily (see Overton [9]). But in case the infimum is equal to zero, there may arise problems. The value is approached from above, and it is an unstable problem to numerically check $\sigma(P(\gamma)) \le 0$.

As in example (4.1), where $p = 1$, $H = [h_1 \cdots h_5]$, and all the components of $G = [g_1 \cdots g_5]^T$ are nonzero, we specify a handy basis $P_1, \cdots, P_{10}$ for the affine space $\mathcal{P}$. Choose

$$P_0 = \text{diag}\{h_1/g_1, \cdots, h_5/g_5\} \tag{4.11}$$

and define to each pair of indexes (j, k) $(j, k = 1(1)5, j < k)$ a matrix P_i having g_k/g_j and g_j/g_k as (j, j) and (k, k) entries, respectively, -1 as (j, k) and (k, j) entries, and zero elsewhere. Using the initial value $\gamma = 0$, a desired solution in example (4.1) is immediately given by $P = P_0$.

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Guaranteed Characterization of Stability Domains Via Set Inversion

Eric Walter and Luc Jaulin

***Abstract*—A new method is presented for characterizing the set of all values of the parameters of a linear time-invariant model that are associated with a stable behavior. A formal Routh table is used to formulate the problem as one of set inversion, which is solved approximately but globally with tools borrowed from interval analysis. The method readily extends to the design of controllers stabilizing all models in a given class.**

I. INTRODUCTION

It is well known that the stability properties of a linear time-invariant system are determined by those of its characteristic polynomial. A continuous-time system, for instance, is asymptotically stable if and only if all roots of its characteristic polynomial have strictly negative real parts. This polynomial is then said to be stable or Hurwitz. In many engineering problems, the characteristic polynomial depends on uncertain parameters of the model of the process and it is useful to characterize the stability domain, i.e., the set $\mathbb{S}$ of all values of these uncertain parameters that correspond to an asymptotically stable system (see, eg, [1] for a survey). This will be performed here in the context of continuous-time systems, but transposition to discrete-time systems is trivial. The first main feature of the approach to be proposed is that it produces guaranteed results, contrary to methods based on systematic exploration over a grid or on random scanning of the parameter space. Let $P_{\boldsymbol{p}}(s)$ be the parametrized characteristic polynomial of the system considered

$$P_{\boldsymbol{p}}(s) = \sum_{k=0}^{n} a_k(\boldsymbol{p})s^k, \tag{1}$$

where $a_n(\boldsymbol{p}) = 1$ for simplicity. The second main feature of the new approach to be described is that a_k may be any computable function of the uncertain parameters $\boldsymbol{p}$, so that the situation considered here is much more general than in Kharitonov's theorem [2]. Section II formulates the characterization of the stability domain

$$\mathbb{S} = \{\boldsymbol{p} | P_{\boldsymbol{p}} \text{ Hurwitz}\} \tag{2}$$

as a problem of *set inversion*, which can be solved with the tools of interval analysis. Section III describes an algorithm for set inversion that makes it possible to bracket $\mathbb{S}$ between two subpavings, i.e., unions of boxes in the parameter space. Two examples are used to illustrate the approach and demonstrate the efficiency of the procedure suggested.

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II. Formulation of the Problem in Terms of Set Inversion

The use of the Routh criterion to study the stability of uncertain polynomials can be traced back at least to [3], where Faedo developed an early version of interval analysis to derive sufficient conditions for stability. A formal Routh table can be constructed from $P_{\mathbf{p}}(s)$. Application of the Routh criterion then results in n inequalities that correspond to necessary and sufficient conditions for asymptotic stability in the generic case

$$f_i(\mathbf{p}) > 0, \qquad i = 1, \cdots, n. \tag{3}$$

Define $\mathbb{Y} = (]0, \infty[)^n$ and let $\mathbf{f}$ be a vector function, the coordinates of which are the f_is. The stability set is then given by

$$\mathbb{S} = \mathbf{f}^{-1}(\mathbb{Y}). \tag{4}$$

Computing $\mathbb{S}$ can thus be seen as a problem of set inversion.

Example 1: Consider the polynomial

$$P_{\mathbf{p}}(s) = s^3 + \sin(p_1p_2)s^2 + p_1^2 s + p_1p_2. \tag{5}$$

Necessary and sufficient conditions for its asymptotic stability are provided by the Routh table under the form

$$\begin{cases} a_0 > 0, \\ a_2 > 0, \\ a_1a_2 - a_0 > 0 \end{cases}, \tag{6}$$

which translates into

$$\begin{cases} p_1p_2 > 0, \\ \sin(p_1p_2) > 0, \\ p_1^2 \sin(p_1p_2) - p_1p_2 > 0. \end{cases} \tag{7}$$

Set $\mathbb{Y} = (]0, \infty[)^3$ and

$$\mathbf{f}: \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \to \begin{pmatrix} p_1p_2 \\ \sin(p_1p_2) \\ p_1^2\sin(p_1p_2) - p_1p_2 \end{pmatrix}. \tag{8}$$

The set to be characterized is then $\mathbb{S} = \mathbf{f}^{-1}(\mathbb{Y})$. ◊

Example 2: Consider a process described by the transfer function

$$\frac{B(s)}{A(s)} = \frac{k\omega_0^2}{(1+Ts)(s^2 + 2z\omega_0 s + \omega_0^2)}, \tag{9}$$

where the nominal values for the parameters are $k = 1$, $T = -1$, $z = 1$, and $\omega_0 = 1$. The nominal roots of $A(s)$ are 1, -1, and -1, so that the nominal system is open-loop unstable. The process is to be controlled with a PID controller in the forward path

$$\frac{R(s)}{S(s)} = \frac{c_1 + c_2 s + c_3 s^2}{s} \tag{10}$$

and a negative unity feedback. The characteristic polynomial of the closed-loop system is then

$$P(s) = s^4 + [2z\omega_0 + T^{-1}]s^3 + \left[\frac{2z\omega_0}{T} + \omega_0^2\left(1 + \frac{kc_3}{T}\right)\right]s^2 + \frac{\omega_0^2.(1+c_2k)}{T}s + \frac{\omega_0^2 kc_1}{T}, \tag{11}$$

and its nominal value is

$$P_n(s) = s^4 + s^3 - (1+c_3)s^2 - (1+c_2)s - c_1. \tag{12}$$

The nominal system is stabilized by setting $c_1 = -1$, $c_2 = -2$, and $c_3 = -6$. The problem to be considered is the robustness of the resulting control to an uncertainty on the value of the time constant T (assumed to differ from zero) and of the damping coefficient z. For this purpose, we want to characterize the set $\mathbb{S}$ of all pairs (z, T) such that the closed-loop system remains stable, i.e., such that

$$P_{\mathbf{p}}(s) = s^4 + (2z + T^{-1})s^3 + \left(\frac{2z-6}{T} + 1\right)s^2 - \frac{1}{T}s - \frac{1}{T} \tag{13}$$

is Hurwitz. From the Routh table, necessary and sufficient conditions for asymptotic stability are

$$\begin{cases} a_0 > 0, \\ a_3 > 0, \\ a_2a_3 - a_1 > 0, \\ a_1a_2a_3 - a_1^2 - a_0a_3^2 > 0. \end{cases} \tag{14}$$

As in Example 1, each a_i is a function of the uncertain coefficients, and characterizing $\mathbb{S}$ is a problem of set inversion. ◊

Remarks:

i) For such problems with two parameters, one can plot all curves in the parameter space on which an entry of the first column of the Routh table vanishes. These curves partition the domain of interest into regions associated with a given number of unstable roots, and the determination of the stability region is thus easy. Similar results could also be obtained by D-decomposition [4]. However, such methods do not extend easily to problems with more than two parameters, contrary to the method proposed here.

ii) The complexity of the computation of $\mathbf{f}$ increases quickly with the degree n of $P_{\mathbf{p}}(s)$, so that computer algebra [5] may be required to deal with complex cases.

iii) By performing the change of variable $s' = s + \lambda$, where λ is some known positive real, it is possible to characterize all values of the uncertain coefficients such that the degree of stability is larger than λ, i.e., that the impulse response of the system converges to zero faster than $\exp(-\lambda t)$. ◊

III. Algorithm for Set Inversion

Although $\mathbb{S}$ is exactly described by the n inequalities (3), such a description cannot be conveniently manipulated on a computer. We shall rather approximate $\mathbb{S}$ by *subpavings*, consisting of unions of axis-aligned parallelepipeds (or boxes). A prior feasible set for the parameters will be defined under the form of a box $[\mathbf{p}_{\text{init}}]$ in the parameter space. The portion of the stability set $\mathbb{S}$ contained in this prior box will then be enclosed between two subpavings $\mathbb{K}_{\text{in}}$ and $\mathbb{K}_{\text{out}}$ in the sense that

$$\mathbb{K}_{\text{in}} \subset [\mathbf{p}_{\text{init}}] \cap \mathbb{S} \subset \mathbb{K}_{\text{out}}. \tag{15}$$

Interval analysis [6] provides [7] a *sufficient* condition for all the parameters of a given box $[\mathbf{p}]$ of the parameter space to belong to $\mathbb{S}$, and a *sufficient* condition for none of them to belong to $\mathbb{S}$. For that purpose, an *inclusion function* $\mathbb{F}$ is used, i.e., a function such that for any box $[\mathbf{p}]$, $\mathbb{F}([\mathbf{p}])$ is a box that contains $\mathbf{f}([\mathbf{p}])$. From this property, one has

$$\mathbb{F}([\mathbf{p}]) \subset \mathbb{Y} \Rightarrow [\mathbf{p}] \subset \mathbb{S}, \tag{16}$$

and

$$\mathbb{F}([\mathbf{p}]) \cap \mathbb{Y} = \varnothing \Rightarrow [\mathbf{p}] \cap \mathbb{S} = \varnothing. \tag{17}$$

Using (16) and (17), the following algorithm generates two subpavings $\mathbb{K}_{\text{in}}$ and $\mathbb{K}_{\text{out}}$ that bracket $\mathbb{S}$ in the sense of (15). It uses a *stack*, which is a structure on which three operations only are possible, namely i) stacking, i.e., putting an element on top of the stack, ii) unstacking, i.e., removing the element located on top of the stack, and iii) testing the stack for emptiness. In the description of the algorithm, $w([\mathbf{p}])$ stands for the *width* of the box $[\mathbf{p}]$, i.e., the length of its largest edge(s). The program inputs are the inclusion function $\mathbb{F}$, the set to be inverted $\mathbb{Y}$, the prior box $[\mathbf{p}_{\text{init}}]$ and the maximum width ϵ_r allowed for an indeterminate box upon completion of the algorithm.

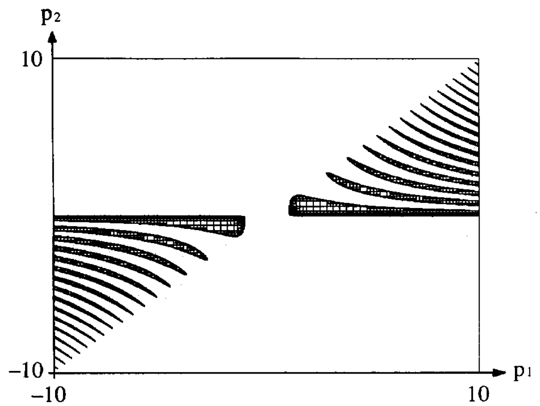

Fig. 1. Subpaving $\mathbb{K}_{\text{in}}$ of all boxes guaranteed to be stable for Example 1.

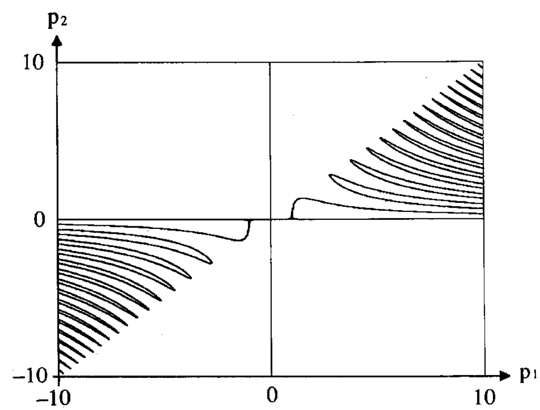

Fig. 2. Subpaving $\mathbb{K}_{\epsilon}$ of all indeterminate boxes upon completion of the algorithm for Example 1.

Initialization:

$$\text{stack} = \varnothing, \quad \mathbb{K}_{\text{in}} = \varnothing, \quad \mathbb{K}_{\text{out}} = \varnothing, \quad [\boldsymbol{p}] = [\boldsymbol{p}_{\text{init}}].$$

Iteration k:

Step 1: If $\mathbb{F}([\boldsymbol{p}]) \subset \mathbb{Y}$, then $\mathbb{K}_{\text{in}} = \mathbb{K}_{\text{in}} \cup [\boldsymbol{p}]$, $\mathbb{K}_{\text{out}} = \mathbb{K}_{\text{out}} \cup [\boldsymbol{p}]$. Go to Step 5.

Step 2: If $\mathbb{F}([\boldsymbol{p}]) \cap \mathbb{Y} = \varnothing$, then go to Step 5.

Step 3: If $w([\boldsymbol{p}]) \leq \epsilon_r$, then $\mathbb{K}_{\text{out}} = \mathbb{K}_{\text{out}} \cup [\boldsymbol{p}]$. Go to Step 5.

Step 4: Bisect $[\boldsymbol{p}]$ perpendicularly to one of its largest edges and stack the two resulting boxes.

Step 5: If the stack is not empty, then unstack into $[\boldsymbol{p}]$, and go to Step 1.
End. ◊

The convergence properties of $\mathbb{K}_{\text{in}}$ and $\mathbb{K}_{\text{out}}$ towards $\mathbb{S}$ (in Hausdorff's sense) when ϵ_r tends to zero have been studied in [7], in the case where

$$w([\boldsymbol{p}]) \to 0 \Rightarrow w(\mathbb{F}([\boldsymbol{p}])) \to 0. \qquad (18)$$

Remarks:

iv) For any vector function $\boldsymbol{f}$ obtained by composition of elementary operators such as $+, -, \times, /$, sin, cos, exp, $\cdots$, it is easy to obtain an inclusion function $\mathbb{F}$ by replacing each of these elementary operators by its minimal inclusion function in the formal expression of $\boldsymbol{f}$ [6]. As long as machine precision does not become a limiting factor, the inclusion function thus obtained satisfies (18), so that one gets better approximations by considering smaller boxes. Hence the interest of splitting indeterminate boxes into subboxes as performed by the algorithm. $\mathbb{F}$ is not uniquely defined, and its efficiency obviously depends on how tight the bounds for $\boldsymbol{f}([\boldsymbol{p}])$ given by $\mathbb{F}([\boldsymbol{p}])$ are. Tighter bounds can be obtained by taking the intersection of several inclusion functions.

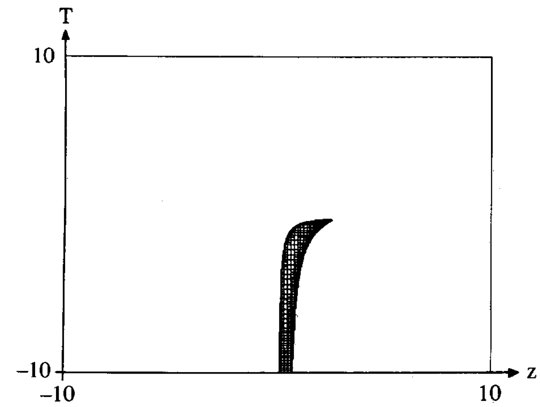

Fig. 3. Subpaving $\mathbb{K}_{\text{in}}$ of all boxes guaranteed to be stable for Example 2.

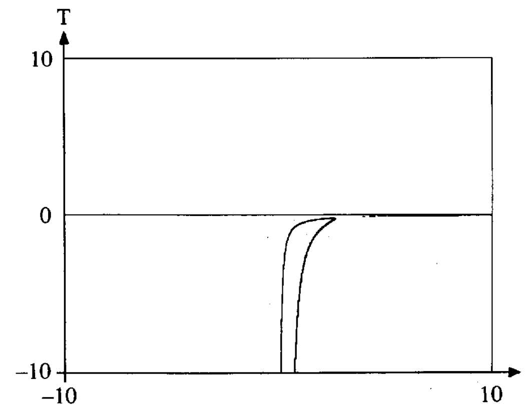

Fig. 4. Subpaving $\mathbb{K}_{\epsilon}$ of all indeterminate boxes upon completion of the algorithm for Example 2.

v) The test of Step 1 could be replaced by an application of Kharitonov's theorem on a box $\mathbb{A}([\boldsymbol{p}])$, where $\mathbb{A}$ is an inclusion function for $\boldsymbol{a}(\boldsymbol{p})$, the vector function with coordinates $a_i(\boldsymbol{p})$. The resulting test would be much stronger, which would speed up the algorithm. However, the improvement would be really significant only if the test of Step 2 could also be replaced by a test similar to that of Kharitonov which would provide simple necessary and sufficient conditions for the *instability* of the polynomial $\sum_{k=0}^{n} a_k s^k$ for all $\boldsymbol{a}$ in $[\boldsymbol{a}]$. To the best of our knowledge, such a test does not exist yet, and this calls for further research.

vi) The idea of decomposing a prior feasible set for the parameters into subboxes to be tested for stability can be found in the work by Kiendl and coworkers (see, eg, [8]), where a Lyapunov function is used to establish the stability of subboxes. ◊

Example 1 (continued): The subpaving $\mathbb{K}_{in}$ of all boxes guaranteed to be stable is presented in Fig. 1. Fig. 2 presents the subpaving $\mathbb{K}_\epsilon$ of all boxes that remain indeterminate upon completion of the algorithm. $\mathbb{K}_{out}$ is therefore the union of $\mathbb{K}_{in}$ and $\mathbb{K}_\epsilon$. These figures were drawn on-line, without storing the subpavings, for $\epsilon_r = 0.01$. The size of the stack, and thus of the memory required for the algorithm was no more than 20 boxes. The initial box $[\boldsymbol{p}_{init}]$, i.e., the search domain, was taken equal to $[-10, 10] \times [-10, 10]$. The actual computing time on a Compaq 386/33 was less than 8 minutes. Note that $\mathbb{K}_{out}$ contains the axes of the parameter space, contrary to $\mathbb{K}_{in}$, although these axes do not belong to $\mathbb{S}$. When ϵ_r tends to zero, $\mathbb{K}_{out}$ does not converge (in Hausdorff sense) to $\mathbb{S}$ because of a discontinuity of $\boldsymbol{f}^{-1}$ around $\mathbb{Y} = (]0, \infty[)^3$ [7]. If, for instance, p_1 is zero, then $\boldsymbol{p} \notin \mathbb{S}$, and it is easy to prove that $\boldsymbol{p}$ has a neighborhood that does not intersect $\mathbb{S}$. However, $\boldsymbol{f}(\boldsymbol{p})$ is infinitely close to $\mathbb{Y}$, as, for instance, the polynomial $s^3 + \epsilon s^2 + 2\epsilon s + \epsilon^2$ is stable for any $\epsilon > 0$. The image manifold of $\boldsymbol{f}$ comes tangent to $\mathbb{Y}$ without penetrating it locally.

Example 2 (continued): The subpavings $\mathbb{K}_{in}$ and $\mathbb{K}_\epsilon$ obtained in 145 seconds for $\epsilon_r = 0.01$ are presented on Figs. 3 and 4. The initial search domain was the box $[-10, 10] \times [-10, 10]$. As for Example 1, $\mathbb{K}_{out}$, the union of $\mathbb{K}_{in}$ and $\mathbb{K}_\epsilon$, does not converge to $\mathbb{S}$. It contains the axis $T = 0$, on which $\boldsymbol{f}$ is not defined. The trail just below the axis $T = 0$ on the right of Fig. 4 disappears when the precision is increased.

IV. Conclusions

Finding the stability region of a parameterized polynomial is a key problem of automatic control. It has received considerable attention in recent years, especially after the seminal work of Kharitonov. What has been proposed in this note is a new approach to that problem, combining formal treatments to obtain inequalities that define $\mathbb{S}$ and the use of interval analysis to bracket it. This approach has the advantage over Kharitonov-based approaches (see, e.g., [9] for recent results) of not being limited to very special types of dependency of the coefficients of the characteristic polynomial with respect to the uncertain parameters. It also has the definite advantage over methods based on random (or systematic) sampling of the parameter space of providing guaranteed results. Upon completion of the algorithm, the domain of interest has been partitioned into a region where the system is guaranteed to be stable, one where it is guaranteed to be unstable and an indeterminate region that can be reduced by decreasing the maximum width acceptable for an indeterminate box, at the cost of course of more intensive computation. Systematic use of the notion of inclusion function has thus made it possible to achieve grid avoidance, one of the main reasons for the success of Kharitonov's theorem among control engineers. The method readily extends to the design of controllers stabilizing all models in a given class, by considering boxes in the cartesian product of the model and controller parameter spaces.

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Production Control of Manufacturing Systems with Production Rate-Dependent Failure Rates

George Liberopoulos and Michael Caramanis

***Abstract*—It is known that for single-part-type production systems with homogeneous Markovian machine failure rates, special single threshold feedback policies, called hedging point policies, are optimal, and the stationary probability distribution of the part-type surplus, for given tentative hedging point values, can be obtained analytically. This approach is extended to multiple threshold policies with production rate-dependent machine failure rates. It is shown that the stationary distribution of the part-type surplus can be obtained under the extended policy and in the presence of production rate-dependent failure rates. The advantage of multiple threshold policies is that they can provide a piecewise constant (step function) approximation of any feedback policy. It is observed that hedging point policies are not always optimal and, in fact, feedback policies are not always optimal either.**

I. Introduction

There has been increasing interest in control theoretic approaches to the production scheduling of failure-prone manufacturing systems. Kimemia and Gershwin [10] were among the first to formulate the production control of a manufacturing system with stochastic capacity

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